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by

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ON PARACOMPACT REMAINDERS

HEIKKI JUNNILA

Abstract. We study Tihonov spaces X which are *paracompact at infinity* (i.e., $\beta X \setminus X$ is paracompact). We characterize paracompactness at infinity of a nowhere locally compact Tihonov space, and we give several examples relating to paracompactness at infinity. We also consider strong paracompactness at infinity. We construct a space which is strongly paracompact at infinity but which also has a non-strongly paracompact remainder. We use this space to solve a problem on "paracompactly placed" sets posed by V.I. Ponomarev in 1962.

1. Introduction and notation

A *space* in this paper means a Tihonov space.

The *remainder* of a space X in a compactification K of X is the subspace $K \setminus X$ of K. We say that a space Z is a *remainder* of X provided that Z is the remainder of X in some compactification of X. The *Čech-Stone remainder* of X is the remainder of X in the Čech-Stone compactification βX of X; this remainder of X is denoted by X^*.

According to terminology introduced by Henriksen and Isbell in [16], a space X has *property P at infinity* if X^* has property P. The paper [16] contains the following characterization of Lindelöfness at infinity: X is Lindelöf at infinity if, and only if, every compact set $K \subset X$ is contained in a compact set $C \subset X$ such that C has a countable outer neighborhood base in X. As a consequence, every metrizable space is Lindelöf at infinity.

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Any remainder of X is the image of X^* under a perfect mapping. It follows that if X has property P at infinity and if P is preserved by perfect mappings, then every remainder of X has P. On the other hand, X has property P at infinity provided that some remainder of X has P and P is inversely preserved by perfect mappings.

The previous observations apply best to those properties which are both preserved and inversely preserved under perfect mappings. Many covering properties, such as (countable) compactness, Lindelöfness, paracompactness and metacompactness are preserved both ways under perfect mappings. Thus we see, for instance, that all remainders of X are paracompact provided that some remainder of X is paracompact

With some other properties, we have to be more careful. For example, it is well known and easy to see that strong paracompactness is inversely preserved by perfect mappings. It follows that X is strongly paracompact at infinity if some remainder of X is strongly paracompact. However, strong paracompactness is not preserved by perfect mappings, and it is not obvious whether strong paracompactness of infinity of X is enough to make all remainders of X strongly paracompact. We shall consider this question below.

Many of our results deal with spaces in which no point has a compact neighborhood. Previous work on remainders has shown that "nowhere locally compact" spaces are often easier to handle than general spaces. The main reason for this is that the remainder of a nowhere locally compact space in a compactification is dense in the compactification. As a consequence of this we see, for example, that a nowhere locally compact space X is ccc if, and only if, X is ccc at infinity.

In Section 2, we consider the property of paracompactness at infinity. We give an internal characterization of this property for nowhere locally compact spaces. We consider the preservation of paracompactness at infinity under topological operations. In Section 3, we give examples of spaces which are paracompact at infinity and of spaces which lack this property. In Section 4, we consider strong paracompactness of remainders. We construct an example of a space X which is strongly paracompact at infinity but which also has a non-strongly paracompact remainder. We use this example to solve a problem of Ponomarev [24] on sets which are "paracompactly placed" in compactifications.

Notation and terminology. In the terminology of Arhangel'skii ([1]) a space X is *of countable type* provided that every compact subset of X is contained in a compact subset which has a countable outer neighborhood base in X, and X is *of pointwise countable type* provided that every point of X belongs to a compact subset which has a countable outer neighborhood base in X. With this terminology, a result of Henriksen and Isbell mentioned above can be stated as follows: *a space is Lindelöf at infinity if, and only if, the space is of countable type.*

A family $\mathcal{L}$ of sets is a *partial refinement* of a family $\mathcal{N}$ provided that, for every $L \in \mathcal{L}$, there exists $N \in \mathcal{N}$ such that $L \subset N$.

For the definition of a "stratifiable" space, see [11] (where these spaces are called "M_3-spaces") and [8]. For definitions of other undefined terms, see [13], [10] and [15].

2. Paracompactness at infinity

In this section, we study paracompactness at infinity. The stronger property of Lindelöfness at infinity has been extensively studied after Henriksen and Isbell characterized the property in 1958. We note here that in many circumstances paracompactness at infinity implies Lindelöfness at infinity. For example, Arhangel'skii and Tokgoz observed in [7] that if X is a ccc nowhere locally compact space, which is paracompact at infinity, then X is Lindelöf at infinity.

In our first result, we give a sufficient condition for paracompactness at infinity.

Let $\mathcal{L}$ be a family of subsets of a space X, and let $F \subset X$. We write $\mathcal{L} \hookrightarrow F$ provided that, for every open set G with $F \subset G$, we have $L \subset G$ for all but finitely many $L \in \mathcal{L}$. The condition $\mathcal{L} \hookrightarrow F$ is trivial if $\mathcal{L}$ is finite, but if $\mathcal{L}$ is an infinite family of open sets, then $\mathcal{L} \hookrightarrow F$ if, and only if, every infinite subfamily of $\mathcal{L}$ is a π-network at F. When $\mathcal{L}$ is an infinite family of open sets, then $\mathcal{L} \hookrightarrow F$ if, and only if, $\mathcal{L}$ is a "strong π-base at F", in the terminology introduced by Arhangel'skii in [4].

Lemma 2.1. *Assume that, for every compact subset K of X, there exists a compact subset C of X such that $K \subset C$ and a cover $\mathcal{H}$ of $X \setminus C$ such that $\mathcal{H} \hookrightarrow C$ and $\overline{H} \cap K = \emptyset$ for every $H \in \mathcal{H}$. Then X is paracompact at infinity.*

Proof. In the following proof, $\overline{A}$ denotes the closure of a set $A \subset \beta X$ in the space βX. To show that X^* is paracompact, let $\mathcal{N}$ be an open cover of X^*. By regularity, there exists an open cover $\mathcal{L}$ of X^* such that the family $\{\mathrm{Cl}_{X^*}(L) : L \in \mathcal{L}\}$ refines $\mathcal{N}$. For each $L \in \mathcal{L}$, let U_L be an open subset of βX with $U_L \cap X^* = L$. Note that $K = \beta X \setminus \bigcup_{L \in \mathcal{L}} U_L$ is a compact subset of X. By our assumption, there exists a compact subset C of X with $K \subset C$ and a cover $\mathcal{H}$ of $X \setminus C$ such that $\mathcal{H} \hookrightarrow C$ and, for every $H \in \mathcal{H}$, we have $\mathrm{Cl}_X(H) \cap K = \emptyset$. Note that, since $K \subset X$, we have $\overline{H} \cap K = \emptyset$ for each $H \in \mathcal{H}$.

The family $\mathcal{H}$ is locally finite at every point of $\beta X \setminus C$. To see this, let $z \in \beta X \setminus C$. The point z has an open neighborhood V in βX such that $\overline{V} \cap C = \emptyset$. Since $\mathcal{H} \hookrightarrow C$, the family $\{H \in \mathcal{H} : H \not\subset \beta X \setminus \overline{V}\}$ is finite. This shows that $\mathcal{H}$ is locally finite at z.

By the foregoing, the family $\overline{\mathcal{H}} = \{\overline{H} : H \in \mathcal{H}\}$ is locally finite, and hence closure-preserving, at every point of $\beta X \setminus C$. It follows that $\overline{\mathcal{H}}$ covers $\beta X \setminus C$, because for every $z \in \beta X \setminus C$, we have that

$$z \in \beta X \setminus C \subset \overline{X \setminus C} = \overline{\bigcup \mathcal{H}} = \overline{\bigcup \overline{\mathcal{H}}}\,.$$

For every $H \in \mathcal{H}$, we have $\overline{H} \subset \beta X \setminus K$. It follows that every $F \in \overline{\mathcal{H}}$ is contained in the set $\beta X \setminus K = \bigcup_{L \in \mathcal{L}} U_L$. Since each $F \in \overline{\mathcal{H}}$ is closed and each U_L, $L \in \mathcal{L}$, is open (in βX), it follows that every $F \in \overline{\mathcal{H}}$ is covered by finitely many sets U_L, $L \in \mathcal{L}$.

The family $\mathcal{J} = \{F \cap X^* : F \in \overline{\mathcal{H}}\}$ is a locally finite closed cover of X^* and for every $J \in \mathcal{J}$, there exists a finite subfamily $\mathcal{L}_J$ of $\mathcal{L}$ such that $J \subset \bigcup \mathcal{L}_J$. It is easy to see that the family $\{J \cap \mathrm{Cl}_{X^*}(L) : J \in \mathcal{J} \text{ and } L \in \mathcal{L}_J\}$ is a locally finite and closed (in X^*) refinement of $\mathcal{N}$.

We have shown that every open cover of X^* has a locally finite closed refinement. By [20, Lemma 1], the space X^* is paracompact. □

Remark. If the space X in the lemma has ccc, then the proof shows that X^* is Lindelöf: the locally finite closed cover $\{\overline{\mathcal{H}} \setminus C : H \in \mathcal{H}\}$ of the open subspace $X \setminus C$ of X has a countable subcover.

It has been remarked in [3] and [7] that there does not exist an internal characterization of paracompactness at infinity in the literature. We shall now give such a characterization for those spaces in which no point has a compact neighborhood.

Theorem 2.2. *The following are equivalent for a nowhere locally compact space X:*

1. *X is paracompact at infinity.*
2. *For every compact $K \subset X$, there exists a compact $C \subset X$ such that $K \subset C$ and an open cover $\mathcal{H}$ of $X \setminus C$ such that $\mathcal{H} \hookrightarrow C$ and $\overline{H} \cap K = \emptyset$ for every $H \in \mathcal{H}$.*
3. *For every compact $K \subset X$, there exists a compact $C \subset X$ such that $K \subset C$ and a cover $\mathcal{H}$ of $X \setminus C$ such that $\mathcal{H} \hookrightarrow C$ and $\overline{H} \cap K = \emptyset$ for every $H \in \mathcal{H}$.*

Proof. We have (3)⇒(1) by Lemma 2.1. To prove that (1)⇒(2), let X be a nowhere locally compact space which is paracompact at infinity. It is a consequence of nowhere local compactness of X that X^* is dense in βX. Let K be a compact subset of X. For every $z \in X^*$, there exists an open neighborhood U_z of z in βX such that $\overline{U_z} \cap K = \emptyset$. Let $\mathcal{U} = \{U_z : z \in X^*\}$. Since X^* is a paracompact and dense subspace of βX, [28, Theorem 2.8] shows that there exists an open (in βX) partial refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ covers X^* and $\mathcal{V}$ is locally finite at every point of $\bigcup \mathcal{V}$.

Let $C = \beta X \setminus \bigcup \mathcal{V}$, and note that C is a closed subset of βX contained in X. Hence C is a compact subset of X. Also note that $K \subset C$. We show that $\mathcal{V} \hookrightarrow C$ in βX. Let G be an open subset of βX with $C \subset G$. Let $F = \beta X \setminus G$, and note that F is closed, and hence compact, and $F \subset \bigcup \mathcal{V}$. The family $\mathcal{V}$ is locally finite at every point of F, and it follows by compactness that F meets only finitely many sets of $\mathcal{V}$. As a consequence, we have $V \subset G$ for all but finitely many $V \in \mathcal{V}$. We have shown that $\mathcal{V} \hookrightarrow C$ in βX. From this it follows that we have $\mathcal{W} \hookrightarrow C$ in X, where $\mathcal{W} = \{V \cap X : V \in \mathcal{V}\}$. Moreover, $\mathcal{W}$ is an open cover of $X \setminus C$ and we have $\mathrm{Cl}_X(W) \cap K = \emptyset$ for every $W \in \mathcal{W}$. We have shown that X satisfies condition (2). □

Example 3.4 below shows that the theorem does not hold with "nowhere locally compact" omitted.

Since a non-locally compact topological group is nowhere locally compact, Theorem 2.2 provides one solution to [3, Problem 4.16].

We close this section with some remarks on the preservation of paracompactness at infinity in topological operations. The following three preservation results are special cases of [16, Theorems 2.7, 2.8 and 3.10]: If a space is paracompact at infinity, then every closed subspace of the space shares this property; If Y is the image of X under a perfect mapping, then X is paracompact at infinity if, and only if, Y is paracompact at infinity; A direct sum of spaces is paracompact at infinity provided that each summand has this property.

In their study of Lindelöfness at infinity, Henriksen and Isbell showed that the product of countably many Lindelöf at infinity spaces is Lindelöf at infinity. For paracompactness at infinity, the corresponding result fails, even with products of two factors.

We denote by H the subspace $\{0\} \cup \{\frac{1}{n} + \frac{1}{k} : n, k \in \mathbb{N} \text{ and } k > n^2\}$ of $\mathbb{R}$. Note that the subspace $K = H \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of $\mathbb{R}$ is a compactification of H.

Proposition 2.3. *Assume that $Y \times H$ is paracompact at infinity. Then Y is Lindelöf at infinity.*

Proof. Let K be the compactification of H mentioned above, and denote by L the (paracompact) remainder of $Y \times H$ in $\beta Y \times K$. The closed subspace $Y^* \times \{0\}$ of L is homeomorphic with Y^*. As a consequence, Y^* is paracompact. To show that Y is Lindelöf at infinity, assume on the contrary that Y^* is not Lindelöf. Since Y^* is paracompact and non-Lindelöf, Y^* has an uncountable closed discrete subset D. The set $D \times \{0\}$ is closed and discrete in L and it follows, since L is paracompact, that there exists a discrete family $\{G_d : d \in D\}$ of open subsets of L such that $d \in G_d$ for each $d \in D$. For every $d \in D$, there exists $\ell_d \in \mathbb{N}$ such that

$(d, \frac{1}{\ell_d}) \in G_d$. Since D is uncountable, there exists $\ell \in \mathbb{N}$ such that the set $E = \{d \in D : \ell_d = \ell\}$ is infinite. The infinite set $M = \{(e, \frac{1}{\ell}) : e \in E\}$ is a closed and discrete subset of L, but this is a contradiction, since M is contained in the compact subset $\beta Y \times \{\frac{1}{\ell}\}$ of L. □

In Example 3.4 below, we construct a separable stratifiable space X such that X is strongly paracompact at infinity but not Lindelöf at infinity. It is a consequence of Proposition 2.3 that the product $X \times H$ fails to be paracompact at infinity even though both factors have this property.

For nowhere locally compact spaces, we can obtain some sufficient conditions for the preservation of paracompactness at infinity in finite and countable products.

Recall that a space X is *σ-metacompact* if every open cover of X has a σ-point finite open refinement.

Proposition 2.4. *Let I be a finite (a countable) set and let $X = \prod_{i \in I} X_i$, where each X_i is a nowhere locally compact space which is paracompact at infinity. Then X is (σ-)metacompact at infinity. If X is normal (and countably paracompact) at infinity, then X is paracompact at infinity.*

Proof. For each $i \in I$, let C_i be a Hausdorff compactification of X_i, and denote by Z_i the paracompact remainder $C_i \setminus X_i$; note that, by nowhere local compactness of X_i, the remainder Z_i is dense in C_i. For all $i, j \in I$, let $Y_{i,j} = C_i$ if $i \neq j$ and $Y_{i,j} = Z_i$ if $i = j$. Note that, for each $j \in I$, the subspace $W_j = \prod_{i \in I} Y_{i,j}$ of $\prod_{i \in I} X_i$ is paracompact and dense. Moreover, the remainder of $\prod_{i \in I} X_i$ in the compactification $\prod_{i \in I} C_i$ can be written as $\bigcup_{j \in I} W_j$. The desired conclusions now follow from [6, Theorems 2.1, 2.2, 2.3 and 2.5] (note that [6, proof of Theorem 2.3] actually establishes σ-metacompactness of X instead of meta-Lindelöfness). □

3. Examples

In this section, we indicate (classes of) spaces which are paracompact at infinity and spaces which fail to have this property.

Theorem 3.1. ([7]) *Every space with a point-countable base and every p-space is Lindelöf at infinity.*

These results, due to Arhangel'skii and Tokgoz, generalize earlier results of Henriksen and Isbell: in [16] it was shown that metrizable spaces, and their perfect preimages (i.e., paracompact p-spaces), are Lindelöf at infinity.

Henriksen and Isbell showed in [16] that every first countable linearly orderable space is paracompact at infinity. By modifying the proof given in [16] and applying a deep result of M.E. Rudin, Junnila and Nyikos established the following strengthening of Henriksen's and Isbell's result.

Theorem 3.2. ([19]) *Let X be a space of pointwise countable type. If X has a monotonically normal compactification, then X is strongly paracompact at infinity.*

Every suborderable space has a linearly orderable compactification, and such a compactification is monotonically normal. It follows that every suborderable space of pointwise countable type is strongly paracompact at infinity.

With the help of an important theorem of Balogh and Rudin, Junnila and Nyikos extended the result of Theorem 3.2, with "strongly" omitted, to all spaces which have a monotonically normal remainder.

Theorem 3.3. ([19]) *Let X be a space of pointwise countable type. Then every monotonically normal closed subspace of a remainder of X is paracompact. In particular, if X has a monotonically normal remainder, then X is paracompact at infinity.*

Theorem 2.2 above characterized paracompactness at infinity of a nowhere locally compact space. We now give an example to show that the assumption of nowhere local compactness cannot be omitted in Theorem 2.2.

Example 3.4. There exists a separable first countable stratifiable space which is strongly paracompact at infinity but not Lindelöf at infinity.

Construction. Topologize $\mathbb{R}$ as follows. Make every $q \in \mathbb{Q}$ isolated. For every x in the set $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$, choose a sequence $\langle q_n(x) \rangle_{n=1}^{\infty}$ from $\mathbb{Q}$ converging to x and let $S(x) = \{q_n(x) : n \in \mathbb{N}\}$; basic neighborhoods of x are the sets $G \setminus S(x)$ where G is a usual Euclidean neighborhood of x.

Denote the above space by X, and note that X is a separable first countable stratifiable space (see [17, Example 4.6]).

Let $\mathbb{P}' = \{x' : x \in \mathbb{P}\}$ be a disjoint "copy" of $\mathbb{P}$, and for each $A \subset \mathbb{R}$, let $A' = \{x' : x \in A \cap \mathbb{P}\}$. Topologize $Z = X \cup \mathbb{P}'$ so that every $q \in \mathbb{Q}$ is isolated, basic neighborhoods of $x \in \mathbb{P}$ are the sets $V \cup \big(V' \setminus \{x'\}\big)$, where V is a neighborhood of x in X, and a point $x' \in \mathbb{P}'$ has a neighborhood base by sets $\{x'\} \cup \{q_n(x) : n \geq k\}$, where $k \in \mathbb{N}$.

The mapping $\varphi : Z \to \mathbb{R}$, which is the identity on X and sends each x' to x, is a perfect mapping from Z onto the rational extension of $\mathbb{R}$. Hence Z is a Lindelöf p-space, and the Henriksen-Isbell result shows that Z^* is Lindelöf. The space βZ is a compactification of X and the remainder $\beta Z \setminus X$ is the subspace $Z^* \cup \mathbb{P}'$ of βZ. Every point of $\mathbb{P}'$ is isolated in $Z^* \cup \mathbb{P}'$ since the point has a compact neighborhood in Z. It follows, since Z^* is Lindelöf, that the remainder $Z^* \cup \mathbb{P}'$ of X is strongly paracompact.

To see that X is not Lindelöf at infinity, let C be an uncountable compact subset of $\mathbb{P}$. Then the set $C \cup C'$ is compact in Z and it follows that C' is an uncountable closed discrete subset of the remainder $\beta Z \setminus X$.

Since X is separable but not Lindelöf at infinity, the remark made after the proof of Lemma 2.1 shows that X does not satisfy the condition of the lemma. This shows that the assumption of nowhere local compactness of the space is essential in Theorem 2.2. □

Next we give some examples of spaces which fail to be paracompact at infinity. The first examples are stratifiable spaces.

Example 3.5. The following countable spaces are non-paracompact at infinity:

(1) For every $p \in \mathbb{N}^*$, the subspace $\{p\} \cup \mathbb{N}$ of $\beta\mathbb{N}$. (see [16])
(2) The quotient of $\mathbb{Q}$ obtained by identifying the points of the subset $\mathbb{N}$. (see [7])
(3) The "sequential fan" $S(\omega)$. (see Example 3.9 below)

Example 3.6. There exist stratifiable topological groups which are non-metacompact at infinity.

Proof. Let G be a topological group. Arhangelskii showed in [3] that G is a paracompact p-space provided that G has a Lindelöf remainder. In [4], he showed that G has either a Lindelöf remainder or a pseudocompact remainder. Recall that a stratifiable p-space is metrizable (see [8, Section 8]) and a pseudocompact metacompact space is compact ([25] and [27]). It follows from the stated results that whenever G is a non-metrizable stratifiable topological group, then G is not metacompact at infinity. For examples of non-metrizable stratifiable topological groups and topological linear spaces, see [9] and [26]; in [14] it is shown that the (non-metrizable) topological linear space $C_k(\mathbb{P})$ is stratifiable. □

Example 3.7. A separable first countable stratifiable space which is not paracompact at infinity.

Proof. We consider the "bow-tie" space X (see [13, Exercise 3.1.I]), and we note that X is separable. In [23], it is shown that X is first countable and stratifiable (see also [11, Example 9.2]). The ground-set of X is the plane, the points off the x-axis have their usual Euclidean neighborhoods, and a point $(x, 0)$ has a neighborhood base by "bow-tie" sets

$$G_r(x,0) = \{(x,0)\} \cup \left[B((x,0),r) \setminus \left(\bar{B}((x,\tfrac{1}{r}),\tfrac{1}{r}) \cup \bar{B}((x,-\tfrac{1}{r}),\tfrac{1}{r})\right)\right],$$

where $r > 0$, and $B(\cdot,\cdot)$ and $\bar{B}(\cdot,\cdot)$ denote open and closed circular disks, respectively.

We let $Y = \{(z, u) \in X : u \in \mathbb{Q}\}$, and we note that Y is nowhere locally compact. We show that Y is not Lindelöf at infinity. Assume on the contrary that Y^* is Lindelöf. Then the compact subset $K = [0,1] \times \{0\}$ is contained in a compact subset C of Y such that C has a countable outer base in Y. Since Y is stratifiable, the compact subspace C is metrizable. It follows, by [1, Proposition 3.3], that K has a countable outer base $\{G_n : n \in \mathbb{N}\}$ in Y. For every $n \in \mathbb{N}$, let U_n be the interior of G_n in the Euclidean topology of Y, and note that $U_n \cap K$ is dense in K. By the Baire Category Theorem, there exists a point $(a, 0) \in \bigcap_{n \in \mathbb{N}} U_n \cap K$. Now every U_n is a Euclidean neighborhood of $(a, 0)$, but this is a contradiction, since one of the sets $U_1, U_2, \dots$ must be contained in the neighborhood $G_2(a, 0)$ of the set K.

By the foregoing, Y fails to be Lindelöf at infinity. Since Y is separable and nowhere locally compact, [7, Proposition 2.1] shows that Y is not paracompact at infinity.

Another example of a separable first countable stratifiable space which is not paracompact at infinity is indicated after Example 4.3 below. □

In Example 3.6 above, the reason for non-metacompactness at infinity was pseudocompactness at infinity. We can obtain more examples of non-metacompactness at infinity when we recall other situations indicated in the literature where remainders are pseudocompact, or even ω-bounded. Recall that a space is *ω-bounded* if the closure of every countable subset is compact; this is a strong form of countable compactness.

Assume that a space X has a closed subspace T which is not locally compact but is countably compact at infinity. Let R be a remainder of X in some compactification. Then R contains a closed copy of a remainder of T, and hence R has a closed countably compact non-compact subspace. As a consequence, R fails to have many familiar covering properties, such as paracompactness, metacompactness, weak submetaLindelöfness,... Also, R fails to be a D-space, in the sense of van Douwen [12].

Let Y be a non-locally compact space which is countably compact at infinity. By [2, Theorem 2.17], Y is not first countable; the same proof shows that Y is not of pointwise countable type. The paper [2] also gives sufficient conditions for extremally disconnected spaces to be countably compact at infinity.

A point y of a space X is a *P-point* of X provided that any intersection of countably many neighborhoods of y is a neighborhood of y.

Arhangel'skii and Bella have made the following observation in [5].

Proposition 3.8. *Assume that every point of X either is a P-point or has a compact neighborhood. Then X is ω-bounded at infinity.*

Note that every non-metrizable ω_μ-metrizable space satisfies the condition above. Hence such a space is ω-bounded at infinity. As a particular case, we see that $L(\omega_1)$, the one-point Lindelöfication of ω_1 with discrete topology, is ω-bounded at infinity. The last fact also follows from the observation that $L(\omega_1)$ is homeomorphic with the subspace of the compact ordinal space $\omega_1 + 1$ which consists of all successor ordinals and the ordinal ω_1; the remainder of this subspace in $\omega_1 + 1$ is the subspace of the ordinal space ω_1 consisting of all limit ordinals; moreover, the latter subspace is homeomorphic with the ω-bounded ordinal space ω_1.

The above observation on $L(\omega_1)$ shows that this space is suborderable, and a simple modification of the order on ω_1 shows that $L(\omega_1)$ is actually linearly orderable. Hence we see that first countability is an essential assumption in the result of [16] that linearly ordered first countable spaces are paracompact at infinity.

Since a linearly orderable space has a linearly orderable compactification, $L(\omega_1)$ is an example of a space which is not metacompact at infinity even though it has a monotonically normal and orthocompact remainder.

Yet another instance of ω-boundedness at infinity is provided by the "sequential fan" $S(\omega)$. The space $S(\omega)$ is the quotient of the direct sum Z of countably many copies of the convergent sequence $\omega + 1$, obtained by identifying all non-isolated points of Z. This is a well-known example of a countable space which is not first countable.

Here we represent $S(\omega)$ as the space $\omega \times \omega \cup \{\infty\}$, where each point of $\omega \times \omega$ is isolated, and the point ∞ has a neighborhood base $\{U_f : f \in \omega^\omega\}$, where $U_f = \{\infty\} \cup \{(n,k) \in \omega \times \omega : k > f(n) \text{ for every } n \in \omega\}$.

Example 3.9. The space $S(\omega)$ is ω-bounded at infinity.

Proof. Every $p \in S(\omega) \setminus \{\infty\}$ is isolated in $\beta S(\omega)$, and hence ω-boundedness of $S(\omega)^*$ follows when we show that ∞ is a P-point in the subspace $S(\omega)^* \cup \{\infty\}$ of $\beta S(\omega)$.

Let V_n be a neighborhood of ∞ in $S(\omega)^* \cup \{\infty\}$ for every $n \in \omega$. For every $n \in \omega$, there exists $f_n \in \omega^\omega$ such that $\mathrm{Cl}_{\beta S(\omega)}(U_{f_n}) \setminus S(\omega) \subset V_n$. Let $f \in \omega^\omega$ be such that $f_n \prec f$ for every $n \in \omega$. It is easy to see that the set $W = \big(\mathrm{Cl}_{\beta S(\omega)}(U_f) \setminus S(\omega)\big) \cup \{\infty\}$ is a neighborhood of ∞ in $S(\omega)^* \cup \{\infty\}$ and $W \subset \bigcap_{n\in\omega} V_n$. $\square$

4. Strong paracompactnes at infinity and Ponomarev's problem

The conditions for paracompactness at infinity appearing in Theorem 2.2 are essentially weaker than the Henriksen-Isbell condition for Lindelöfness at infinity. To see this, let $C \supset K$ be a compact set with a countable outer neighborhood base. Then C has an outer neighborhood base $\{V_n : n = 1, 2, \ldots\}$ such that $\overline{V_{n+1}} \subset V_n$ for each n.

If we set $V_0 = X$, then the family $\mathcal{H} = \{V_n \setminus \overline{V_{n+2}} : n = 0, 1, 2, \ldots\}$ is an open cover of $X \setminus C$, and it is easy to see that $\mathcal{H} \hookrightarrow C$. Moreover, we have $\overline{H} \cap C = \emptyset$ for each $H \in \mathcal{H}$. This raises the question whether it would be possible to strengthen "$\overline{H} \cap K = \emptyset$" to "$\overline{H} \cap C = \emptyset$" also in Conditions B and C of Theorem 2.2. However, this is not possible: we shall see below that the strengthened conditions hold exactly in the situation where the remainders of X are "paracompactly placed" in the compactifications.

In the terminology introduced by Ponomarev in [24], a subset A of a space Z is *paracompactly placed* in Z provided that every open subset of Z containing A contains a paracompact open set containing A.

Proposition 4.1. *The following conditions are mutually equivalent for a space X:*

(1) *For every compactification T of X, the remainder $T \setminus X$ is paracompactly placed in T.*
(2) *There exists a compactification T of X such that $T \setminus X$ is paracompactly placed in T.*
(3) *For every compact subset K of X, there exists a compact subset C of X such that $K \subset C$ and an open cover $\mathcal{H}$ of $X \setminus C$ such that $\mathcal{H} \hookrightarrow C$ and $\overline{H} \cap C = \emptyset$ for every $H \in \mathcal{H}$.*
(4) *For every compact subset K of X, there exists a compact subset C of X such that $K \subset C$ and a cover $\mathcal{H}$ of $X \setminus C$ such that $\mathcal{H} \hookrightarrow C$ and $\overline{H} \cap C = \emptyset$ for every $H \in \mathcal{H}$.*

Proof. (2)⇒(3): Assume that (2) holds. To show that (3) holds, let $K \subset X$ be compact. Then the set $O = T \setminus K$ is an open subset of T containing $T \setminus X$. Since $T \setminus X$ is paracompactly placed in T, there exists a paracompact open subspace U of T such that $T \setminus X \subset U \subset O$. Let $C = T \setminus U$, and note that C is a compact subset of X and $K \subset C$. By paracompactness of U and regularity of T, there exists a locally finite open cover $\mathcal{J}$ of U such that $\mathrm{Cl}_T J \subset U$ for every $J \in \mathcal{J}$. Let $\mathcal{H} = \{J \cap X : J \in \mathcal{J}\}$, and note that $\mathcal{H}$ is an open cover of $X \setminus C$ and $\mathrm{Cl}_X H \cap C = \emptyset$ for each $H \in \mathcal{H}$. To show that $\mathcal{H} \hookrightarrow C$, let G be open subset of X such that $C \subset G$. Let $\hat{G}$ be an open subset of T such that $\hat{G} \cap X = G$. Then $T \setminus \hat{G}$ is a compact subset of U and it follows, since $\mathcal{J}$ is locally finite in U, that the family $\mathcal{J}' = \{J \in \mathcal{J} : J \cap (T \setminus \hat{G}) \neq \emptyset\}$ is finite. For every $J \in \mathcal{J} \setminus \mathcal{J}'$, we have $J \subset \hat{G}$ and hence $J \cap X \subset G$. As a consequence, $H \subset G$ for all but finitely many $H \in \mathcal{H}$.

(4)⇒(1): Assume that X satisfies Condition (4). To show that (1) holds, let T be a compactification of X. We show that the set $T \setminus X$ is paracompactly placed in T. Let G be an open subset of T such that $T \setminus X \subset G$.

Then $K = T \setminus G$ is a compact subset of X and hence there exists a compact subset C of X and a cover $\mathcal{H}$ of $X \setminus C$ such that $\mathcal{H} \hookrightarrow C$ and $\mathrm{Cl}_X H \cap C = \emptyset$ for every $H \in \mathcal{H}$. Since $C \subset X$, we have $\mathrm{Cl}_T H \cap C = \emptyset$ for each $H \in \mathcal{H}$.

Let $\overline{\mathcal{H}} = \{\mathrm{Cl}_T H : H \in \mathcal{H}\}$. Since $\bigcup \overline{\mathcal{H}} \subset T \setminus C$, arguments from the proof of Lemma 2.1 (with T substituted for βX) show that $\overline{\mathcal{H}}$ is a locally finite cover of the subspace $T \setminus C$. Hence $T \setminus C$ has a locally finite cover by compact closed sets, and it follows by [20, Lemma 1] that $T \setminus C$ is paracompact. The set $T \setminus C$ is a paracompact open subset of T containing $T \setminus X$ and contained in the set G. □

Morita has shown in [22] that every locally compact paracompact space is strongly paracompact. Hence, as observed by Ponomarev in [24], every paracompactly placed subset of a compact space is strongly paracompact. We therefore have the following consequence of Proposition 4.1.

Corollary 4.2. *Assume that some remainder of X is paracompactly placed in the respective compactification. Then every remainder of X is strongly paracompact.*

As noted in the introduction, a space is strongly paracompact at infinity whenever the space has some compactification with strongly paracompact remainder. We shall next give an example to show that strong paracompactness at infinity does not imply that all remainders would have to be strongly paracompact. It turns out that our example, together with the previous results, gives the solution to a problem raised by Ponomarev. He showed in [24] that a strongly paracompact space Y is paracompactly placed in βY, and he asked whether βY can be replaced by an arbitary compactification of Y.

Before giving the promised example, we present a “non-example”:

Example 4.3. A space X with a strongly paracompact remainder $C \setminus X$ which is not paracompactly placed in the compactification C of X.

Proof. In Example 3.4, we constructed a stratifiable space X and a compactification βZ of X such that the remainder $\beta Z \setminus X$ is strongly paracompact. We also showed that X does not satisfy the condition of Lemma 2.1. As a consequence, X does not satisfy Condition C of Proposition 4.1; hence remainders are not paracompactly placed in compactifications of X.

Note that, even though the strongly paracompact remainder $\beta Z \setminus X$ of X is not paracompactly placed in βZ, this does not solve Ponomarev's problem, because βZ is not a compactification of $\beta Z \setminus X$. The closure of $\beta Z \setminus X$ in βZ is the subset $\beta Z \setminus \mathbb{Q}$, and $\beta Z \setminus X$ is paracompactly placed in $\beta Z \setminus \mathbb{Q}$. This follows once we observe that the points of $\mathbb{P}'$ are isolated in $\beta Z \setminus \mathbb{Q}$ and that the set $A \setminus \mathbb{P}' = Z^* \cup (A \cap \mathbb{P})$ is Lindelöf whenever $\beta Z \setminus X \subset A \subset \beta Z \setminus \mathbb{Q}$. □

The problem with the space X above is that it has isolated points. We could try to modify X to get a nowhere locally compact example. We could, for instance, replace each isolated point by a copy of $\mathbb{Q}$; unfortunately, the resulting space is not strongly paracompact (or even metacompact) at infinity. Instead, we consider a completely different space.

Example 4.4. *A space which is strongly paracompact at infinity but which has a non-strongly paracompact remainder in some compactification.*

Construction. Let D be an uncountable discrete space, and let L be the product $D \times \mathbb{I} \times \mathbb{I}$. Define an equivalence relation $\sim$ on L by setting $(e, r, s) \sim (f, t, u)$ if either $(e, r, s) = (f, t, u)$ or $r = 0 = t$ and $s = u$. For every $(e, r, s) \in L$, denote by $\overline{(e, r, s)}$ the $\sim$-equivalence class of (e, r, s). Denote by K the quotient $L/{\sim} = \{\overline{(e, r, s)} : (e, r, s) \in L\}$.

The formula

$$d\big(\overline{(e, r, s)}, \overline{(f, t, u)}\big) = \begin{cases} |r - t| + |s - u| & \text{if } e = f \\ r + t + |s - u| & \text{if } e \neq f \end{cases}$$

defines a metric d on K. The metric space (K, d) is a two-dimensional analogue of a standard metric hedgehog (see, e.g., [13, Example 4.1.5]): instead of a *hedgehog* with one-dimensional *spines*, we have a *book* with two-dimensional *pages*. Note that the subset $K_0 = \{\overline{(e, 0, s)} : e \in D \text{ and } s \in \mathbb{I}\}$ (the *spine of the book*), with the relative d-metric, is isometric with $\mathbb{I}$ with its standard metric.

Let g be a point of D. We shall topologize K by modifying the metric topology τ_d of K. The other points of K shall retain their τ_d-neighborhoods, but we change neighborhoods of points $\overline{(g, r, s)} \in K$, where $r \neq 0$. New basic neighborhoods of the point $\overline{(g, r, s)}$ have the form $\bigcup_{e \in D \setminus F} B_d\big(\overline{(e, r, s)}, \epsilon\big)$, where $\epsilon > 0$ and F is a finite subset of $D \setminus \{g\}$. We denote by τ this new topology of K and from now on, we consider K as a topological space, equipped with the topology τ. Note that K is a compact Hausdorff space.

Consider the subspaces $Y = \big\{\overline{(e, r, s)} \in K : e \neq g \text{ and } r \text{ or } s \text{ is rational}\big\}$ and $X = K \setminus Y$ of K. Note that both X and Y are dense in K. Moreover, Y is a non-separable connected metrizable space, and hence Y is not strongly paracompact. As a consequence, the space X has a remainder, which fails to be strongly paracompact.

Even though X has a non-strongly paracompact remainder, it turns out that X is strongly paracompact at infinity. Since strong paracompactness is inversely preserved under perfect mappings, strong paracompactness at infinity of X follows once we show that some remainder of X is strongly paracompact.

For every $n \in \mathbb{N}$, let $A_n = \{\overline{(e,r,s)} \in K : e \neq g \text{ and } r = \frac{1}{n}\}$, and let $A = \bigcup_{n=1}^{\infty} A_n$. We shall modify the space K by "splitting" each point $\overline{(e,r,s)}$ of A into two new points $(e,r,s)^\ell$ and $(e,r,s)^u$. For every $B \subset A$, we set $B^\ell = \{(e,r,s)^\ell : \overline{(e,r,s)} \in B\}$ and $B^u = \{(e,r,s)^u : \overline{(e,r,s)} \in B\}$. We let $C = (K \setminus A) \cup A^\ell \cup A^u$, and we define a mapping $\varphi : C \to K$ by setting $\varphi(p) = p$ for each $p \in K \setminus A$ and $\varphi((e,r,s)^\ell) = \overline{(e,r,s)} = \varphi((e,r,s)^u)$ for every $\overline{(e,r,s)} \in A$.

For every $n \in \mathbb{N}$, set $L_n = \{\overline{(e,r,s)}) \in K : e \neq g \text{ and } r < \frac{1}{n}\}$ and $U_n = \{\overline{(e,r,s)}) \in K : e \neq g \text{ and } r > \frac{1}{n}\}$; further, let $\hat{L}_n = \varphi^{-1}(L_n) \cup A_n^\ell$ and $\hat{U}_n = \varphi^{-1}(U_n) \cup A_n^u$. Let π be the topology of C which has the family $\{\varphi^{-1}(G) : G \in \tau\} \cup \{\hat{L}_n : n \in \mathbb{N}\} \cup \{\hat{U}_n : n \in \mathbb{N}\}$ as a subbase. We consider C as a topological space, equipped with the topology π. Note that the mapping $\varphi : C \to K$ is continuous, closed and finite-to-one. As a consequence, C is compact. It is easy to see that C is a Hausdorff space.

Note that $X \subset C$ and the relative topology of X in C is the same as the relative topology of X in K. Moreover, X is dense in C, and hence C is a compactification of X. Denote by Z the remainder $C \setminus X$. To prove that Z is strongly paracompact, we show first that Z is metrizable. Note that $\varphi(Z) = Y$ and $\varphi^{-1}(Y) = Z$. The metrizable space Y has a σ-locally finite base $\mathcal{B}$. The definition of the topology π shows that the family $\mathcal{E} = \{\varphi^{-1}(B) \cap \hat{L}_n : B \in \mathcal{B} \text{ and } n \in \mathbb{N}\} \cup \{\varphi^{-1}(B) \cap \hat{U}_n : B \in \mathcal{B} \text{ and } n \in \mathbb{N}\}$ is a base of Z. It is easy to see that the family $\mathcal{E}$ is σ-locally finite in Z. As a consequence, Z is metrizable. A similar base argument shows that, for each $e \in D \setminus \{g\}$, the subspace $Z \cap \varphi^{-1}(\{e\} \times \mathbb{I} \times \mathbb{I})$ of Z is second countable. It follows that the subspace $Z_+ = \varphi^{-1}\big\{\overline{(e,r,s)} \in Y : r \neq 0\big\}$ of Z is locally separable. Note that the subspace $C \setminus Z_+$ of Z coincides with the compact subspace K_0 of K. Also note that $K_0 = \varphi^{-1}(K_0)$. Now, let $\mathcal{U}$ be an open cover of Z. Let $\mathcal{V}$ be a finite subfamily of $\mathcal{U}$ which covers the compact set K_0. Then $G = K \setminus \varphi^{-1}(Z \setminus \bigcup \mathcal{V})$ is a τ-open set containing K_0. We have $d(K_0, K \setminus G) > 0$, and hence there exists $n \in \mathbb{N}$ with $\frac{1}{n} < d(K_0, K \setminus G)$. The set $\hat{L}_n \cap Z$ is clopen in Z and $\hat{L}_n \cap Z \subset \bigcup \mathcal{V}$. The clopen subspace $\hat{U}_n \cap Z = Z \setminus \hat{L}_n$ of the metrizable space Z is locally separable and hence strongly paracompact. It follows that there exists a star-finite open partial refinement $\mathcal{W}$ of $\mathcal{U}$ with $\bigcup \mathcal{W} = \hat{U}_n \cap Z$. Now $\{V \cap \hat{L}_n : V \in \mathcal{V}\} \cup \mathcal{W}$ is a star-finite open refinement of $\mathcal{U}$. We have shown that the remainder Z of X is strongly paracompact. It follows that also the Čech-Stone remainder X^* of X is strongly paracompact.

Since the homeomorphic copy Z of X has a non-strongly paracompact remainder, it follows from Corollary 4.2 that no remainder of X is paracompactly placed in the compactification. The space X is nowhere locally compact and hence every compactification of X is also a compactification of the remainder. It follows that either one of the strongly paracompact remainders Z and X^* of X can be used to solve Ponomarev's problem. In particular, the strongly paracompact metrizable space Z fails to be paracompactly placed in its compactification C. $\square$

In the example above, both the set $\{\overline{(g,s,t)} : s,t \in \mathbb{I}\}$ and its complement are metrizable in the compactifications K and C of X. It follows by [21, Theorem 1.1] that the spaces K and C are Eberlein compact.

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