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by

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# CAUCHY $sn$-SYMMETRIC SPACES WITH A $cs$-NETWORK ($cs^*$-NETWORK) HAVING PROPERTY $\sigma$-$(P)$

TRAN VAN AN AND LUONG QUOC TUYEN

Abstract. In this paper, we introduce the concept of Cauchy $sn$-symmetric spaces, consider properties of Cauchy $sn$-symmetric spaces with $cs$-networks ($cs^*$-networks) having certain $\sigma$-(P) properties, and give some characterizations of images of metric spaces under certain sequence-covering $\pi$-maps. Then, we give affirmative answers to the problems posed by Y. Tanaka and Y. Ge in [18], and give some partial answers to the problems posed by Y. Ikeda, C. Liu and Y. Tanaka in [6].

## 1. Introduction and Preliminaries

In 2002, Y. Ikeda, C. Liu and Y. Tanaka introduced the notion of $\sigma$-strong networks as a generalization of "development" in developable spaces, and consider certain quotient images of metric spaces in terms of $\sigma$-strong networks. By means of $\sigma$-strong networks, some characterizations for the quotient compact images of metric spaces are obtained (see in [6], [18], for example). It is known that if $X$ is a quotient compact image of a metric space, then $X$ is a symmetric space having a $\sigma$-point-finite $cs^*$-network, see in [6]. Then, the following question was posed by Y. Ikeda, C. Liu and Y. Tanaka.

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*Key words and phrases.* $cs$-network; $cs^*$-network; Cauchy $sn$-symmetric space; $\sigma$-(P)-strong network; property $\sigma$-$(P)$; $\alpha(P)$-map.

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**Question 1.1** ([6])**.** *Let $X$ be a symmetric space having a $\sigma$-point-finite cs-network. Is $X$ a quotient, compact image of a metric space?*

Recently, Y. Tanaka and Y. Ge introduced the concept of strongly $g$-developable spaces. It was shown that every strongly $g$-developable space is a sequence-covering quotient compact, $\sigma$-image of a metric space, and every sequence-covering quotient $\pi$, $\sigma$-image of a metric space is a Cauchy symmetric, $\aleph$-space, see in [18]. Then, Y. Tanaka and Y. Ge posed the question:

**Question 1.2** (Question 3.5, [18])**.** *Is every Cauchy symmetric, $\aleph$-space a strongly $g$-developable space?*

In this paper, we introduce the concept of Cauchy $sn$-symmetric spaces as a generalization of "Cauchy symmetric spaces", consider properties of Cauchy $sn$-symmetric spaces with $cs$-networks ($cs^*$-networks) having certain $\sigma$-(P) properties, and give some characterizations of images of metric spaces under certain sequence-covering $\pi$-maps. As an application of this result, we give partial answers to the Question 1.1, and give affirmative answers to the Question 1.2.

Throughout this paper, all spaces are assumed to be $T_1$ and regular, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of subsets of $X$, we denote $(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$ and $\mathcal{P} \bigwedge \mathcal{Q} = \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}$. We say that $\mathcal{P}$ is a *network at* $x$ in $X$, if $x \in \bigcap \mathcal{P}$, and whenever $x \in U$ with $U$ is open in $X$, then $x \in P \subset U$ for some $P \in \mathcal{P}$; $\mathcal{P}$ is a *network* for $X$, if for each $x \in X$, $(\mathcal{P})_x$ is a network at $x$. For a sequence $\{x_n\}$ converging to $x$ and $P \subset X$, we say that $\{x_n\}$ is *eventually* in $P$ if $\{x\} \bigcup \{x_n : n \geq m\} \subset P$ for some $m \in \mathbb{N}$, and $\{x_n\}$ is *frequently* in $P$ if some subsequence of $\{x_n\}$ is eventually in $P$.

**Definition 1.3** ([19])**.** For a cover $\mathcal{P}$ of a space $X$, let $(P)$ be a (certain) covering-property of $\mathcal{P}$. Let us say that $\mathcal{P}$ has *property* $\sigma$-$(P)$, if $\mathcal{P}$ can be expressed as $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$, where each $\mathcal{P}_n$ is a cover of $X$ having the property $(P)$, and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$.

**Definition 1.4.** Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space $X$ such that $\mathcal{P}_x$ is a network at $x$, and if $P_1$, $P_2 \in \mathcal{P}_x$, then $P \subset P_1 \cap P_2$ for some $P \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is a *weak base* [2], if for $G \subset X$, $G$ is open in $X$ iff for every $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$; $\mathcal{P}_x$ is said to be a *weak neighborhood base* at $x$.

(2) $\mathcal{P}$ is an *sn-network* [11], if every element of $\mathcal{P}_x$ is a sequential neighborhood of $x$ for every $x \in X$; $\mathcal{P}_x$ is said to be an *sn-network* at $x$.

**Remark 1.5.** (1) weak bases $\Longrightarrow$ $sn$-networks.
(2) In a sequential space, weak bases $\Longleftrightarrow$ $sn$-networks.

**Definition 1.6.** Let $f : X \longrightarrow Y$ be a map.

(1) $f$ is *weak-open* [21], if there exists a weak base $\mathcal{B} = \bigcup\{\mathcal{B}_y : y \in Y\}$ for $Y$, and for every $y \in Y$, there exists $x \in f^{-1}(y)$ such that for each open neighborhood $U$ of $x$, $B \subset f(U)$ for some $B \in \mathcal{B}_y$.
(2) $f$ is *sequence-covering* [16] (resp., *pseudo-sequence-covering* [6]), if every convergent sequence of $Y$ is the image of some convergent sequence (resp., compact subset) of $X$.
(3) $f$ is *sequentially-quotient* [3], if for every convergent sequence $S$ of $Y$, there exists a convergent sequence $L$ of $X$ such that $f(L)$ is a subsequence of $S$.
(4) $f$ is an *msss-map* (resp., *mssc-map*) [10], if $X$ is a subspace of the product space $\prod_{i\in\mathbb{N}} X_i$ of a family $\{X_i : i \in \mathbb{N}\}$ of metric spaces and for each $y \in Y$, there is a sequence $\{V_i\}$ of open neighborhoods of $y$ such that each $p_i f^{-1}(V_i)$ is separable in $X_i$ (resp., each $\mathtt{cl}\big(p_i f^{-1}(V_i)\big)$ is compact in $X_i$).
(5) $f$ is an *msk-map* [8], if $X$ is a subspace of the product space $\prod_{i\in\mathbb{N}} X_i$ of a family $\{X_i : i \in \mathbb{N}\}$ of metric spaces and for each compact subset $K$ of $Y$ and $i \in \mathbb{N}$, $\mathtt{cl}\big(p_i f^{-1}(K)\big)$ is compact in $X_i$.

**Definition 1.7** ([5])**.** Let $d$ be a $d$-function on a space $X$.

(1) For each $x \in X$, $n \in \mathbb{N}$, let $S_n(x) = \{y \in X : d(x, y) < 1/n\}$.
(2) For every $P \subset X$, put $d(P) = \sup\{d(x, y) : x, y \in P\}$.
(3) $X$ is *symmetric*, if $\{S_n(x) : n \in \mathbb{N}\}$ is a weak neighborhood base at $x$ for each $x \in X$.
(4) $X$ is *sn-symmetric*, if $\{S_n(x) : n \in \mathbb{N}\}$ is an $sn$-network at $x$ for each $x \in X$.

**Definition 1.8.** (1) A symmetric space $(X, d)$ is called a *Cauchy symmetric* space ([20]), if every convergent sequence is $d$-Cauchy.
(2) An $sn$-symmetric space $(X, d)$ is called a *Cauchy sn-symmetric* space, if every convergent sequence is $d$-Cauchy.

**Remark 1.9.** (1) symmetric spaces $\Longleftrightarrow$ sequential and $sn$-symmetric spaces.
(2) Cauchy symmetric spaces $\Longleftrightarrow$ sequential and Cauchy $sn$-symmetric spaces.

**Definition 1.10.** Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a sequence of covers of a space $X$ such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for every $n \in \mathbb{N}$.

(1) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *σ-strong network* for $X$ [6], if $\{\mathtt{St}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at each $x \in X$.

(2) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *$\sigma$-$(P)$-strong network* for $X$, if it is a $\sigma$-strong network and each $\mathcal{P}_n$ has property $(P)$.

**Definition 1.11.** Let $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a $\sigma$-$(P)$-strong network for a space $X$.

(1) $\mathcal{P}$ is a *$\sigma$-$(P)$-strong weak base* (resp., *sn-network*, *cs-network*, *$cs^*$-network*), if $\mathcal{P}$ is a weak base (resp., $sn$-network, $cs$-network, $cs^*$-network).
(2) $\mathcal{P}$ is a *$\sigma$-$(P)$-strong network consisting of sn-covers* (resp., *cs-covers*, *$cs^*$-covers*), if each $\mathcal{P}_n$ is an $sn$-cover (resp., $cs$-cover, $cs^*$-cover).

**Notation 1.12.** Let $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a $\sigma$-strong network for a space $X$. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in \Lambda_n\}$ and endow $\Lambda_n$ with the discrete topology. Then,

$$M = \Big\{\alpha = (\alpha_n) \in \prod_{n\in\mathbb{N}} \Lambda_n : \{P_{\alpha_n}\} \text{ forms a network at some point } x_\alpha \in X\Big\}$$

is a metric space and the point $x_\alpha$ is unique in $X$ for every $\alpha \in M$. Define $f : M \to X$ by $f(\alpha) = x_\alpha$. Let us call $(f, M, X, \mathcal{P}_n)$ a *Ponomarev's system*, following [14].

From now on, let us restrict the properties $(P)$, $(P_1)$, $(P_2)$, $(P_3)$ and $(P_4)$ to the following

(1) $(P)$ are point-finite, compact-finite, locally finite, point-countable, compact-countable, and locally countable.
(2) $(P_1)$ are point-finite, compact-finite, and locally finite.
(3) $(P_2)$ are point-countable, compact-countable, and locally countable.
(4) $(P_3)$ are point-finite, compact-finite, locally finite, and locally countable.
(5) $(P_4)$ are point-countable, and compact-countable.

And, let us restrict the prefixes $\alpha(P_1)$ and $\alpha(P_2)$ to the following

(6) $\alpha(P_1)$ is compact if $(P_1)$ is point-finite, $\alpha(P_1)$ is *mssc* if $(P_1)$ is locally finite, and $\alpha(P_1)$ is *msk* if $(P_1)$ is compact-finite.
(7) $\alpha(P_2)$ is *s* if $(P_2)$ is point-countable, $\alpha(P_2)$ is *cs* if $(P_2)$ is compact-countable, and $\alpha(P_2)$ is *msss* if $(P_2)$ is locally countable.

For some undefined or related concepts, we refer the reader to [4], [17] and [18].

## 2. Cauchy $sn$-symmetric spaces with a $cs$-network having property $\sigma$-$(P)$

**Lemma 2.1.** *The following statements hold for an sn-symmetric space $X$.*

(1) *If $X$ has a cs-network with property $\sigma$-$(P)$, then $X$ has an sn-network with property $\sigma$-$(P)$.*
(2) *If $X$ has a $\sigma$-$(P)$-strong network consisting of cs-covers, then $X$ has a $\sigma$-$(P)$-strong network consisting of sn-covers.*
(3) *If $\mathcal{P}_x$ is a countable sn-network at $x$, then for each $n \in \mathbb{N}$, there exists $P \in \mathcal{P}_x$ such that $P \subset S_n(x)$.*
(4) *If $P$ is a sequential neighborhood at $x$, then $S_n(x) \subset P$ for some $n \in \mathbb{N}$.*

*Proof.* (1) Let $\mathcal{F} = \bigcup\{\mathcal{F}_n : n \in \mathbb{N}\}$ be a $cs$-network with property $\sigma$-$(P)$ for $X$. We can assume that each $\mathcal{F}_n$ is closed under finite intersections. Since $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \in \mathbb{N}$, $\mathcal{F}$ is closed under finite intersections. For each $x \in X$, let $\mathcal{P}_x = \{P \in \mathcal{F} : S_n(x) \subset P \text{ for some } n \in \mathbb{N}\}$. Then, each element of $\mathcal{P}_x$ is a sequential neighborhood at $x$ and for $P_1, P_2 \in \mathcal{P}_x$, there exists $P \in \mathcal{P}_x$ such that $P \subset P_1 \cap P_2$. On the other hand, by using proof of [13, Lemma 7], we obtain $\mathcal{P}_x$ is a network at $x$. Now, we define $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$, and for each $n \in \mathbb{N}$, let $\mathcal{P}_n = \mathcal{F}_n \cap \mathcal{P}$. Then, $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$, and $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is an $sn$-network having property $\sigma$-$(P)$.

(2) Let $\bigcup\{\mathcal{F}_i : i \in \mathbb{N}\}$ be a $\sigma$-$(P)$-strong network consisting of $cs$-covers for $X$. For each $i \in \mathbb{N}$, put $\mathcal{P}_i = \{P \in \mathcal{F}_i : \text{ there exist } x \in X, n \in \mathbb{N} \text{ such that } S_n(x) \subset P\}$. Then,

(a) For each $x \in X$, by using the proof of [13, Lemma 7], there exist $P \in \mathcal{P}_i$ and $n \in \mathbb{N}$ such that $S_n(x) \subset P$. This implies that $P$ is a sequential neighborhood at $x$.
(b) For each $P \in \mathcal{P}_i$, there exist $x \in X$ and $n \in \mathbb{N}$ such that $S_n(x) \subset P$. This implies that $P$ is a sequential neighborhood at $x$.
(c) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a $\sigma$-$(P)$-strong network.

Therefore, $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a $\sigma$-$(P)$-strong network consisting of $sn$-covers.

(3) Since $\mathcal{P}_x$ is countable, we can put $\mathcal{P}_x = \{P_n(x) : n \in \mathbb{N}\}$. On the other hand, because $\mathcal{P}_x$ is an $sn$-network at $x$, we can choose a sequence $\{n_i : i \in \mathbb{N}\}$ such that $\{P_{n_i}(x) : i \in \mathbb{N}\}$ is a decreasing network at $x$. Then, there exists $i \in \mathbb{N}$ such that $P_{n_i}(x) \subset S_n(x)$.

(4) If not, for each $n \in \mathbb{N}$, there exists $x_n \in S_n(x) - P$. Then, $\{x_n\}$ converges to $x$. Hence, there exists $m \in \mathbb{N}$ such that $x_n \in P$ for every $n \geq m$. This is a contradiction. $\square$

**Lemma 2.2.** (1) *Let $X$ be an sn-symmetric space. Then, sequence $\{d(S_n(x))\}$ converges to 0 for every $x \in X$ iff every convergent sequence in $X$ is $d$-Cauchy.*

(2) *$X$ has a $\sigma$-strong network consisting of cs-covers iff $X$ is Cauchy sn-symmetric.*

*Proof.* (1) Assume that $\{d(S_n(x))\}$ converges to 0 for every $x \in X$ and $\{y_n\}$ is a sequence converging to $y \in X$. Then, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(S_{n_0}(y)) < \varepsilon$. Thus, there exists $m_0 \in \mathbb{N}$ such that $\{y\} \bigcup \{y_n : n \geq m_0\} \subset S_{n_0}(y)$. Therefore, $d(y_i, y_j) < \varepsilon$ for all $i, j \geq m_0$.

Conversely, assume that every convergent sequence in $X$ is $d$-Cauchy and $x \in X$. We shall show that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(S_n(x)) \leq \varepsilon$ for every $n \geq n_0$. Otherwise, there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, there exists $i_n \geq n$ such that $d(S_{i_n}(x)) > \varepsilon$. We can assume that $i_n < i_{n+1}$ for every $n \in \mathbb{N}$. This follows that for each $n \in \mathbb{N}$, there exist $x_n, y_n \in S_{i_n}(x)$ such that $d(x_n, y_n) > \varepsilon$. Then, the sequence $\{x_n, y_n : n \in \mathbb{N}\}$ converges to $x$. By assumption, this implies that there exists $k \in \mathbb{N}$ such that $d(x_n, y_n) < \varepsilon$ for all $n \geq k$. This is a contradiction.

(2) Let $X$ have a $\sigma$-strong network $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ consisting of $cs$-covers. For each $x, y \in X$ such that $x \neq y$, let $\delta(x, y) = \min\{n : y \notin \mathtt{St}(x, \mathcal{P}_n)\}$, we define

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1/\delta(x,y) & \text{if } x \neq y. \end{cases}$$

Then, $d$ is a $d$-function on $X$ and $\mathtt{St}(x, \mathcal{P}_n) = S_n(x)$ for all $n \in \mathbb{N}$. Since $\mathcal{P}$ is a $\sigma$-strong network consisting of $cs$-covers, $\{S_n(x) : n \in \mathbb{N}\}$ is an $sn$-network at each $x \in X$. Therefore, $(X, d)$ is $sn$-symmetric. Now, we shall show that every convergent sequence in $X$ is $d$-Cauchy. Indeed, let $\{x_i\}$ be a sequence converging to $x \in X$. Then, for any $\varepsilon > 0$, choose $k \in \mathbb{N}$ such that $1/k < \varepsilon$. Since $\mathcal{P}_k$ is a $cs$-cover, there exist $P \in \mathcal{P}_k$, and $m \in \mathbb{N}$ such that $x_i \in P$ for all $i \geq m$. This implies that $d(x_i, x_j) < \varepsilon$ for all $i, j \geq m$.

Conversely, let $X$ be Cauchy $sn$-symmetric. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P \subset X : d(P) < 1/n\}$. Then, $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a $\sigma$-strong network. Furthermore, each $\mathcal{P}_n$ is a $cs$-cover. In fact, let $\{x_i\}$ be a sequence converging to $x \in U$ with $U$ is open in $X$. Since $X$ is Cauchy $sn$-symmetric, there exists $m \in \mathbb{N}$ such that $d(x, x_i) < 1/2n$ and $d(x_i, x_j) < 1/2n$ for all $i, j \geq m$. By putting $P = \{x\} \bigcup \{x_i : i \geq m\}$, we have $P \in \mathcal{P}_n$ and $\{x_i\}$ is eventually in $P$.

Therefore, $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a $\sigma$-strong network consisting of $cs$-covers for $X$. ∎

**Theorem 2.3.** *For a space $X$, consider the below statements. Then, the following implications $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4) \Rightarrow (5)$ hold.*

(1) *$X$ is a Cauchy sn-symmetric space with a cs-network having property $\sigma$-$(P)$;*
(2) *$X$ has a $\sigma$-$(P)$-strong network consisting of cs-covers;*
(3) *$X$ has a $\sigma$-$(P)$-strong network consisting of sn-covers;*
(4) *$X$ has a $\sigma$-$(P)$-strong sn-network;*
(5) *$X$ has a $\sigma$-$(P)$-strong cs-network.*

*If the property $(P)$ is replaced by $(P_1)$, then we have $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$.*

*Proof.* $(1) \Rightarrow (2)$. Let $X$ be a Cauchy $sn$-symmetric space with a $cs$-network having property $\sigma$-$(P)$. Then, by Lemma 2.1(1), $X$ has an $sn$-network $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ such that each $\mathcal{P}_n$ has property $(P)$ and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$. Denote $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ with each $\mathcal{P}_x$ is an $sn$-network at $x$. For each $m, n \in \mathbb{N}$, put

$$\mathcal{Q}_{m,n}(x) = \Big\{P \in \mathcal{P}_m \cap \mathcal{P}_x : S_m(x) \subset P, \text{ and } d(P) < 1/n\Big\};$$

$$A_{m,n} = \{x \in X : \mathcal{Q}_{m,n}(x) = \emptyset\}; \quad B_{m,n} = X - A_{m,n};$$

$$\mathcal{Q}_{m,n} = \bigcup\{\mathcal{Q}_{m,n}(x) : x \in B_{m,n}\}; \text{ and } \mathcal{F}_{m,n} = \mathcal{Q}_{m,n} \bigcup\{A_{m,n}\}.$$

Then, each $\mathcal{F}_{m,n}$ has property $(P)$. Furthermore, we have

(i) *Each $\mathcal{F}_{m,n}$ is a cs-cover.* Let $x \in X$ and $L = \{x_i : i \in \mathbb{N}\}$ be a sequence converging to $x$, then

*Case 1.* If $x \in B_{m,n}$, then there is $P \in \mathcal{Q}_{m,n}(x)$ such that $S_m(x) \subset P$. Hence, $L$ is eventually in $P \in \mathcal{F}_{m,n}$.

*Case 2.* If $x \notin B_{m,n}$ and $L \cap B_{m,n}$ is finite, then $L$ is eventually in $A_{m,n} \in \mathcal{F}_{m,n}$.

*Case 3.* If $x \notin B_{m,n}$ and $L \cap B_{m,n}$ is infinite, then we can assume that $L \cap B_{m,n} = \{x_{i_k} : k \in \mathbb{N}\}$. Since $X$ is Cauchy $sn$-symmetric and $L$ converges to $x$, there exists $n_0 \in \mathbb{N}$ such that $d(x, x_i) < 1/m$ and $d(x_i, x_j) < 1/m$ for all $i, j \geq n_0$. Now, we pick $k_0 \in \mathbb{N}$ such that $i_{k_0} \geq n_0$. Since $d(x_{i_{k_0}}, x) < 1/m$ and $d(x_{i_{k_0}}, x_i) < 1/m$ for all $i \geq n_0$, $L$ is eventually in $S_m(x_{i_{k_0}})$. Furthermore, since $x_{i_{k_0}} \in B_{m,n}$, $S_m(x_{i_{k_0}}) \subset P$ for some $P \in Q_{m,n}(x_{i_{k_0}})$. Hence, $P \in \mathcal{F}_{m,n}$ and $L$ is eventually in $P$.

Therefore, each $\mathcal{F}_{m,n}$ is a $cs$-cover for $X$.

(ii) $\{\mathtt{St}(x, \mathcal{F}_{m,n}) : m, n \in \mathbb{N}\}$ *is a network at $x$.* Assume that $x \in U$ with $U$ is open in $X$. Then, $S_n(x) \subset U$ for some $n \in \mathbb{N}$. Since $X$ is Cauchy $sn$-symmetric, by Lemma 2.2(1), it implies that there exists $j \in \mathbb{N}$ such that $d\big(S_j(x)\big) < 1/n$. On the other hand, since $\mathcal{P}$ is a point-countable $sn$-network, it follows from Lemma 2.1(3) that $P \subset S_j(x)$ for some $P \in \mathcal{P}_x$.

Thus, $P \in \mathcal{P}_k$ for some $k \in \mathbb{N}$. Since $P$ is a sequential neighborhood at $x$, it follows from Lemma 2.1(4) that there exists $i \in \mathbb{N}$ such that $S_i(x) \subset P$. Put $m = \max\{i, k\}$, we get $S_m(x) \subset S_i(x) \subset P \in \mathcal{P}_k \subset \mathcal{P}_m$. Because $d(P) < 1/n$, it implies that $P \in \mathcal{Q}_{m,n} \subset \mathcal{F}_{m,n}$. Then, we have $\mathtt{St}(x, \mathcal{F}_{m,n}) \subset S_n(x)$. Therefore, $\{\mathtt{St}(x, \mathcal{F}_{m,n}) : m, n \in \mathbb{N}\}$ is a network at $x$.

Next, we write $\{\mathcal{F}_{m,n} : m, n \in \mathbb{N}\} = \{\mathcal{H}_i : i \in \mathbb{N}\}$, and for each $n \in \mathbb{N}$, put $\mathcal{G}_n = \bigwedge\{\mathcal{H}_i : i \leq n\}$. Then, $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a $\sigma$-$(P)$-strong network consisting of *cs*-covers for $X$.

(2) $\Rightarrow$ (3). By Lemma 2.2(2) and Lemma 2.1(2).

(3) $\Rightarrow$ (1). By Lemma 2.2(2).

(3) $\Rightarrow$ (4) $\Rightarrow$ (5). It is obvious.

If the property $(P)$ is replaced by $(P_1)$, then (5) $\Rightarrow$ (2) holds by [18, Lemma 3.3(1)]. □

**Lemma 2.4.** *For a Ponomarev's system $(f, M, X, \mathcal{P}_n)$, the following statements hold.*

(1) *$f$ is a $\pi$-map.*
(2) *$f$ is a $\alpha(P)$-map, if each $\mathcal{P}_n$ having property $(P)$.*
(3) *$f$ is pseudo-sequence-covering, if each $\mathcal{P}_n$ is a point-countable $cs^*$-cover.*
(4) *$f$ is a 1-sequence-covering map, if each $\mathcal{P}_n$ is a point-countable sn-cover.*
(5) *$f$ is a compact-covering map, if each $\mathcal{P}_n$ is an sn-cover and each compact subset of $X$ is metrizable.*

*Proof.* By [18, Lemma 2.2], (1) and (3) hold. For (2), see in the proof of [7, Theorem 4], [8, Theorem 2.2], [9, Theorem 2.1], [12, Theorem 1.1], and by [18, Lemma 2.2].

For (4), since each $\mathcal{P}_n$ is an *sn*-cover, it follows from [18, Lemma 2.2] that $f$ is sequence-covering. Furthermore, by (1) and (2), $f$ is a $\pi$- and $s$-map. It follows from [1, Theorem 2.5] that $f$ is 1-sequence-covering.

For (5), since each compact subset of $X$ is metrizable and each $\mathcal{P}_n$ is an *sn*-cover, by using the proof of [18, Lemma 3.10], it follows that each $\mathcal{P}_n$ is a $cfp$-cover for $X$. By [18, Lemma 2.2(2)] this implies that $f$ is compact-covering. □

**Lemma 2.5.** *Let $f : M \to X$ be a sequence-covering map, and $M$ be a metric space. Then, the following statements hold.*

(1) *$X$ has a cs-network hawing property $\sigma$-$(P)$, if $f$ is a $\alpha(P)$-map.*
(2) *$X$ is Cauchy sn-symmetric, if $f$ is a $\pi$-map.*

*Proof.* For (1), by using the proof of [7, Theorem 4], [8, Theorem 4.1], [9, Theorem 5.1], [12, Theorem 1.1] and by [6, Proposition 16(2b)].

Now, let $f$ be a $\pi$-map, it follows from [6, Proposition 16(3b)] that $X$ has a $\sigma$-strong network consisting of $cs$-covers. So, by Lemma 2.2(2), $X$ is Cauchy $sn$-symmetric, and (2) holds. $\square$

**Theorem 2.6.** *The following are equivalent for a space $X$.*

(1) *$X$ is a Cauchy sn-symmetric space with a cs-network having property $\sigma$-$(P_1)$;*
(2) *$X$ has a $\sigma$-$(P_1)$-strong sn-network;*
(3) *$X$ has a $\sigma$-$(P_1)$-strong network consisting of sn-covers;*
(4) *$X$ is a 1-sequence-covering compact, $\alpha(P_1)$-image of a metric space;*
(5) *$X$ is a sequence-covering $\pi$, $\alpha(P_1)$-image of a metric space.*

*It is possible to add the prefix "compact-covering" before "1-sequence-covering" in (4) if we restrict $(P_1)$ to locally finite or compact-finite.*

*Proof.* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). By Theorem 2.3.

(3) $\Rightarrow$ (4). Let $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a $\sigma$-$(P_1)$-strong network consisting of $sn$-covers. Consider the Ponomarev's system $(f, M, X, \mathcal{P}_n)$. Because each $\mathcal{P}_n$ is an $sn$-cover having property $(P_1)$, it follows from Lemma 2.4 that $f$ is a 1-sequence-covering compact, $\alpha(P_1)$-map.

(4) $\Rightarrow$ (5). It is obvious.

(5) $\Rightarrow$ (1). By Lemma 2.5.

Now, if property $(P_1)$ are locally finite or compact-finite, then each compact subset of $X$ is metrizable. Hence, by Lemma 2.4(5), $f$ is compact-covering. $\square$

**Corollary 2.7.** *The following are equivalent for a space $X$.*

(1) *$X$ is a Cauchy symmetric space with a cs-network having property $\sigma$-$(P_1)$;*
(2) *$X$ has a $\sigma$-$(P_1)$-strong weak base;*
(3) *$X$ is a sequential space with a $\sigma$-$(P_1)$-strong network consisting of sn-covers;*
(4) *$X$ is a weak-open compact-covering compact, $\alpha(P_1)$-image of a metric space;*
(5) *$X$ is a weak-open $\pi$, $\alpha(P_1)$-image of a metric space.*

*Proof.* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5). By Remark 1.9, Theorem 2.6 and [1, Corollary 2.8].

(4) $\Rightarrow$ (5). It is obvious.

(3) $\Rightarrow$ (4). Let $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a $\sigma$-$(P_1)$-strong network consisting of $sn$-covers for a sequential space $X$. By Lemma 2.2(2), $X$ is $sn$-symmetric. Since $X$ is sequential, it implies that $X$ is symmetric. Then, every compact subset of $X$ is metrizable ([2]). Consider the Ponomarev's system

$(f, M, X, \mathcal{P}_n)$. Since each $\mathcal{P}_n$ is an $sn$-cover having property $(P_1)$ and every compact subset of $X$ is metrizable, it follows from Lemma 2.4 that $f$ is a 1-sequence-covering compact-covering compact, $\alpha(P_1)$-map. Since $X$ is sequential, $f$ is weak-open by [1, Corollary 2.8]. Hence, (4) holds. $\square$

**Remark 2.8.** By Corollary 2.7, in case that the property $(P_1)$ is locally finite, we get an affirmative answer to the Question 1.2.

Similar to the proof of Theorem 2.6, we have the following theorem.

**Theorem 2.9.** *The following are equivalent for a space $X$.*

(1) *$X$ is a Cauchy sn-symmetric space with a cs-network having property $\sigma$-$(P_2)$;*
(2) *$X$ is a Cauchy sn-symmetric space has a $\sigma$-$(P_2)$-strong sn-network;*
(3) *$X$ has a $\sigma$-$(P_2)$-strong network consisting of sn-covers;*
(4) *$X$ is a 1-sequence-covering $\pi$, $\alpha(P_2)$-image of a metric space;*
(5) *$X$ is a sequence-covering $\pi$, $\alpha(P_2)$-image of a metric space.*

*It is possible to add the prefix "compact-covering" before "1-sequence-covering" in* (4) *if we restrict* $(P_2)$ *to compact-countable or locally countable.*

By Theorem 2.9 and similar to the proof of Corollary 2.7, we obtained the following.

**Corollary 2.10.** *The following are equivalent for a space $X$.*

(1) *$X$ is a Cauchy symmetric space with a cs-network having property $\sigma$-$(P_2)$;*
(2) *$X$ is a Cauchy symmetric space with a $\sigma$-$(P_2)$-strong weak base;*
(3) *$X$ is a sequential space with a $\sigma$-$(P_2)$-strong network consisting of sn-covers;*
(4) *$X$ is a weak-open compact-covering $\pi$, $\alpha(P_2)$-image of a metric space;*
(5) *$X$ is a weak-open $\pi$, $\alpha(P_2)$-image of a metric space.*

## 3. Cauchy $sn$-symmetric spaces with a $cs^*$-network having property $\sigma$-$(P)$

**Lemma 3.1.** *Let $\mathcal{P}$ be a point-countable cs*-network for an sn-symmetric space $X$. Then, the following statements hold.*

(1) *For each $x \in X$, there exist a finite subfamily $\mathcal{H} \subset (\mathcal{P})_x$ and $k \in \mathbb{N}$ such that $S_k(x) \subset \bigcup \mathcal{H}$.*
(2) *Let $\{x_n\}$ be a sequence converging to $x \in X$. For each $n \in \mathbb{N}$, there is a finite subfamily $\mathcal{H} \subset (\mathcal{P})_x$ such that $\{x_n\}$ is eventually in $\bigcup \mathcal{H} \subset S_n(x)$.*

*Proof.* (1) Assume conversely that there exists $x \in X$ such that $S_n(x) \not\subset \bigcup \mathcal{H}$ for every $n \in \mathbb{N}$ and for every finite subfamily $\mathcal{H} \subset (\mathcal{P})_x$. Since $(\mathcal{P})_x$ is countable, we can write $\{\mathcal{H} : \mathcal{H}$ is a finite subfamily of $(\mathcal{P})_x\} = \{\mathcal{H}_i : i \in \mathbb{N}\}$. Then, for each $m, n \in \mathbb{N}$, there exists $x_{n,m} \in S_n(x) - \bigcup \mathcal{H}_m$. For each $n \geq m$, let $x_{n,m} = y_k$, where $k = m + n(n-1)/2$. Then, the sequence $\{y_k\}$ converges to $x$. By [19, Lemma 3], there exist $m, i \in \mathbb{N}$ such that $\{x\} \bigcup \{y_k : k \geq i\} \subset \bigcup \mathcal{H}_m$. Take $j \geq i$ with $y_j = x_{n,m}$ for some $n \geq m$. Then, $x_{n,m} \in \bigcup \mathcal{H}_m$. This is a contradiction.

(2) Since $(\mathcal{P})_x$ is a countable $cs^*$-network at $x$, it follows from [15, Lemma 2.2] that $\mathcal{G} = \{\bigcup \mathcal{F} : \mathcal{F} \subset (\mathcal{P})_x, \mathcal{F}$ is finite$\}$ is a countable $cs$-network at $x$. Furthermore, by using the proof in [13, Lemma 7], there exists a countable subfamily $\mathcal{Q} \subset \mathcal{G}$ such that $\mathcal{Q}$ is a countable $sn$-network at $x$. By Lemma 2.1(3), there exists a finite subfamily $\mathcal{H} \subset (\mathcal{P})_x$ such that $\bigcup \mathcal{H} \subset S_n(x)$. □

**Lemma 3.2.** *If $\mathcal{P}$ is a point-countable $cs^*$-network for a Cauchy sn-symmetric space $X$, then for each $n \in \mathbb{N}$, the collection $\mathcal{F}_n = \{P \in \mathcal{P} : d(P) < 1/n\}$ is a point-countable $cs^*$-network for $X$.*

*Proof.* Let $\{x_i\}$ be a sequence converging to $x \in U$ with $U$ open in $X$. For each $n \in \mathbb{N}$, it follows from Lemma 2.2(1) that there exists $m \in \mathbb{N}$ such that $S_m(x) \subset U$, and $d\big(S_m(x)\big) < 1/n$. It follows from Lemma 3.1(2) that there is a finite subfamily $\mathcal{H} \subset (\mathcal{P})_x$ such that $\bigcup \mathcal{H} \subset S_m(x)$ and $\{x_i\}$ is eventually in $\bigcup \mathcal{H}$. Thus, there exists $P \in \mathcal{H}$ such that $\{x_i\}$ is frequently in $P$. Since $P \subset \bigcup \mathcal{H} \subset S_m(x)$, we have $d(P) < 1/n$. This implies that $P \in \mathcal{F}_n$. Therefore, $\mathcal{F}_n$ is a point-countable $cs^*$-network for $X$. □

**Theorem 3.3.** *The following are equivalent for a Cauchy sn-symmetric space $X$.*

(1) *$X$ has a $cs^*$-network having property $\sigma$-$(P_3)$;*
(2) *$X$ has a $\sigma$-$(P_3)$-strong $cs^*$-network;*
(3) *$X$ has a $\sigma$-$(P_3)$-strong network consisting of $cs^*$-covers.*

*Proof.* (1) ⇒ (3). Let $\mathcal{P} = \bigcup \{\mathcal{P}_n : n \in \mathbb{N}\}$ be a $cs^*$-network having property $\sigma$-$(P_3)$ for a Cauchy $sn$-symmetric space $X$. We can assume that each $\mathcal{P}_n$ is closed under finite intersections. Since $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$, $\mathcal{P}$ is closed under finite intersections. In case $(P_3)$ is locally countable, we can assume that each element of $\mathcal{P}$ is closed. Now, for each $m, n, k \in \mathbb{N}$, put $\mathcal{Q}_{m,n} = \{P \in \mathcal{P}_m : d(P) < 1/n\}$,

$$A_{m,n,k} = \Big\{x \in X : \text{ there exists a finite subfamily } \mathcal{H}_x \subset (Q_{m,n})_x \text{ such that } S_k(x) \subset \bigcup \mathcal{H}_x\Big\},$$

$$B_{m,n,k} = X - A_{m,n,k}, \text{ and } \mathcal{F}_{m,n,k} = \mathcal{Q}_{m,n} \bigcup \{B_{m,n,k}\}.$$

Then, each $\mathcal{F}_{m,n,k}$ has the property $(P_3)$. Furthermore, we have

(i) *Each* $\mathcal{F}_{m,n,k}$ *is a* $cs^*$*-cover for* $X$. Let $L = \{x_i : i \in \mathbb{N}\}$ be a sequence converging to $x \in X$, then

*Case 1.* If $x \in A_{m,n,k}$, then $S_k(x) \subset \bigcup \mathcal{H}_x$ for some finite subfamily $\mathcal{H}_x \subset (\mathcal{Q}_{m,n})_x$. Since $L$ is eventually in $S_k(x)$, $L$ is eventually in $\bigcup \mathcal{H}_x$. Because $\mathcal{H}_x$ is finite, $L$ is frequently in $P$ for some $P \in \mathcal{Q}_{m,n}$. Therefore, $L$ is frequently in $P$ for some $P \in \mathcal{F}_{m,n,k}$.

*Case 2.* If $x \notin A_{m,n,k}$ and $L \cap B_{m,n,k}$ is infinite, then $L$ is frequently in $B_{m,n,k} \in \mathcal{F}_{m,n,k}$.

*Case 3.* If $x \notin A_{m,n,k}$ and $L \cap B_{m,n,k}$ is finite, then there exists $i_0 \in \mathbb{N}$ such that $\{x_i : i \geq i_0\} \subset L \cap A_{m,n,k}$. This implies that for each $i \geq i_0$, there exists a finite subfamily $\mathcal{H}_{x_i} \subset (\mathcal{Q}_{m,n})_{x_i}$ such that $x_i \in S_k(x_i) \subset \bigcup \mathcal{H}_{x_i}$. On the other hand, since $L$ converges to $x$ and $X$ is Cauchy $sn$-symmetric, there exists $j_0 \geq i_0$ such that $d(x, x_i) < 1/k$ and $d(x_i, x_j) < 1/k$ for every $i, j \geq j_0$. Then, we have

$*$ If $(P_3)$ is point-finite, compact-finite or locally finite, then $\mathcal{Q}_{m,n}$ is point-finite. Since $d(x, x_i) < 1/k$ for all $i \geq j_0$, it implies that for each $i \geq j_0$, there exists $P_i \in \mathcal{H}_{x_i}$ such that $\{x, x_i\} \subset P_i$. Furthermore, since $\mathcal{Q}_{m,n}$ is point-finite, the set $\{P_i : i \geq j_0\}$ is finite. Thus, $L$ is frequently in $P_i \in \mathcal{F}_{m,n,k}$ for some $i \geq j_0$.

$*$ If $(P_3)$ is locally countable, then we pick $k_0 \geq j_0$. Since $d(x, x_{k_0}) < 1/k$ and $d(x_i, x_{k_0}) < 1/k$ for all $i \geq j_0$, $\{x, x_i, x_{k_0}\} \subset \bigcup \mathcal{H}_{x_{k_0}}$ for every $i \geq j_0$. Since $\mathcal{H}_{x_{k_0}}$ is finite, there exists a subsequence $K$ of $L$ such that $K \subset P$ for some $P \in \mathcal{H}_{x_{k_0}}$. Furthermore, since $P$ is closed and $K$ converges to $x$, it implies that $x \in P$. Hence, $L$ is frequently in $P$ for some $P \in \mathcal{F}_{m,n,k}$.

Therefore, each $\mathcal{F}_{m,n,k}$ is a $cs^*$-cover for $X$.

(ii) $\{\mathtt{St}(x, \mathcal{F}_{m,n,k}) : m, n, k \in \mathbb{N}\}$ *is a network at* $x$. Let $x \in U$ with $U$ is open in $X$. Since $U$ is a neighborhood of $x$, there exists $n_0 \in \mathbb{N}$ such that $S_{n_0}(x) \subset U$. Furthermore, since $X$ is Cauchy $sn$-symmetric, it follows from Lemma 3.2 that $\mathcal{F}_{n_0} = \{P \in \mathcal{P} : d(P) < 1/n_0\}$ is a point-countable $cs^*$-network for $X$. By Lemma 3.1(1), there exist a finite subfamily $\mathcal{H} \subset (\mathcal{F}_{n_0})_x$ and $k_0 \in \mathbb{N}$ such that $S_{k_0}(x) \subset \bigcup \mathcal{H}$. On the other hand, since $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$, it follows that $\mathcal{H} \subset \mathcal{P}_{m_0}$ for some $m_0 \in \mathbb{N}$. This implies that $\mathcal{H} \subset \mathcal{Q}_{m_0,n_0}$. Because $S_{k_0}(x) \subset \bigcup \mathcal{H}$, it implies that $x \in A_{m_0,n_0,k_0}$. Then, we have $\mathtt{St}(x, \mathcal{F}_{m_0,n_0,k_0}) \subset U$. Therefore, $\{\mathtt{St}(x, \mathcal{F}_{m,n,k}) : m, n, k \in \mathbb{N}\}$ is a network at $x$.

Next, we write $\{\mathcal{F}_{m,n,k} : m, n, k \in \mathbb{N}\} = \{\mathcal{H}_i : i \in \mathbb{N}\}$, and for each $i \in \mathbb{N}$, put $\mathcal{G}_i = \bigwedge\{\mathcal{H}_j : j \leq i\}$. Then, $\bigcup\{\mathcal{G}_i : i \in \mathbb{N}\}$ is a $\sigma$-$(P_3)$-strong network consisting of $cs^*$-covers for $X$.

(3) $\Rightarrow$ (2) $\Rightarrow$ (1). It is obvious. $\square$

**Theorem 3.4.** *The following are equivalent for a Cauchy sn-symmetric space $X$.*

(1) *$X$ has a $cs^*$-network having property $(P_4)$;*
(2) *$X$ has a $\sigma$-$(P_4)$-strong $cs^*$-network;*
(3) *$X$ has a $\sigma$-$(P_4)$-strong network consisting of $cs^*$-covers.*

*Proof.* (1) $\Rightarrow$ (3). Let $\mathcal{P}$ be a $cs^*$-network having property $(P_4)$. For each $n \in \mathbb{N}$, denote $\mathcal{G}_n = \{P \in \mathcal{P} : d(P) < 1/n\}$. Then, $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}$ is a $\sigma$-$(P_4)$-strong network. Furthermore, by Lemma 3.2, each $\mathcal{G}_n$ is a $cs^*$-cover. Therefore, $\bigcup\{\mathcal{G}_i : i \in \mathbb{N}\}$ is a $\sigma$-$(P_4)$-strong network consisting of $cs^*$-covers for $X$.

(3) $\Rightarrow$ (2) $\Rightarrow$ (1). It is obvious. □

**Lemma 3.5.** *Let $f : M \to X$ be a sequentially-quotient $\alpha(P)$-map, and $M$ be a metric space. Then, $X$ has a $cs^*$-network having property $\sigma$-$(P)$.*

*Proof.* Since $f$ is a $\alpha(P)$-map, by using the proof of [6, Theorem 9], [7, Theorem 4], [8, Lemma 2.1], [9, Theorem 2.1], and [12, Theorem 1.1] there exists a base $\mathcal{B}$ for $X$ such that $f(\mathcal{B})$ is a network having property $\sigma$-$(P)$. On the other hand, since every $cs^*$-network is preserved by a sequentially-quotient map, we have $f(\mathcal{B})$ is a $cs^*$-network. □

By Theorem 3.3, Theorem 3.4, Lemma 2.4 and Lemma 3.5, we have:

**Corollary 3.6.** *The following are equivalent for a Cauchy sn-symmetric space $X$.*

(1) *$X$ has a $cs^*$-network having property $\sigma$-$(P_1)$;*
(2) *$X$ has a $\sigma$-$(P_1)$-strong $cs^*$-network;*
(3) *$X$ has a $\sigma$-$(P_1)$-strong network consisting of $cs^*$-covers;*
(4) *$X$ is a pseudo-sequence-covering compact, $\alpha(P_1)$-image of a metric space;*
(5) *$X$ is a sequentially-quotient $\alpha(P_1)$-image of a metric space.*

**Remark 3.7.** By Corollary 3.6, in case that the property $(P_1)$ is point-finite and $X$ is Cauchy symmetric, we get a partial answer to the question in [18, Question 3.9], and get an another partial answer to Question 1.1.

**Corollary 3.8.** *The following are equivalent for a Cauchy sn-symmetric space $X$.*

(1) *$X$ has a $cs^*$-network having property $\sigma$-$(P_2)$;*
(2) *$X$ has a $\sigma$-$(P_2)$-strong $cs^*$-network;*
(3) *$X$ has a $\sigma$-$(P_2)$-strong network consisting of $cs^*$-covers;*
(4) *$X$ is a pseudo-sequence-covering $\pi$, $\alpha(P_2)$-image of a metric space;*
(5) *$X$ is a sequentially-quotient $\alpha(P_2)$-image of a metric space.*

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