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SOME CHARACTERIZATIONS OF PRE-METRIZABILITY

A. GARCÍA-MÁYNEZ† AND M. A. LÓPEZ-RAMÍREZ

Abstract. The class of pre-metrizable spaces (i.e., perfect preimages of metrizable spaces) coincides with the class of paracompact p-spaces. In this paper we give three additional characterizations. One of them is the following:

(1) A Tychonoff space X is pre-metrizable if and only if there exists a zero set H in $X \times \beta X$ such that $\Delta(X) \subseteq H \subseteq X \times X$, where βX is the Stone-Čech compactification of X and $\Delta(X) = \{(x, x) : x \in X\}$.

Another one depends on the existence of a countable family of normal covers of X satisfying a certain property.

The final characterization requires X to be in the class of pseudoparacompact spaces, which includes both pseudocompact and paracompact spaces, together with an additional property which requires every open cover of X to be semi-normal.

1. Definitions and preliminary results

All spaces considered in this paper are completely regular and Hausdorff. As usual, βX denotes the Stone-Čech compactification of a space X. The p-spaces were originally defined by A. V. Arhangel'skii in [1]. Čech-complete and Moore spaces, and hence, locally compact and metrizable spaces, are examples of p-spaces. An interesting subclass of Čech-complete spaces are *ultracomplete* spaces:

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Definition 1.1. A space X is *ultracomplete* if X has a countable local basis as a subspace of βX (see [7]).

We have the following characterizations [11]:

Theorem 1.2. *A space X is a p-space if and only if there exists a G_δ set L in $X \times \beta X$ such that $\Delta(X) = \{(x, x) : x \in X\} \subseteq L \subseteq X \times X$.*

We quote the following characterization of pre-metrizable spaces [1]:

Theorem 1.3. *A space X is* pre-metrizable *if and only if X is a paracompact p-space.*

Among the multiple characterizarions of paracompactness ([21] and [20]), we state the following:

Theorem 1.4. *X is paracompact if and only if $X \times \beta X$ is a normal space.*

Theorem 1.5. *X is paracompact if and only if every open cover α of X has an open barycentric refinement β, i.e., $\beta^\Delta = \{St(x, \beta) : x \in X\}$ refines α (see* [19]*).*

Corollary 1.6. *If α is an open cover of a paracompact space X, there exists a sequence $\alpha_1, \alpha_2, \ldots,$ of open covers of X such that α_1^Δ refines α and for every natural number n, α_{n+1}^Δ refines α_n.*

Definition 1.7. Dropping the assumption of paracompactness, we say a sequence of open covers $\alpha_1, \alpha_2, \ldots,$ of a space X is *normal* if for every $n \in \mathbb{N}$, α_{n+1}^Δ refines α_n. An open cover γ of a space X is said to be *normal* if γ belongs to a normal sequence of covers.

Therefore, Theorem 1.5 may be re-stated as follows:

Theorem 1.8. *A space X is paracompact if and only if every open cover of X is normal.*

If A is a closed subset of a space X, we consider two types of embeddings:

Definition 1.9. A is C_1-*embedded* in X if for every zero set K in X disjoint from A, there exists a zero set H in X such that $A \subseteq H \subseteq X \backslash K$.

For instance, every pseudocompact subset of a space X is C_1-*embedded* in X (see [12]). It is obvious that every zero set in X is C_1-embedded in X.

Definition 1.10. A is C_2-embedded in X if for every closed set L disjoint from A, there exist zero sets H, K in X such that $A \subseteq H \subseteq X \backslash K \subseteq X \backslash L$.

Clearly, every C_2-*embedded* set A in X is C_1-embedded in X. By Urysohn's lemma, a space X is normal if and only if every closed subset of X is C_2-embedded in X.

Definition 1.11. For every subset A of X, we define:

$$A^* = \beta X \backslash Cl_{\beta X}(X \backslash A).$$

Clearly A^* is open in βX and it is the largest open set in βX whose intersection with X is $int_X A$. We always have $A^* \subseteq Cl_{\beta X} A$.

For every pair of subsets A, B of X we have $(A \cap B)^* = A^* \cap B^*$. If A and B are cozero sets we also have $(A \cup B)^* = A^* \cup B^*$ (see [14]).

Definition 1.12. For every open cover α of a space X we define:

$$L(\alpha) = \bigcup \{A^* : A \in \alpha\};$$

$$E(\alpha) = \bigcup \{A \times A^* : A \in \alpha\}.$$

It is easy to see that $E(\alpha)$ is an open set in $X \times \beta X$ containing $\Delta(X)$ and $L(\alpha)$ is an open neighborhood of X in βX.

Definition 1.13. Let $A \subseteq X$ and let V be a neighborhood of A. We say V is a *strong neighborhood* of A if there exists a zero set H in X and a cozero set U in X such that $A \subseteq H \subseteq U \subseteq V$.

The following characterization of normal covers is in [13]:

Theorem 1.14. *An open cover α of a space X is normal if and only if $E(\alpha)$ is a strong neighborhood of $\Delta(X)$.*

Therefore, using Tamano's Theorem 1.4, we have:

Theorem 1.15. *A space X is paracompact if and only if $\Delta(X)$ is C_2-embedded in $X \times \beta X$.*

The next characterization of normal covers of a topological space is well know (see, for example, [3], p. 122, [16], Theorems 1.2 and 1.4 and [17], Theorem 1.2, among others):

Theorem 1.16. *An open cover α of a space X is normal if and only if α has a locally finite cozero refinement.*

We close this section by defining a class of spaces containing all paracompact and all pseudocompact spaces and a class of open covers containing all normal covers.

Definition 1.17. A space X is *pseudo-paracompact* if $\Delta(X)$ is C_1-embedded in $X \times \beta X$.

Definition 1.18. An open cover α of a space X is *semi-normal* if there exists a cozero set U in $X \times \beta X$ such that $\Delta(X) \subseteq U \subseteq E(\alpha)$.

2. Characterizations via frames

Definition 2.1. A *frame* of a space X is simply a non-empty family $\{\alpha_i : i \in J\}$ of open covers of X. If every α_i is normal (resp., seminormal) we say that the frame $\{\alpha_i : i \in J\}$ is normal (resp., semi-normal).

Definition 2.2. Let $\{\alpha_i : i \in J\}$ be a frame in X and let η be a filter in X.

(1) η is *Cauchy* with respect to $\{\alpha_i : i \in J\}$ if for every $i \in J$, we have $\eta \cap \alpha_i \neq \emptyset$.
(2) η is *cofinally Cauchy* with respect to $\{\alpha_i : i \in J\}$ if for every $i \in J$, we can find an element $V_i \in \alpha_i$ such that $V_i \cap N \neq \emptyset$ for every $N \in \eta$.

Definition 2.3. For every filter η in a space X, we define its adherence set as

$$A_\eta = \bigcap\{Cl_{\beta X} N : N \in \eta\}.$$

Definition 2.4.

(1) A frame $\{\alpha_i : i \in J\}$ is *ultracomplete* if every cofinally Cauchy filter η in X satisfies $X \cap A_\eta \neq \emptyset$.
(2) A frame $\{\alpha_i : i \in J\}$ is of *Čech − type* if every Cauchy filter η satisfies $A_\eta \subseteq X$.
(3) A frame $\{\alpha_i : i \in J\}$ is of *p-type* if every fixed Cauchy filter η in X satisfies $A_\eta \subseteq X$.

Definition 2.5. Let W be an open neighborhood of X in βX. Select an open cover $\alpha_W = \{V_x : x \in X\}$, where $x \in V_x \subseteq Cl_{\beta X} V_x \subseteq W$. We say then that α_W is *induced* by W. Likewise, if T is an open neighborhood of $\Delta(X)$ in βX, $\{V_x : x \in X\}$ is induced by T if $\Delta(X) \subseteq \bigcup\{V_x \times Cl_{\beta X} V_x : x \in X\} \subseteq T$.

The next lemma can be found in [9, 3.1.5]:

Lemma 2.6. *Let $\{K_i : i \in J\}$ be a family of compact sets in a space T_2 with the PIF. If $\bigcap_{i\in J} K_i \subseteq U$, U an open set of X, then there exists $J_0 \subseteq J$, J_0 finite such that $\bigcap_{i\in J_0} K_i \subseteq U$.*

We give now the following characterizations:

Theorem 2.7. *Let $\{\alpha_i : i \in J\}$ be an ultracomplete frame of a space X. Then $\{L(\alpha_i) : i \in J\}$ is a local basis of X in βX. Conversely, if $\{W_i : i \in J\}$ is a local basis of X in βX and if α_{W_i} is a cover of X induced by W_i $(i \in J)$, then $\{\alpha_{W_i} : i \in J\}$ is an ultracomplete frame of X.*

Proof. (Necessity) Let T be an open set in βX such that $X \subseteq T \neq \beta X$. Define $K = \beta X \setminus T$ and let η be the filter in X consisting of all possible intersections $X \cap U$, where U is a neighborhood of K in βX. Since $A_\eta = K \subseteq \beta X \setminus X$, η cannot be cofinally Cauchy with respect to $\{\alpha_i : i \in J\}$. Therefore, for some $i \in J$ and for every $G \in \alpha_i$, there exists an open neighborhood L_G of K in βX such that $G \cap L_G = \emptyset$. Therefore, if $G \in \alpha_i$, we have $G \subseteq G^* \subseteq Cl_{\beta X} G \subseteq \beta X \setminus L_G$. We deduce then that $L(\alpha_i) \cap K = \emptyset$, or $L(\alpha_i) \subseteq T$.

(Sufficiency) Let $\{W_i : i \in J\}$ be a local basis of X in βX and for each $i \in J$, let α_{W_i} be a cover of X induced by W_i. We prove that $\{\alpha_{W_i} : i \in J\}$ is an ultracomplete frame of X. Let η be a cofinally Cauchy filter with respect to $\{\alpha_{W_i} : i \in J\}$. For each $i \in J$, select an element $L_i \in \alpha_{W_i}$ such that $L_i \cap N \neq \emptyset$ for each $N \in \eta$. We must prove that $X \cap A_\eta \neq \emptyset$. Suppose, on the contrary, that $K = \bigcap\{Cl_{\beta X} N : N \in \eta\} \subseteq \beta X \setminus X$. By hypotesis, there exists an index $i \in J$ such that $W_i \subseteq \beta X \setminus K$. Therefore, for each $L \in \alpha_{W_i}$, we have:

$$L \subseteq L^* \subseteq Cl_{\beta X} L \subseteq W_i \subseteq \beta X \setminus K.$$

The set $U = \beta X \setminus Cl_{\beta X} L_i$ is a neighborhood of K. By Lemma 2.6, we deduce the existence of an element $N \in \eta$ such that $N \subseteq \beta X \setminus Cl_{\beta X} L_i$, in contradiction with $L_i \cap N \neq \emptyset$. □

Corollary 2.8. [15] *The character $\chi(X, \beta X)$ coincides with the least infinite cardinal number κ such that X has an ultracomplete frame of cardinality lower than or equal to κ. Therefore, X is ultracomplete if and only if X has a countable ultracomplete frame.*

Theorem 2.9. *Let $\{\alpha_i : i \in J\}$ be a frame of X of* Čech-*type. Then $X = \bigcap\{L(\alpha_i) : i \in J\}$. Conversely, if $\{W_i : i \in J\}$ is a family of open neighborhoods of X in βX such that $X = \bigcap\{W_i : i \in J\}$ and if α_{W_i} is a cover of X induced by W_i $(i \in J)$, then $\{\alpha_{W_i} : i \in J\}$ is a frame of X of* Čech-*type.*

Proof. (Necessity) We must prove that $X = \bigcap\{L(\alpha_i) : i \in J\}$. Suppose, on the contrary, that there exists a point $z \in \beta X \setminus X$ such that $z \in L(\alpha_i)$ for every $i \in J$. Select $V_i \in \alpha_i$ such that $z \in V_i^*$. Therefore the family:

$$\eta_0 = \{V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_k} : k \in \mathbb{N}\}$$

is a filterbase in X, (each element of η_o is non-empty, because $V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_k} = \emptyset$ would imply that $\emptyset = (V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_k})^* = V_{i_1}^* \cap V_{i_2}^* \cap \cdots \cap V_{i_k}^*$, contradicting the fact that $z \in V_{i_1}^* \cap V_{i_2}^* \cap \cdots \cap V_{i_k}^*$). The filter η of X with basis η_0 is then Cauchy with respect to $\{\alpha_i : i \in J\}$. Our hypothesis implies that $A_\eta \subseteq X$. But also $z \in A_\eta$ since $z \in (V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_k})^* \subseteq Cl_{\beta X}(V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_k})$ for each choice $i_i, i_2, \ldots, i_k \in J$ and this is a contradiction.

(Sufficiency) Let η be a Cauchy filter with respect to $\{\alpha_{W_i} : i \in J\}$. Proceeding by contradiction, suppose there exists a point $z \in A_\eta \cap (\beta X \backslash X)$. Choose $L_i \in \eta \cap \alpha_{W_i}$. By construction, $Cl_{\beta X} L_i \subseteq W_i$. Therefore, $z \in \bigcap\{W_i : i \in J\} = X$, a contradiction. $\square$

Corollary 2.10. *The pseudocharacter $\chi(X, \beta X)$ coincides with the least infinite cardinal number κ such that X has a frame of* Čech-*type of cardinality lower than or equal to κ. Therefore, X is* Čech-*complete if and only if X has a countable frame of* Čech-*type.*

Theorem 2.11. *Let $\{\alpha_i : i \in J\}$ be a frame of X of p-type. Then $L = \bigcap\{E(\alpha_i) : i \in J\}$ is a G_δ set in $X \times \beta X$ such that $\Delta(X) \subseteq L \subseteq X \times X$. Conversely, if $\{T_i : i \in J\}$ is a family of open sets in $X \times \beta X$ such that $\Delta(X) \subseteq \bigcap\{T_i : i \in J\} \subseteq X \times X$ and if α_{T_i} is a cover of X induced by T_i $(i \in J)$, then $\{\alpha_{T_i} : i \in J\}$ is a frame of X of p-type.*

Proof. (Necessity) Take a pair $(p, z) \in \bigcap\{E(\alpha_i) : i \in J\}$, $p \in X$, $z \in \beta X$. Choose an element $V_i \in \alpha_i$ such that $(p, z) \in V_i \times V_i^*$. We prove as in 2.9, that

$$\eta_0 = \{V_{i_1} \cap V_{i_2} \cap \cdots \cap V_{i_k} : k \in \mathbb{N}, i_1, i_2, \ldots i_k \in J\}$$

is a fixed filterbase in X which is Cauchy with respect to $\{\alpha_i : i \in J\}$. Let η be the filter in X with basis η_0. By hypothesis, $A_\eta \subseteq X$. Hence $z \in N^* \subseteq Cl_{\beta X} N$ for every $N \in \eta$ and this implies that $z \in A_\eta \subseteq X$, i.e., $\bigcap\{E(\alpha_i) : i \in J\} \subseteq X \times X$.

(Sufficiency) Let η be a fixed filter in X which is Cauchy with respect to $\{\alpha_{T_i} : i \in J\}$. Take a point $p \in X$ such that $p \in N$ for every $N \in \eta$ and let $z \in \bigcap\{Cl_{\beta X} N : N \in \eta\}$. For each $i \in J$, select an element $L_i \in \eta \cap \alpha_{T_i}$. We have then $(p, z) \in L_i \times Cl_{\beta X} L_i \subseteq T_i$. Since $\bigcap\{T_i : i \in J\} \subseteq X \times X$, we conclude that $z \in X$. $\square$

Corollary 2.12. *X is a p-space if and only if X has a countable frame of p-type.*

3. Main results

Definition 3.1. Let $\alpha_1, \alpha_2, \ldots,$ be a sequence of open covers of a space X. We say $\alpha_1, \alpha_2, \ldots$ is a $\omega\Delta$-*sequence* if whenever $x_n \in St(x, \alpha_n)$, with $x \in X$, the sequence $\{x_n : n \in \mathbb{N}\}$ has a cluster point.

We quote the following theorem (see [5]):

Theorem 3.2. *Let $\alpha_1, \alpha_2, \ldots$ be a $\omega\Delta$-sequence of a space X. If each cover α_i is normal, then there exists a metrizable space Y and a closed continuous surjective map $\varphi : X \to Y$ such that $\varphi^{-1}\varphi(x) = \bigcap\{St(x, \alpha_n) : n \in \mathbb{N}\}$ and $\varphi^{-1}\varphi(x)$ is countably compact for each $x \in X$. Additionally, if each $\bigcap\{St(x, \alpha_n) : n \in \mathbb{N}\}$ is compact, then X is a paracompact p-space.*

Definition 3.3. A space X is *definitely* p if there exists a sequence $W_1, W_2, \dots$ of cozero sets in $X \times \beta X$ such that $\Delta(X) \subseteq \bigcap_{n=1}^{\infty} W_n \subseteq X \times X$.

We prove now our main result:

Theorem 3.4. *The following conditions on a space X are equivalent:*

(1) *X is pre-metrizable.*
(2) *X is a paracompact p-space.*
(3) *X is a pseudoparacompact p-space and every open cover of X is semi-normal.*
(4) *X is a pseudoparacompact definitely p-space.*
(5) *There exists a zero set K in $X \times \beta X$ such that $\Delta(X) \subseteq K \subseteq X \times X$.*
(6) *X has a countable p-frame consisting of normal covers.*

Proof. The equivalence of (1) and (2) is well known (see [1]). (2)⇒(3) This implication is clear because every paracompact space is pseudoparacompact and every open cover of a paracompact space is normal.

(3)⇒(4) Let $\alpha_1, \alpha_2, \dots$ be a p-frame of X. Since every cover α_i is seminormal, there exists, for each $i \in \mathbb{N}$, a cozero set W_i in $X \times \beta X$ such that $\Delta(X) \subseteq W_i \subseteq E(\alpha_i)$. By Theorem 2.11 we have $\Delta(X) \subseteq \bigcap_{i=1}^{\infty} W_i \subseteq \bigcap_{i=1}^{\infty} E(\alpha_i) \subseteq X \times X$ and X is definitely p.

(4)⇒(5) Let $W_1, W_2, \dots$ be cozero sets in $X \times \beta X$ such that $\Delta(X) \subseteq \bigcap_{i=1}^{\infty} W_n \subseteq X \times X$. Since $\Delta(X)$ is C_1-embedded in $X \times \beta X$, for each $i \in \mathbb{N}$ we may find a zero set H_i in $X \times \beta X$ such that $\Delta(X) \subseteq H_i \subseteq W_i$. Hence $H = \bigcap_{i=i}^{\infty} H_i$ is a zero set in $X \times \beta X$ such that $\Delta(X) \subseteq H \subseteq X \times X$.

(5)⇒(6) Let H be a zero set in $X \times \beta X$ such that $\Delta(X) \subseteq H \subseteq X \times X$. Define a sequence $U_1, U_2, \dots$ of cozero sets in $X \times \beta X$ such that $H = \bigcap_{n=1}^{\infty} U_n$ and $Cl_{X \times \beta X} U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$. Since H and $X \times \beta X - U_n$ are disjoints zero sets in $X \times \beta X$, we can find a continuous map $\varphi_n : X \times \beta X \to [0, 1 - 2^{-n}]$ such that $\varphi_n^{-1}(0) = H$ and $\varphi_n^{-1}(1 - 2^{-n}) = X \times \beta X - U_n$. Define $g_n : X \times X \to [0,1]$ by means of the formula:

$$g_n(x,y) = \sup\{\{|\varphi_n(x,p) - \varphi_n(y,p)| : p \in \beta X\}\} \, x, y \in X.$$

It is easy to prove that g_n is a continuous pseudo-metric in X. For each $n \in \mathbb{N}$, let $\alpha_n = \{V_{2^{-n-1}}^{g_n}(x) : x \in X\}$. Each α_n is a normal cover of X. Besides:

$$V_{2^{-n-1}}^{g_n}(x) \times V_{2^{-n-1}}^{g_n}(x) \subseteq U_n$$

for each $x \in X$ and $n \in \mathbb{N}$. To prove this inclusion, take two points $x', x'' \in V_{2^{-n-1}}^{g_n}(x)$ and suppose, on the contrary, that $(x', x'') \notin U_n$.

Then $\varphi(x', x'') = 1 - 2^{-n}$. On the other hand, by definition of g_n, we have:

$$2^{-n} = 2^{-n-1} + 2^{-n-1} > g_n(x, x') + g_n(x, x'') \geq g_n(x', x'') \geq \\ \varphi(x', x'') - \varphi(x'', x'') = \varphi(x', x'') = 1 - 2^{-n}.$$

From here we obtain $1 - 2^{-n} < 2^{-n}$, a contradiction. Therefore:

$$\bigcup \{V^{g_n}_{2^{-n-1}}(x) \times Cl_{\beta X} V^{g_n}_{2^{-n-1}}(x) : x \in X\} \subseteq Cl_{X \times \beta X} U_n \subseteq U_{n-1}.$$

We deduce then that the normal covers $\alpha_1, \alpha_2, \ldots$ constitute a countable p-frame of the space X.

(6) $\Rightarrow$ (2) Let $\alpha_1, \alpha_2, \ldots$ be a countable p-frame of X consisting of normal covers. By Theorem 1.16, α_1 has a cozero and locally finite refinement β_1. We may further require that for each $x \in X$, there exists an element $A_x \in \alpha_1$ such that $St(x, \beta_1) \subseteq Cl_{\beta X} St(x, \beta_1) \subseteq A^*_x$ (see definition 1.11). Inductively, suppose the covers $\beta_1, \beta_2, \ldots, \beta_{n-1}$ have already been defined. Let β_n be a cozero and locally finite cover of X such that $St(x, \beta_n) \subseteq Cl_{\beta X} St(x, \beta_n) \subseteq A^*_x \cap B^*_x$, $(A_x \in \alpha_n, B_x \in \beta_{n-1})$. Once all the covers β_n have been constructed, define:

$$K_x = \bigcap_{n=1}^{\infty} St(x, \beta_n)^*, \quad x \in X.$$

Therefore, the set K_x is a compact G_δ in βX and hence $\{St(x, \beta_n)^* : n \in \mathbb{N}\}$ is a local basis of K_x in βX. However, $\eta_x = \{St(x, \beta_n) : n \in \mathbb{N}\}$ is a fixed filterbase in X which is Cauchy with respect to the frame $\alpha_1, \alpha_2, \ldots$. Therefore, the adherence $K_x = \bigcap_{n=1}^{\infty} St(x, \beta_n)^* = \bigcap_{n=1}^{\infty} Cl_{\beta X} St(x, \beta_n)$ of η_x is contained in X. Therefore, $\{St(x, \beta_n) : n \in \mathbb{N}\}$ is a local basis of K_x in X and $\beta_1, \beta_2 \ldots$ is a normal $\omega\Delta$-sequence in X. By Theorem 3.2, X is pre-metrizable. □

4. Open problems

We finish this paper with some open problems:

Q_1: Is there an example of a semi-normal cover which is not normal?

Q_2 : Is every definitely p-space paracompact?

Q_3 : Is there an example of a pseudo-paracompact space which is not paracompact nor pseudocompact?

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