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ULTRA STRONG S-SPACES

JOAN E. HART AND KENNETH KUNEN

Abstract. A strong S-space is an S-space X such that X^n is HS for all finite n. We consider replacing the "n" here with something infinite.

1. Introduction

All topological spaces considered in this paper are T_2 (Hausdorff).

A space X is *hereditarily separable* (*HS*) iff all subspaces of X are separable, and *hereditarily Lindelöf* (*HL*) iff all subspaces of X are Lindelöf. Also, X is *strongly* HS/HL iff X^n is HS/HL for all $n \in \omega$. Then, X is an *S-space* iff X is T_3 and HS but not HL, and X is a *strong S-space* iff in addition X is strongly HS.

S-spaces are consistent with $\mathsf{MA}(\aleph_1)$ [13], but are refuted by PFA [14]. On the other hand, strong S-spaces are refuted by $\mathsf{MA}(\aleph_1)$ [8], but exist under CH [11, 3]. For more background, see [12].

In this paper, we use $\diamondsuit$ to prove the existence of *ultra strong* S-spaces, satisfying a natural strengthening of strongly HS:

Definition 1.1. For topological spaces Q and X, let X^Q denote the space $C(Q, X)$ of continuous functions with the compact-open topology. Call X an *ultra strong S-space* iff X is an S-space and in addition X^Q is HS for all second countable compact Q.

Equivalently, an S-space X is ultra strong if X^Q is HS, where Q is the Cantor set; see Proposition 5.15.

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For discrete spaces Q, the space X^Q is just the Tychonov product topology, so Definition 1.1 applied with finite Q shows that ultra strong implies strong. The converse implication is false; the simplest infinite compact space is $\omega+1$, and in Section 7, we shall (assuming CH) produce a strong S-space X for which $X^{\omega+1}$ is not HS.

We note that one cannot simply delete the "second countable", since if Q is compact and infinite and zero-dimensional and $|X| \geq 2$, then X^Q contains a closed copy of $\{0,1\}^Q$, which is discrete and of size $w(Q)$. If we had considered instead discrete Q (so we would have the Tychonov product), then there would be nothing new to be said here. If X is strongly HS, then X^ω is always HS, while for $\kappa > \aleph_0$, the Tychonov product X^κ is never HS unless $|X| < 2$.

The results of Sections 4 and 5 will yield our ultra strong S-space X. We shall also show (Lemma 5.3) that for any of our S-spaces X, if $P \subseteq Q$ and X^Q is HS then X^P is HS. In particular, if X^Q is HS for some infinite compact second countable Q, then X is a strong S-space. All our S-spaces will be locally compact and locally countable, so that X^Q will be non-trivial; for example, see our strong but not ultra strong S-space in Theorem 7.2. If we considered instead an extremally disconnected strong S-space X, then every continuous function from Q into X would have finite range, so X would trivially be ultra strong. (An extremally disconnected S-space exists under $\diamondsuit$ [16]; we do not know whether it is consistent that a strongly HS one exists.)

Our ultra strong S-space X also gives us a homogeneous ultra strong S-space; our X is zero-dimensional and first countable, so X^ω is homogeneous by Dow and Pearl [4]. For Q compact and $n \in \omega$, $(X^Q)^n \cong X^{Q\times n}$; thus, for Q also second countable, X^Q is strongly HS, and hence, as remarked above, $(X^Q)^\omega$ is HS. Since ω is locally compact, $(X^\omega)^Q \cong X^{Q\times\omega} \cong (X^Q)^\omega$ (see Engelking [5], Theorem 3.4.8), so X^ω is ultra strong.

The actual construction of our S-space will be done in Section 4, but it will be done in terms of the Vietoris topology, rather than the compact-open topology. Let $\mathcal{K}(X)$ denote the set of all compact non-empty subsets of X, given the Vietoris topology; this topology is described in more detail in Section 2. In Section 4, we shall construct our S-space X with the property that $\mathcal{K}(X)$ is HS. Then, in Section 5, we shall explain why this implies that X is ultra strong.

Our basic constructions involve $\mathcal{K}(X)$ because it seems more tractable than X^Q. In particular, we shall use the fact that $\mathcal{K}(X)$ is compact whenever X is compact. Note that $X^{\omega+1}$ is never compact when $|X| \geq 2$, since it contains a closed copy of $\{0,1\}^{\omega+1} = C(\omega+1, \{0,1\})$, which is discrete and infinite.

Section 5 contains further details on the compact-open topology and describes its relation to the Vietoris topology. This relationship will let us prove that for our S-spaces X, if $\mathcal{K}(X)$ is HS then $\mathcal{K}(\mathcal{K}(X))$ is also HS. The corresponding statement for the compact-open topology is trivial; that is, if X is an ultra strong S-space and P, Q are second countable compact spaces, then $(X^Q)^P$ is HS because $(X^Q)^P \cong X^{P\times Q}$ (Engelking [5], Theorem 3.4.8).

This homeomorphism of function spaces $(X^Q)^P$, $X^{P\times Q}$, and $(X^P)^Q$ holds for every space X with locally compact spaces P and Q. In contrast, no such identification exists for the related subspaces of $\mathcal{K}(\mathcal{K}(X))$. Section 9 shows that even for countable compact P, Q, the two spaces $[[X]^{\leq Q}]^{\leq P}$ and $[[X]^{\leq P}]^{\leq Q}$ can differ significantly.

Section 6 describes an application of Section 4 results to properties we introduced in paper [6]. Section 3 provides a brief note on the Section 4 use of elementary submodels, and Section 8 comments on compact ultra strong S-spaces.

All of our S-space constructions follow the general pattern of [7, 12, 11, 3]: Start with an HS space (our *fundamental space*), and refine the topology so that it remains HS but fails to be HL. To get an S-space X such that $\mathcal{K}(X)$ is HS, our fundamental space can be the Cantor set or ω^ω. To produce intermediate results, such as a strong S-space X such that $X^{\omega+1}$ is not HS, the fundamental space will be some variant of the Sorgenfrey line.

We shall use the well-known ([12], Theorem 3.1) characterization that X is HS iff X has no *left separated* ω_1*-sequences*. Sequences $\langle x_\alpha : \alpha < \omega_1\rangle$ are left separated provided that each $x_\alpha \notin \mathrm{cl}(\{x_\xi : \xi < \alpha\})$. As in many constructions in the literature, we use CH or $\diamondsuit$ to "capture and kill" all potential initial segments of such sequences.

2. Remarks on the Vietoris Topology

Definition 2.1. Let X be any T_3 space. Then $\mathcal{K}(X)$ denotes the family of all non-empty *compact* subsets of X. We make this into a topological space by giving it the *Vietoris topology*, whose subbase consists of all subsets of $\mathcal{K}(X)$ of the forms $\{K : K \subseteq U\}$ and $\{K : K \cap U \neq \emptyset\}$, where U is an open subset of X.

The Vietoris topology is usually defined on the family of non-empty *closed* subsets of X, called the *hyperspace* or *exponential space* (see Engelking [5], page 120), so our $\mathcal{K}(X)$ is a subspace of the hyperspace. Using the hyperspace instead in our construction would yield left separated ω_1-sequences, so that our space would not be HS (see Lemma 8.1).

Lemmas 2.3 and 2.6 give us convenient ways of describing basic open sets in $\mathcal{K}(X)$.

Definition 2.2. Let $\mathcal{N}(V_0, \ldots, V_\ell) = \{K \in \mathcal{K}(X) : K \subseteq \bigcup_i V_i \ \& \ \forall i \ K \cap V_i \neq \emptyset\}$, where $\ell \in \omega$ and the V_i are open subsets of X. This $\mathcal{N}(V_0, \ldots, V_\ell)$ is in *standard form* iff the V_i are clopen and non-empty and pairwise disjoint.

Lemma 2.3. *For each $B \in \mathcal{K}(X)$, a local base at B is the set of all $\mathcal{N}(V_0, \ldots, V_\ell)$ such that $B \in \mathcal{N}(V_0, \ldots, V_\ell)$. Furthermore, if X is zero-dimensional, then one can require also that $\mathcal{N}(V_0, \ldots, V_\ell)$ be in standard form; then these $\mathcal{N}(V_0, \ldots, V_\ell)$ are clopen in $\mathcal{K}(X)$, so that $\mathcal{K}(X)$ is also zero-dimensional.*

Proof. To prove the last sentence, let $\mathcal{N} = \mathcal{N}(V_0, \ldots, V_\ell)$, and consider any B in $\mathcal{K}(X)$ such that $B \notin \mathcal{N}$. There are two possible cases: 1. If $B \not\subseteq \bigcup_i V_i$, then, since $\bigcup_i V_i$ is clopen, $\{C \in \mathcal{K}(X) : C \cap (X \backslash \bigcup_i V_i) \neq \emptyset\}$ is a Vietoris neighborhood of B that is disjoint from $\mathcal{N}$. 2. If $B \subseteq \bigcup_i V_i$ but $B \cap V_i = \emptyset$ for some i, then, since V_i is clopen, $\{C \in \mathcal{K}(X) : C \subseteq X \backslash V_i\}$ is a neighborhood of B that is disjoint from $\mathcal{N}$. $\square$

Note that the value of $\mathcal{N}(V_0, \ldots, V_\ell)$ only depends on the *set* $\{V_0, \ldots, V_\ell\}$, not the order in which the V_i are listed.

Lemma 2.4. *Let X be zero-dimensional, and let $\mathcal{N}(U_0, \ldots, U_k)$ and $\mathcal{N}(V_0, \ldots, V_\ell)$ be standard form basic sets. Then*

1. *$\mathcal{N}(U_0, \ldots, U_k) \subseteq \mathcal{N}(V_0, \ldots, V_\ell)$ iff both ① $U_0 \cup \cdots \cup U_k \subseteq V_0 \cup \cdots \cup V_\ell$ and ② $\forall j \ \exists i \ U_i \subseteq V_j$.*
2. *If $\mathcal{G} = \mathcal{N}(U_0, \ldots, U_k)$ then $\bigcup \mathcal{G} = U_0 \cup \cdots \cup U_k$.*
3. *$\mathcal{N}(U_0, \ldots, U_k) = \mathcal{N}(V_0, \ldots, V_\ell)$ iff $k = \ell$ and $\{U_0, \ldots, U_k\} = \{V_0, \ldots, V_\ell\}$.*

Proof. For (1): the $\leftarrow$ direction is clear from Definition 2.2. For the $\rightarrow$ direction: If either ① or ② is false, then we can find a $k+1$ element set K in $\mathcal{N}(U_0, \ldots, U_k) \setminus \mathcal{N}(V_0, \ldots, V_\ell)$; K contains one element from each U_i. When $\neg$①, include in K an element of $(U_0 \cup \cdots \cup U_k) \setminus (V_0 \cup \cdots \cup V_\ell)$.

(2) is also easily proved by considering the finite elements of $\mathcal{G}$. Note that $\bigcup \mathcal{G}$ has its usual set-theoretic meaning — that is, the union of all the elements $K \in \mathcal{G}$.

(3) follows easily from (1). $\square$

Definition 2.5. For each standard form $\mathcal{G} = \mathcal{N}(U_0, \ldots, U_k)$, let $\mathcal{G}^*$ denote the corresponding set $\{U_0, \ldots, U_k\}$.

This definition makes sense in view of Lemma 2.4 (3). It is sometimes notationally convenient to refer to $\mathcal{G}^*$ or $\bigcup \mathcal{G}$ without mentioning the $U_0, \ldots, U_k$.

The next lemma shows that one may build a local base at B in $\mathcal{K}(X)$ just by using the basic neighborhoods assigned to the various points of B:

Lemma 2.6. *Assume that X is zero-dimensional and $B \in \mathcal{K}(X)$. For each $b \in B$, let $\mathcal{L}_b$ be a local clopen base at b. Then a local base at B in $\mathcal{K}(X)$ is the set of all $\widetilde{\mathcal{N}}(U_0, \ldots, U_m) := \mathcal{N}\big(U_0\ ,\ U_1 \backslash U_0\ ,\ U_2 \backslash (U_0 \cup U_1)\ ,\ \ldots\ ,\ U_m \setminus \bigcup_{\ell<m} U_\ell\big)$ such that for some* distinct *$b_0, \ldots, b_m \in B$: each $U_i \in \mathcal{L}_{b_i}$, each set $B \cap (U_i \backslash \bigcup_{\ell<i} U_\ell)$ is non-empty, and $B \subseteq \bigcup_{i<m} U_i$.*

Proof. To see that the $\widetilde{\mathcal{N}}(U_0, \ldots, U_m)$ form a local base at B, we shall apply Lemma 2.4 (1). Start with a standard form basic neighborhood $\mathcal{G} = \mathcal{N}(V_0, \ldots, V_n)$ of B. For each $b \in B$, choose $U(b) \in \mathcal{L}_b$ with $U(b)$ a subset of the $V_j \in \mathcal{G}^*$ such that $b \in V_j$. By compactness of B, choose a finite cover of B of the form $\{U(b) : b \in F\}$, where F is minimal. List F as $\{b_0, \ldots, b_m\}$, and let U_i denote $U(b_i)$. Then $B \in \widetilde{\mathcal{N}}(U_0, \ldots, U_m) \subseteq \mathcal{N}(V_0, \ldots, V_n)$. $\square$

This lemma will be used in the S-space construction when we consider multiple topologies on the same set. We shall also apply the following immediate consequence:

Lemma 2.7. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are both zero-dimensional topologies on the set X and $B \in \mathcal{K}(X, \mathcal{T}_1) \cap \mathcal{K}(X, \mathcal{T}_2)$ and $\mathcal{S} \subseteq \mathcal{K}(X, \mathcal{T}_1) \cap \mathcal{K}(X, \mathcal{T}_2)$ and $\mathcal{L}_b$ is the same in $\mathcal{T}_1$ and $\mathcal{T}_2$ for all $b \in B$, then $B \in \mathrm{cl}(\mathcal{S}, \mathcal{T}_1)$ iff $B \in \mathrm{cl}(\mathcal{S}, \mathcal{T}_2)$.*

For $\mathcal{S} \subseteq \mathcal{K}(X, \mathcal{T})$, we write $\mathrm{cl}(\mathcal{S}, \mathcal{T})$ to mean the closure of $\mathcal{S}$ in $\mathcal{K}(X, \mathcal{T})$.

The next definition and lemma involve some subsets of $\mathcal{K}(X)$:

Definition 2.8. Let X be any non-empty T_3 space and Q any non-empty compact space. Then $[X]^Q$ and $[X]^{\leq Q}$ denote the subspaces consisting of all $K \in \mathcal{K}(X)$ such that K is, respectively, homeomorphic to Q and homeomorphic to a continuous image of Q.

In particular, if Q is the natural number $n > 0$ with the discrete topology, then $[X]^n$ has its usual meaning, while our $[X]^{\leq n}$ would usually be called $[X]^{\leq n} \setminus \{\emptyset\}$.

Lemma 2.9. *Assume that X is a T_3 space and $0 < n < \omega$. Then in the following (1) $\leftrightarrow$ (2) $\leftrightarrow$ (3) and (4) $\leftrightarrow$ (5):*

1. *$[X]^n$ is HS.*
2. *$[X]^m$ is HS whenever $0 < m \leq n$.*
3. *$[X]^{\leq n}$ is HS.*
4. *X^n is HS.*
5. *X^m is HS whenever $0 < m \leq n$.*

Also, (1)(2)(3) *follow from* (4)(5)*, and all five are equivalent if* X *is submetrizable.*

Proof. (5) $\to$ (4) is trivial, and (4) $\to$ (5) holds because each X^m embeds into X^n.

(3) $\to$ (2) holds because each $[X]^m$ is a subspace of $[X]^{\leq n}$, and (2) $\to$ (3) holds because $[X]^{\leq n}$ is the *finite* union $\bigcup_{0<m\leq n}[X]^m$.

(2) $\to$ (1) is trivial. To prove (1) $\to$ (2), assume $\neg$(2), and assume that $0 < m \leq n$ and $\langle B_\alpha : \alpha < \omega_1\rangle$ is a left separated sequence in $[X]^m$, and we shall get such a sequence in $[X]^n$; of course, this is trivial unless $m < n$. X must be infinite, so let $W_0, \ldots, W_{2n}$ be non-empty open sets with disjoint closures. For each α, there are at least n values of $i < 2n$ such that $B_\alpha \cap \mathrm{cl}(W_i) = \emptyset$. Thinning the sequence, WLOG these are the same values for each α, so we have $i_1 < \cdots < i_{n-m}$ so that $B_\alpha \cap \mathrm{cl}(W_{i_\mu}) = \emptyset$ whenever $1 \leq \mu \leq n-m$. Then, choose a point $p_\mu \in W_{i_\mu}$, and let $C_\alpha = B_\alpha \cup \{p_\mu : 1 \leq \mu \leq n-m\}$. Then $\langle C_\alpha : \alpha < \omega_1\rangle$ is a left separated sequence in $[X]^n$.

(4) $\to$ (3) holds because $[X]^{\leq n}$ is the continuous image of X^n under the map $(x_0, \ldots, x_{n-1}) \mapsto \{x_0, \ldots, x_{n-1}\}$.

We now assume that X is submetrizable and prove (2) $\to$ (4), whereupon we are done. Let $\widehat{\mathcal{T}}$ denote the topology on X, and recall that submetrizable means that there is a coarser topology $\mathcal{T}$ on X such that $(X, \mathcal{T})$ is a metric space. It is also separable by (2), and hence second countable. Let $\mathcal{B}$ be a countable open base for $(X, \mathcal{T})$. Now, assume $\neg$(4), and we shall prove $\neg$(2).

Let $\langle \vec{x}_\alpha : \alpha < \omega_1\rangle$ be a left separated sequence in X^n. Thinning the sequence, WLOG for each $i, j < n$, either $\forall\alpha\,[x^i_\alpha = x^j_\alpha]$ or $\forall\alpha\,[x^i_\alpha \neq x^j_\alpha]$. Then, permuting the n-tuples, WLOG there is some m with $1 \leq m \leq n$ such that $i < j < m \to \forall\alpha\,[x^i_\alpha \neq x^j_\alpha]$, while $m \leq j < n \to \exists i < m\,\forall\alpha\,[x^i_\alpha = x^j_\alpha]$. Let $\vec{y}_\alpha$ be the m-tuple $(x^0_\alpha, \ldots, x^{m-1}_\alpha)$, and note that $\langle \vec{y}_\alpha : \alpha < \omega_1\rangle$ is a left separated sequence in X^m.

For each α: Choose sets $U_i \in \mathcal{B}$ for $i < m$ with disjoint closures such that each $x^i_\alpha \in U_i$. Since there are only countably many choices for the U_i, we may thin the sequence and assume WLOG that the U_i are independent of α. But now, the sequence $\langle \{x^i_\alpha : i < m\} : \alpha < \omega_1\rangle$ is left separated in $[X]^m$. $\square$

Lemma 5.3 gives a version of this lemma for more general $[X]^Q$ and $[X]^{\leq Q}$.

We shall now prove a "*Reduction Lemma*" (Lemma 2.14). It reduces statements of the form "$B \in \mathrm{cl}(\mathcal{S})$" to statements about proper closed subsets of B. It will be used in Section 4 (see Lemma 4.13) to prove results about B by induction on B.

Definition 2.10. Fix $\mathcal{S} \subseteq \mathcal{K}(X)$. For standard form $\mathcal{G} \subseteq \mathcal{K}(X)$, let $\mathcal{S} \mathbin{\backslash\!\!\backslash} \mathcal{G} = \{K \setminus \bigcup \mathcal{G} : K \in \mathcal{S} \;\&\; \forall U \in \mathcal{G}^* \; [K \cap U \neq \emptyset]\} \setminus \{\emptyset\}$.

Remark. For standard form $\mathcal{G} \subseteq \mathcal{K}(X)$ and $K \in \mathcal{K}(X)$: $\forall U \in \mathcal{G}^*$ $[K \cap U \neq \emptyset]$ iff $K \cap \bigcup \mathcal{G} \in \mathcal{G}$.

Lemma 2.11. *If X is zero-dimensional, $B \in \mathcal{K}(X)$, and W is a clopen subset of X with $B \backslash W \neq \emptyset$, then a local base at $B \backslash W$ in $\mathcal{K}(X)$ is the set of all standard form $\mathcal{H} = \mathcal{N}(V_0, \ldots, V_\ell)$ such that $B \backslash W \in \mathcal{H}$ and $\bigcup \mathcal{H} \cap W = \emptyset$.*

Lemma 2.12. *Assume X is zero-dimensional. Fix $B \in \mathcal{K}(X)$ and $\mathcal{S} \subseteq \mathcal{K}(X)$ with $B \in \mathrm{cl}(\mathcal{S})$. Let $\mathcal{G}$ be a standard form basic set with $B \setminus \bigcup \mathcal{G} \neq \emptyset$ and $\forall U \in \mathcal{G}^*$ $[B \cap U \neq \emptyset]$. Then $B \setminus \bigcup \mathcal{G} \in \mathrm{cl}(\mathcal{S} \mathbin{\backslash\!\!\backslash} \mathcal{G})$.*

Proof. Let $\mathcal{G} = \mathcal{N}(U_0, \ldots, U_k)$. Applying Lemma 2.11 with $W = \bigcup \mathcal{G}$, let $\mathcal{H} = \mathcal{N}(V_0, \ldots, V_\ell)$ be a standard form neighborhood of $B \backslash \bigcup \mathcal{G}$ with $\bigcup \mathcal{H} \cap \bigcup \mathcal{G} = \emptyset$. Then $\mathcal{N}(U_0, \ldots, U_k, V_0, \ldots, V_\ell)$ is a standard form neighborhood of B, so it contains some $K \in \mathcal{S}$. Then $K \setminus \bigcup \mathcal{G} \in \mathcal{H} \cap (\mathcal{S} \mathbin{\backslash\!\!\backslash} \mathcal{G})$. $\square$

For the B and $\mathcal{G}$ of Lemma 2.12, each $U \in \mathcal{G}^*$ *splits* B:

Definition 2.13. For any sets V and K, say V *splits* K (denoted $V \ddagger K$) iff $K \backslash V \neq \emptyset$ and $K \cap V \neq \emptyset$.

In Lemma 2.14, we consider B and $\mathcal{G}$ for which $\bigcup \mathcal{G}$ splits B.

Lemma 2.14. *Assume that X is zero-dimensional, $\mathcal{S} \subseteq \mathcal{K}(X)$, and $B, F \in \mathcal{K}(X)$ with $F \subsetneqq B$. Let $\mathbb{G}$ be a local base for $F \in \mathcal{K}(X)$ consisting of standard form clopen sets. Then $B \in \mathrm{cl}(\mathcal{S})$ iff $B \setminus \bigcup \mathcal{G} \in \mathrm{cl}(\mathcal{S} \mathbin{\backslash\!\!\backslash} \mathcal{G})$ whenever $\mathcal{G} \in \mathbb{G}$ satisfies $B \setminus \bigcup \mathcal{G} \neq \emptyset$.*

Proof. The $\to$ direction follows from Lemma 2.12; $\forall U \in \mathcal{G}^*$ $[B \cap U \neq \emptyset]$ holds here for all $\mathcal{G} \in \mathbb{G}$ because $U \in \mathcal{G}^* \to F \cap U \neq \emptyset$.

For the $\leftarrow$ direction, let $\mathcal{H}$ be any standard form neighborhood of B, and we prove that it contains some $K \in \mathcal{S}$.

Call $\mathcal{H}$ *good* iff $\mathcal{H} = \mathcal{N}(U_0, \ldots, U_k, V_0, \ldots, V_\ell)$, where $\mathcal{G} := \mathcal{N}(U_0, \ldots, U_k) \in \mathbb{G}$. *If* $\mathcal{H}$ is good: $B \setminus \bigcup \mathcal{G} \neq \emptyset$ (because B meets each V_j), so $B \setminus \bigcup \mathcal{G} \in \mathrm{cl}(\mathcal{S} \mathbin{\backslash\!\!\backslash} \mathcal{G})$. Then, since $\mathcal{N}(V_0, \ldots, V_\ell)$ is a neighborhood of $B \setminus \bigcup \mathcal{G}$, it contains some element of $\mathcal{S} \mathbin{\backslash\!\!\backslash} \mathcal{G}$. So, fix $K \in \mathcal{S}$ such that $K \setminus \bigcup \mathcal{G} \neq \emptyset$ and $\forall U \in \mathcal{G}^*$ $[K \cap U \neq \emptyset]$ and $K \setminus \bigcup \mathcal{G} \in \mathcal{N}(V_0, \ldots, V_\ell)$. But then $K \in \mathcal{H}$.

Now, we are done if we can show that given any standard form neighborhood $\mathcal{H}$ of B, we can find a good $\widehat{\mathcal{H}}$ with $B \in \widehat{\mathcal{H}} \subseteq \mathcal{H}$.

First, since $F \subsetneqq B$, we may shrink $\mathcal{H}$ if necessary and assume that some set in $\mathcal{H}^*$ is disjoint from F. Then $\mathcal{H} = \mathcal{N}(U_0, \ldots, U_k, V_0, \ldots, V_\ell)$, where the clopen sets are listed so that all the U_i meet F and all the V_j do not meet F (so they meet $B\backslash F$).

Since $\mathcal{N}(U_0, \ldots, U_k)$ is a neighborhood of F and $\mathbb{G}$ is a local base at F, fix $\mathcal{G} \in \mathbb{G}$ such that $F \in \mathcal{G} \subseteq \mathcal{N}(U_0, \ldots, U_k)$. Say $\mathcal{G} = \mathcal{N}(W_0, \ldots, W_r)$, and define $\widetilde{\mathcal{H}} = \mathcal{N}(W_0, \ldots, W_r, V_0, \ldots, V_\ell)$. Then $\widetilde{\mathcal{H}} \subseteq \mathcal{H}$ and $\widetilde{\mathcal{H}}$ is good, so let $\widehat{\mathcal{H}} = \widetilde{\mathcal{H}}$ *if* $B \in \widetilde{\mathcal{H}}$.

Now, suppose that $B \notin \widetilde{\mathcal{H}}$. This will happen iff $B \not\subseteq \bigcup \widetilde{\mathcal{H}}$; that is, some points of B lie in $\bigcup_i U_i \setminus \bigcup_j W_j$. So, we add these back in. Let $\widehat{U}_i = U_i \setminus \bigcup_j W_j$, and define $\widehat{\mathcal{H}}$ by: $(\widehat{\mathcal{H}})^* = (\widetilde{\mathcal{H}})^* \cup \{\widehat{U}_i : B \cap \widehat{U}_i \neq \emptyset\}$. $\square$

A special case of this is Lemma 2.15, where $F = \{p\}$ with $p \in B$.

Lemma 2.15. *Assume that X is zero-dimensional. Fix $p \in X$ and $\mathcal{L}$ a local clopen base at p and fix $\mathcal{S} \subseteq \mathcal{K}(X)$ and $B \in \mathcal{K}(X)$ with $B \supsetneqq \{p\}$. Let $\mathcal{L}^- = \{V \in \mathcal{L} : V \ddagger B\}$; this is also a local base at p. Let $\mathcal{S}_V = \{K\backslash V : K \in \mathcal{S}\ \&\ V \ddagger K\}$. Then $B \in \mathrm{cl}(\mathcal{S}) \leftrightarrow \forall V \in \mathcal{L}^-\ [B\backslash V \in \mathrm{cl}(\mathcal{S}_V)]$.*

Proof. Apply Lemma 2.14 with $F = \{p\}$ and $\mathbb{G} = \{\mathcal{N}(V) : V \in \mathcal{L}^-\}$. For each $\mathcal{G} = \mathcal{N}(V)$ in $\mathbb{G}$, $B \setminus \bigcup \mathcal{G} = B\backslash V \neq \emptyset$. So, Lemma 2.14 says that $B \in \mathrm{cl}(\mathcal{S})$ iff $\forall V \in \mathcal{L}^-\ [B\backslash V \in \mathrm{cl}(\mathcal{S} \setminus\!\!\setminus \mathcal{N}(V))]$.

But $\mathcal{S} \setminus\!\!\setminus \mathcal{N}(V) = \{K \setminus V : K \in \mathcal{S}\ \&\ [K \cap V \neq \emptyset]\} \setminus \{\emptyset\} = \mathcal{S}_V$. $\square$

Finally, our S-space X will be non-compact but locally compact, and we shall apply "locally" the following theorem of Vietoris, which is Problem 3.12.27 on page 244 of Engelking [5]:

Lemma 2.16. *If X is compact, then $\mathcal{K}(X)$ is compact.*

3. Elementary Submodels, $\diamondsuit$, and CH

Some of our arguments will use elementary submodels to simplify the combinatorics. We shall use the notation from the exposition in [9] (see Definition III.8.14). We fix a suitably large regular θ. Then, we build a *nice chain* $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable elementary submodels. This means that: $0 < \alpha < \beta < \omega_1 \to M_\alpha \prec M_\beta \prec H(\theta)$ *and* $M_0 = \emptyset$ *and* $\alpha < \beta \to M_\alpha \in M_\beta\ \&\ M_\alpha \subseteq M_\beta$ *and* for limit $\beta \leq \omega_1$, $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$; this last item defines M_{ω_1}, which has size $\aleph_1$.

Our theorems will often assume $\diamondsuit$ or CH. Because of the elementary submodels, our arguments will not start out with the conventional "fix a $\diamondsuit$ sequence" or "list $\mathcal{P}(\omega)$ as $\{x_\alpha : \alpha < \omega_1\}$". Instead, we shall just quote the following:

Lemma 3.1. *Fix a suitably large regular θ along with any nice chain $\langle M_\alpha : \alpha < \omega_1\rangle$ of elementary submodels, as described above. Then:*

1. CH *holds iff* $H(\aleph_1) \subseteq M_{\omega_1}$.

2. $\diamondsuit$ *holds iff for all* $X \subseteq H(\aleph_1)$ *the set* $\{\gamma < \omega_1 : X \cap M_\gamma \in M_{\gamma+1}\}$ *is stationary.*

Proof. For 1: The $\rightarrow$ direction is [9] Lemma III.8.24. The $\leftarrow$ direction holds because $|H(\aleph_1)| = \mathfrak{c}$ ([9], Lemma I.13.28).

For 2: The $\rightarrow$ direction holds because $M_1 \prec H(\theta)$ implies that M_1 contains a $\diamondsuit$ sequence, along with a bijection from ω_1 onto $H(\aleph_1)$. The $\leftarrow$ direction holds because $\diamondsuit^- \leftrightarrow \diamondsuit$ ([9], Theorem III.7.8). $\square$

4. The Construction of S-Spaces

As indicated in Section 1, our general procedure will be to start with a "suitable" space $(X, \mathcal{T})$ of size $\aleph_1$, and then refine the topology $\mathcal{T}$ to some $\widehat{\mathcal{T}}$ that satisfies the desired properties. We shall describe the entire construction of $\widehat{\mathcal{T}}$ working in ZFC. This $(X, \widehat{\mathcal{T}})$ will never be HL. It may be HS, or even strongly HS, or even better, $\mathcal{K}(X, \widehat{\mathcal{T}})$ may be HS; this will depend on the properties of $(X, \mathcal{T})$ and our set theory beyond ZFC (whether CH or $\diamondsuit$ holds).

The next definition specifies what "suitable" means:

Definition 4.1. A *fundamental space* is a topological space $(X, \mathcal{T})$ such that $|X| = \aleph_1$, and $(X, \mathcal{T})$ is zero-dimensional and separable and first countable and $\aleph_1$-dense, and such that each point $x \in X$ has been assigned a clopen local base $\{W^x_n : n \in \omega\}$ with $X = W^x_0 \supsetneqq W^x_1 \supsetneqq W^x_2 \cdots$. Then this $(X, \mathcal{T})$ is an *ordered fundamental space* iff in addition $X = \omega_1$ and the set ω is dense in X.

Our assumptions on $(X, \mathcal{T})$ could be weakened somewhat, but the present definition simplifies the combinatorics and is sufficient for all our intended applications.

Recall that "$\aleph_1$-dense" means that every non-empty open set has size $\aleph_1$; so, each $|W^x_n \setminus W^x_{n+1}| = \aleph_1$. Of course, $\bigcap_n W^x_n = \{x\}$ because the W^x_n form a local base. Assuming CH, simple examples of fundamental spaces are ω^ω and 2^ω, where the W^x_n are the standard basic sets, $\{g : g \restriction n = x \restriction n\}$.

Given any fundamental space $(X, \mathcal{T})$, we can list X as $\{x_\xi : \xi < \omega_1\}$ so that, replacing x_ξ by ξ, it becomes an *ordered* fundamental space; we just make sure that $\{x_n : n < \omega\}$ is dense. Our construction of $\widehat{\mathcal{T}}$ will always be done with the *ordered* fundamental space $(\omega_1, \mathcal{T})$, and will proceed by transfinite recursion on $\xi \in \omega_1$. In almost all cases, the choice of the listing $\{x_\xi : \xi < \omega_1\}$ will not matter (Lemma 4.24 is our one exception). Our space $(\omega_1, \widehat{\mathcal{T}})$ will be the refinement of $\mathcal{T}$ that we get by replacing the W^ξ_n by smaller sets $V^\xi_n \subseteq W^\xi_n$.

The next two lemmas will enable us to verify some elementary facts about $\widehat{\mathcal{T}}$. We use the shorthand "$W_n^\xi \searrow_n A$" to abbreviate "$\bigcap_n W_n^\xi = A$ and $W_0^\xi \supseteq W_1^\xi \supseteq W_2^\xi \supseteq \cdots$".

Lemma 4.2. *Suppose that $W_n^\xi \subseteq \omega_1$ for $n < \omega$ and $\xi < \omega_1$, and $W_n^\xi \searrow_n \{\xi\}$ for each $\xi < \omega_1$. Then $\{W_n^\xi \colon n < \omega \ \& \ \xi < \omega_1\}$ is a base for a T_2 zero-dimensional topology $\mathcal{T}$ on ω_1, with each $\{W_n^\xi : n < \omega\}$ a local clopen base at ξ, iff the following three conditions hold:*

1. $\forall \xi, \eta, n\, [\eta \in W_n^\xi \to \exists m\, [W_m^\eta \subseteq W_n^\xi]]$
2. $\forall \xi, \eta [\xi \neq \eta \to \exists m\, [W_m^\xi \cap W_m^\eta = \emptyset]]$
3. $\forall \xi, \eta, n\, [\eta \notin W_n^\xi \to \exists m\, [W_m^\eta \cap W_n^\xi = \emptyset]]$

Proof. Condition (1), plus the fact that the W_n^ξ are nested, implies that $\{W_n^\xi : n < \omega \ \& \ \xi < \omega_1\}$ forms an open base for a topology $\mathcal{T}$. Then (2) implies that $\mathcal{T}$ is Hausdorff, and (3) implies that each W_n^ξ is clopen, since its complement is open. Also, (1) implies that for each η, $\{W_m^\eta : m < \omega\}$ is a local base at the point η. $\square$

Definition 4.3. Whenever $\mathcal{B} \subseteq \mathcal{P}(X)$ and $\bigcup \mathcal{B} = X$, let $\mathfrak{T}(\mathcal{B}, X)$ be the topology on X generated by $\mathcal{B}$, formed by closing $\mathcal{B}$ under finite intersections and arbitrary unions.

So, $\mathfrak{T}(\mathcal{B}, X)$ is the topology with $\mathcal{B}$ as a subbase.

Definition 4.4. Suppose that we have $\{W_n^\xi \colon n < \omega \ \& \ \xi < \omega_1\}$ as in Lemma 4.2, satisfying (1)(2)(3). Suppose that $V_n^\xi \subseteq W_n^\xi$ for $n < \omega$ and $\xi < \omega_1$ with each $V_n^\xi \searrow_n \{\xi\}$. For each $\alpha \leq \omega_1$, define $R_n^\xi(\alpha)$ to be V_n^ξ when $\xi < \alpha$ and W_n^ξ when $\xi \geq \alpha$. Also, define $\mathring{\mathcal{T}}_\alpha$ to be the topology $\mathfrak{T}(\{R_n^\xi(\alpha) : n < \omega \ \& \ \xi < \omega_1\}, \omega_1)$. Let $\widehat{\mathcal{T}} = \mathring{\mathcal{T}}_{\omega_1}$.

The following lemma tells us that in our applications, $\{R_n^\xi(\alpha) : n < \omega \ \& \ \xi < \omega_1\}$ will be a *base* for $\mathring{\mathcal{T}}_\alpha$, which will be a T_2 zero-dimensional topology on ω_1. Note that $\mathring{\mathcal{T}}_0$ is just $\mathcal{T}$, and (using $V_n^\xi \subseteq W_n^\xi$) $\mathring{\mathcal{T}}_\beta$ refines $\mathring{\mathcal{T}}_\alpha$ whenever $\alpha \leq \beta$.

Lemma 4.5. *Let $\{W_n^\xi \colon n < \omega \ \& \ \xi < \omega_1\}$ and $\{V_n^\xi \colon n < \omega \ \& \ \xi < \omega_1\}$ be as in Definition 4.4. Suppose that for each $\alpha < \omega_1$ and $n \in \omega$, $\alpha = \max(V_n^\alpha)$ and in the topology $\mathring{\mathcal{T}}_\alpha$, V_n^α is closed and $\alpha \cap V_n^\alpha$ is open. The set α is also open in $\mathring{\mathcal{T}}_\alpha$, because $\xi = \max(V_n^\xi)$ for $\xi < \alpha$. Moreover, for each $\beta \leq \omega_1$, $\{R_n^\xi(\beta) : n < \omega \ \& \ \xi < \omega_1\}$ satisfies conditions* (1)(2)(3) *of Lemma 4.2.*

Remarks. Lemma 4.2 does not use the ordering, whereas Lemma 4.5 does. The definition of $\mathring{\mathcal{T}}_\alpha$ just uses the V_n^ξ for $\xi < \alpha$. Starting from an ordered fundamental space, we shall construct the V_n^ξ recursively to satisfy Lemma 4.5 and so that in addition the topology $\widehat{\mathcal{T}}$ is locally compact.

Our $\widehat{\mathcal{T}}$ is not Lindelöf, since it is right separated (i.e., each initial segment is open). For $\alpha < \omega_1$, $\mathring{\mathcal{T}}_\alpha$ is HS if the original $\mathcal{T}$ is. Many standard S-space constructions [7, 12] use CH to make $\mathring{\mathcal{T}}_{\omega_1}$ (i.e., $\widehat{\mathcal{T}}$) also HS.

Proof. Let $\Phi(\beta)$ abbreviate the statement that $\{R^\xi_n(\beta) : n < \omega \ \&\ \xi < \omega_1\}$ satisfies (1)(2)(3). We prove $\Phi(\beta)$ by induction on β. $\Phi(0)$ is trivial. Also, condition (2) follows from the fact that each $V^\xi_n \subseteq W^\xi_n$, so we concentrate on (1)(3). Fix β with $0 < \beta \le \omega_1$, and assume inductively that $\Phi(\alpha)$ holds for all $\alpha < \beta$. Now, fix the ξ, η, n as in (1)(3). If $\xi = \eta$, then (1) holds with $m = n$, and (3) holds because $\eta \in R^\eta_n(\beta)$, so assume that $\xi \neq \eta$.

For limit β: Fix $\alpha < \beta$ such that $\xi < \beta \to \xi < \alpha$ and $\eta < \beta \to \eta < \alpha$. Then (1)(3) are immediate from $\Phi(\alpha)$.

Now, assume that $\beta = \alpha + 1$. So $\Phi(\alpha)$ is true, and the only change from α to β is that each $R^\alpha_n(\beta)$ replaces W^α_n by V^α_n. So, (1)(3) are clear using $\Phi(\alpha)$ unless one of ξ, η equals α. So, we consider the four cases: $\xi < \eta = \alpha$; $\eta < \xi = \alpha$; $\alpha = \xi < \eta$; $\alpha = \eta < \xi$. This gives us eight statements to verify:

$\xi < \eta = \alpha$ (1)	$\alpha \in V^\xi_n \to \exists m\ [V^\alpha_m \subseteq V^\xi_n]$	$\alpha \notin V^\xi_n$ because $\max(V^\xi_n) = \xi$
$\xi < \eta = \alpha$ (3)	$\alpha \notin V^\xi_n \to \exists m\ [V^\alpha_m \cap V^\xi_n = \emptyset]$	V^ξ_n is $\mathring{\mathcal{T}}_\xi$ closed and $V^\alpha_m \subseteq W^\alpha_m$
$\eta < \xi = \alpha$ (1)	$\eta \in V^\alpha_n \to \exists m\ [V^\eta_m \subseteq V^\alpha_n]$	$\alpha \cap V^\alpha_n$ is $\mathring{\mathcal{T}}_\alpha$ open
$\eta < \xi = \alpha$ (3)	$\eta \notin V^\alpha_n \to \exists m\ [V^\eta_m \cap V^\alpha_n = \emptyset]$	V^α_n is $\mathring{\mathcal{T}}_\alpha$ closed
$\alpha = \xi < \eta$ (1)	$\eta \in V^\alpha_n \to \exists m\ [W^\eta_m \subseteq V^\alpha_n]$	$\eta \notin V^\alpha_n$ because $\max(V^\alpha_n) = \alpha$
$\alpha = \xi < \eta$ (3)	$\eta \notin V^\alpha_n \to \exists m\ [W^\eta_m \cap V^\alpha_n = \emptyset]$	V^α_n is $\mathring{\mathcal{T}}_\alpha$ closed
$\alpha = \eta < \xi$ (1)	$\alpha \in W^\xi_n \to \exists m\ [V^\alpha_m \subseteq W^\xi_n]$	use $V^\alpha_m \subseteq W^\alpha_m$
$\alpha = \eta < \xi$ (3)	$\alpha \notin W^\xi_n \to \exists m\ [V^\alpha_m \cap W^\xi_n = \emptyset]$	use $V^\alpha_m \subseteq W^\alpha_m$

They are listed here, together with their justifications, some of which implicitly use the fact that the W^ξ_n satisfy (1)(2)(3). The justifications that mention $\mathring{\mathcal{T}}_\alpha$ or $\mathring{\mathcal{T}}_\xi$ for $\xi < \alpha$ implicitly use $\Phi(\alpha)$ and $\Phi(\xi)$, which we are assuming inductively. $\square$

The proof of Lemma 4.13 below will use another useful consequence of the fact that initial segments are open in $\mathring{\mathcal{T}}_\alpha$:

Corollary 4.6. *Each $B \in \mathcal{K}(\omega_1, \widehat{\mathcal{T}})$ has a maximum element.*

Here are two ZFC examples of Lemma 4.5. For both, start with an ordered fundamental space as in Definition 4.1. As indicated in the above "Remarks", we can choose the V^ξ_n recursively so that Lemma 4.5 applies. Trivial example: Let each $V^\alpha_n = \{\alpha\}$. Then $\widehat{\mathcal{T}} = \mathring{\mathcal{T}}_{\omega_1}$ is discrete, and for each α, $\mathring{\mathcal{T}}_\alpha$ can be obtained by using as a base the open sets from $\mathcal{T}$ plus $\{\xi\}$ for all $\xi < \alpha$. Somewhat less trivially, Lemma 4.8 gives us a locally compact and separable *nice refinement*.

Definition 4.7. A *nice refinement* $(\omega_1, \widehat{\mathcal{T}})$ of the ordered fundamental space $(\omega_1, \mathcal{T})$ is a refinement satisfying the conditions of Lemma 4.5, such that $V_n^\alpha = \{\alpha\}$ for all $\alpha < \omega$ (so $(\omega, \widehat{\mathcal{T}})$ is discrete), and when $\omega \le \alpha < \omega_1$, there are $\widehat{\mathcal{T}}$ compact clopen non-empty pairwise disjoint $H_\ell^\alpha \subset \alpha \cap (W_\ell^\alpha \backslash W_{\ell+1}^\alpha)$ for $\ell \in \omega$ such that:

(1) $V_m^\alpha = \{\alpha\} \cup \bigcup_{\ell \ge m} H_\ell^\alpha$, for $m \in \omega$, and
(2) $\{V_m^\alpha : m \in \omega\}$ is a $\widehat{\mathcal{T}}$ clopen neighborhood base at the point α.

Lemma 4.8. *Each ordered fundamental space $(\omega_1, \mathcal{T})$ has a nice fundamental refinement $(\omega_1, \widehat{\mathcal{T}})$. Furthermore,* any *such refinement is separable with dense set ω, and locally compact and locally countable.*

Proof. Given the refinement, it is easily proved by induction that ω is open and dense in $(\alpha, \widehat{\mathcal{T}})$ whenever $\omega \le \alpha < \omega_1$. The V_m^α are compact because the H_ℓ^α are compact, and $\{V_m^\alpha : m \in \omega\}$ forms a local base at α.

To construct such a refinement, proceed by recursion on α. Fix α with $\omega \le \alpha < \omega_1$, and assume that we have defined the V_n^ξ for all $\xi < \alpha$ and $n < \omega$, and that the required properties hold below α. By our assumptions, the sets $\alpha \cap (W_\ell^\alpha \backslash W_{\ell+1}^\alpha)$ are all non-empty and relatively clopen in $\mathring{\mathcal{T}}_\alpha \restriction \alpha$. So, we may choose compact relatively clopen sets $H_\ell^\alpha \subseteq \alpha \cap (W_\ell^\alpha \backslash W_{\ell+1}^\alpha)$; for example, H_ℓ^α could be V_n^ξ for some $\xi \in W_\ell^\alpha \backslash W_{\ell+1}^\alpha$, with n large enough that $V_n^\xi \subseteq (W_\ell^\alpha \backslash W_{\ell+1}^\alpha)$; or, H_ℓ^α could be a finite union of such sets. Then, let $V_n^\alpha = \{\alpha\} \cup \bigcup_{\ell \ge n} H_\ell^\alpha$ for $n < \omega$. ☐

The construction is determined by the choice of the sets H_ℓ^α. The space $(\omega_1, \widehat{\mathcal{T}})$ need not be HS; for example, it is easy to do the above construction so that the set $\omega_1 \backslash \omega$ is closed and discrete; each $H_\ell^\alpha = \{k_\ell^\alpha\}$, with $k_\ell^\alpha \in \omega$. We now describe a method of using a chain of elementary submodels to guide the choice of the H_ℓ^α. Then we shall show, for example, that when $(X, \mathcal{T})$ is second countable, CH implies that $(\omega_1, \widehat{\mathcal{T}})$ is strongly HS, and $\diamondsuit$ implies that $\mathcal{K}(\omega_1, \widehat{\mathcal{T}})$ is HS.

The following lemma is essentially trivial, but it introduces the model notation.

Lemma 4.9. *Fix $(\omega_1, \mathcal{T})$ as in Lemma 4.8, and fix a suitably large regular θ. Then there is a nice chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable elementary submodels of $H(\theta)$ (as described in Section 3) and a nice fundamental refinement $(X, \widehat{\mathcal{T}})$ so that the sequence $\langle V_n^\xi : \xi < \alpha \ \& \ n < \omega \rangle$ lies in $M_{\alpha+1}$, and $\widehat{\mathcal{T}} \restriction \alpha \in M_{\alpha+1}$ for each $\alpha < \omega_1$.*

Proof. We shall build the M_α by recursion. We *always assume* that the map $(\alpha, n) \mapsto W_n^\alpha$ is an element of M_1. Since $\widehat{\mathcal{T}} \restriction \omega$ is discrete, the requirements of the lemma are trivial when $\alpha \le \omega$.

Along with M_α, we shall define $\widehat{\mathcal{T}}\restriction\alpha$ $(= \mathring{\mathcal{T}}_\alpha\restriction\alpha)$; that is, we specify the V_n^ξ for $\xi < \alpha$ and $n < \omega$. For limit $\alpha < \omega_1$, $M_\alpha = \bigcup_{\xi<\alpha} M_\xi$ and $\widehat{\mathcal{T}}\restriction\alpha = \mathring{\mathcal{T}}_\alpha\restriction\alpha$ is defined in the obvious way because we have defined the V_n^ξ for all $\xi < \alpha$. Note that $\alpha \le M_\alpha \cap \omega_1 \in \omega_1$. We assume inductively that each $\widehat{\mathcal{T}}\restriction\alpha = \mathring{\mathcal{T}}_\alpha\restriction\alpha$ is locally compact (the topology $\mathring{\mathcal{T}}_\alpha$ on all of ω_1 need not be locally compact when $\alpha < \omega_1$).

For $\alpha \le \omega$, let $\widehat{\mathcal{T}}\restriction\alpha$ be discrete. Now, assume that $\omega \le \alpha < \omega_1$ and that we have already defined M_α and $\widehat{\mathcal{T}}\restriction\alpha$, and we need to define $M_{\alpha+1}$ and $\widehat{\mathcal{T}}\restriction(\alpha+1)$. First choose $M_{\alpha+1}$ so that $M_\alpha \in M_{\alpha+1}$ (so also $\alpha \in M_{\alpha+1}$) and $\langle V_n^\xi : \xi < \alpha \ \&\ n < \omega\rangle \in M_{\alpha+1}$ (so that also $\widehat{\mathcal{T}}\restriction\alpha \in M_{\alpha+1}$).

Observe that $\langle W_n^\alpha : n \in \omega\rangle \in M_{\alpha+1}$ because the map $(\alpha, n) \mapsto W_n^\alpha$ lies in $M_{\alpha+1}$. Then the rest of the construction, as described in the proof of Lemma 4.8, can also be done within $M_{\alpha+1}$. $\square$

So far the models have not accomplished anything. Their role in specifying the sets H_ℓ^α follows. The next definitions and lemma augment stage α in the recursive construction, as outlined in the proof of Lemma 4.9. Observe that the space $(\alpha+1, \mathring{\mathcal{T}}_\alpha)$ is second countable and T_3, and hence metrizable.

Definition 4.10. Let $d = d^\alpha$ denote a metric on $\alpha + 1$ that induces the topology $\mathring{\mathcal{T}}_\alpha$ there, such that d has the following properties: Fix $\psi : \alpha + 1 \to \mathbb{R}$ defined by $\psi(\alpha) = 0$ and $\psi((W_\ell^\alpha \backslash W_{\ell+1}^\alpha) \cap (\alpha+1)) = \{2^{-\ell}\}$. Then $d(x,y) = |\psi(x) - \psi(y)|$ whenever $\psi(x) \ne \psi(y)$, and $d(x,y) \le 2^{-\ell-3}$ whenever $\psi(x) = \psi(y) = 2^{-\ell}$. Always choose $d^\alpha \in M_{\alpha+1}$.

Note that with this metric, $W_\ell^\alpha \cap (\alpha+1) = B(\alpha, 2^{-\ell} + \varepsilon)$ for any $\varepsilon \in (0, 2^{-\ell})$, and $\operatorname{diam}(W_\ell^\alpha \cap (\alpha+1)) = 2^{-\ell}$.

Definition 4.11. For a countable set $\mathcal{S} \in M_{\alpha+1}$ (where $\omega \le \alpha < \omega_1$) such that $\mathcal{S} \subseteq \mathcal{K}(\alpha, \widehat{\mathcal{T}}\restriction\alpha) = \mathcal{K}(\alpha, \mathring{\mathcal{T}}_\alpha\restriction\alpha)$, and for any $\nu \le \omega$, let

$$\Gamma(\mathcal{S}, \nu) = \Gamma^\alpha(\mathcal{S}, \nu) = \{K \in \mathcal{S} : \forall \ell < \nu\ [K \cap (W_\ell^\alpha \backslash W_{\ell+1}^\alpha) \subseteq H_\ell^\alpha]\} \quad .$$

Some of these sets might be empty. Also, whenever $\mu \le \nu$ we have $\mathcal{S} = \Gamma(\mathcal{S}, 0) \supseteq \Gamma(\mathcal{S}, \mu) \supseteq \Gamma(\mathcal{S}, \nu) \supseteq \Gamma(\mathcal{S}, \omega)$.

Observe that $\Gamma(\mathcal{S}, \nu) = \mathcal{S} \cap \mathcal{P}(\alpha \setminus \bigcup_{\ell<\nu}((W_\ell^\alpha \backslash W_{\ell+1}^\alpha) \backslash H_\ell^\alpha)) = \mathcal{S} \cap \mathcal{P}(W_\nu^\alpha \cup \bigcup_{\ell<\nu} H_\ell^\alpha)$, for $\nu < \omega$. Also, $\Gamma(\mathcal{S}, \omega) = \mathcal{S} \cap \mathcal{P}(\bigcup_{\ell<\omega} H_\ell^\alpha)$. For a given ν, think informally of $\bigcup_{\ell<\nu}((W_\ell^\alpha \backslash W_{\ell+1}^\alpha) \backslash H_\ell^\alpha)$ as being the "danger zone", and $K \in \Gamma(\mathcal{S}, \nu)$ iff K avoids this "danger zone".

Lemma 4.12. *When $\omega \le \alpha < \omega_1$ we may choose the H_ℓ^α so that $\bigstar_\alpha$ holds:*

$$(\bigstar_\alpha) \qquad \begin{aligned} &\forall \mathcal{S}\ \big[\ |\mathcal{S}| \le \aleph_0 \ \&\ \mathcal{S} \in M_{\alpha+1} \ \&\ \mathcal{S} \subseteq \mathcal{K}(\alpha, \widehat{\mathcal{T}}\restriction\alpha) \ \to \\ &\qquad \exists^\infty \nu\ \forall K \in \Gamma(\mathcal{S}, \nu)\ \exists J \in \Gamma(\mathcal{S}, \omega)\ [d(K, J) \le 2^{-\nu}]\ \big] \quad . \end{aligned}$$

Here, "$d(K,J)$" refers to the Hausdorff metric (see Problem 4.5.23 on page 298 of [5]*).*

Proof. List all relevant $\mathcal{S}$ as $\langle \mathcal{S}_\nu : \nu \in \omega\rangle$, with each $\mathcal{S}$ appearing ω times. Then, recursively define $H_0^\alpha, H_1^\alpha, H_2^\alpha, \ldots$. At stage ν, assume that we have defined H_ℓ^α for $\ell < \nu$; and so $\Gamma(\mathcal{S}_\nu, \nu)$ is defined.

Choose $\mathcal{P}_\nu \in [\Gamma(\mathcal{S}_\nu, \nu)]^{<\aleph_0}$ such that $\forall K \in \Gamma(\mathcal{S}_\nu, \nu)\ \exists J \in \mathcal{P}_\nu\ [d(K,J) \leq 2^{-\nu}]$. Such a finite $\mathcal{P}_\nu$ exists because the second countable $\mathcal{K}(\bigcup_{\ell<\nu} H_\ell^\alpha)$ is compact (see Lemma 2.16) (and hence totally bounded), and because $\mathrm{diam}(W_\nu^\alpha \cap (\alpha+1)) = 2^{-\nu}$.

Choose $H_\nu^\alpha \subseteq \alpha \cap (W_\nu^\alpha \backslash W_{\nu+1}^\alpha)$ so that $H_\nu^\alpha \supseteq J \cap (W_\nu^\alpha \backslash W_{\nu+1}^\alpha)$ for all $J \in \bigcup_{\mu \leq \nu} \mathcal{P}_\mu$. This guarantees that for each $J \in \mathcal{P}_\nu$, $J \cap (W_\ell^\alpha \backslash W_{\ell+1}^\alpha) \subseteq H_\ell^\alpha$ for all $\ell \geq \nu$ (replacing (μ,ν) by (ν,ℓ)). For these J, we also have $J \cap (W_\ell^\alpha \backslash W_{\ell+1}^\alpha) \subseteq H_\ell^\alpha$ for all $\ell < \nu$ because $\mathcal{P}_\nu \subseteq \Gamma(\mathcal{S}_\nu, \nu)$. So, $\mathcal{P}_\nu \subseteq \Gamma(\mathcal{S}_\nu, \omega)$.

So, given $\mathcal{S}$, we have $\forall K \in \Gamma(\mathcal{S}, \nu)\ \exists J \in \Gamma(\mathcal{S}, \omega)\ [d(K,J) \leq 2^{-\nu}]$ holding for all ν such that $\mathcal{S}_\nu = \mathcal{S}$, and this implies $(\bigstar_\alpha)$. □

The property $(\bigstar_\alpha)$ is important mainly because of Lemma 4.13. In the following, "assume $(\bigstar)$" is shorthand for "assume that $(\bigstar_\beta)$ holds whenever $\omega \leq \beta < \omega_1$".

Lemma 4.13. *Assume $(\bigstar)$. Fix α with $\omega \leq \alpha < \omega_1$. Suppose that $B \in \mathcal{K}(\omega_1, \widehat{\mathcal{T}})$ satisfies (☹): $\forall \beta \in B\ \forall n\ [W_n^\beta \in M_{\alpha+1}]$ (we do* not *assume that $B \in M_{\alpha+1}$). Then*

$$\forall \mathcal{S}\ \big[\ [|\mathcal{S}| \leq \aleph_0\ \&\ \mathcal{S} \in M_{\alpha+1}\ \&\ \mathcal{S} \subseteq \mathcal{K}(\alpha, \widehat{\mathcal{T}}\restriction\alpha)] \ \to \\ [B \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\alpha) \to B \in \mathrm{cl}(\mathcal{S}, \widehat{\mathcal{T}})]\ \big]\ . \tag{4.13(α)}$$

Remarks. Note that if $B \in \mathcal{K}(\omega_1, \widehat{\mathcal{T}})$, then B is also $\mathring{\mathcal{T}}_\alpha$ compact. In our applications, B will satisfy the hypothesis (☹) in one of two ways: In one, $B \subset M_{\alpha+1}$; equivalently, $\max(B) \in M_{\alpha+1}$. In the other, the fundamental space $(X, \mathcal{T})$ is second countable, with a countable clopen base $\mathcal{B} \in M_1$, and we choose all $W_n^\beta \in \mathcal{B}$.

We first prove a preliminary lemma:

Lemma 4.14. *Assume $(\bigstar)$. Fix β with $\omega \leq \beta < \omega_1$. For all $B \in \mathcal{K}(\omega_1, \widehat{\mathcal{T}})$ with $\max(B) = \beta$, sentence 4.13(β) holds.*

Proof. Suppose $B \in \mathcal{K}(\omega_1, \widehat{\mathcal{T}})$ with $\max(B) = \beta$, and $\mathcal{S}$ is as in 4.13(β) with $B \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\beta)$. Observe that since B is $\widehat{\mathcal{T}}$ compact, there must be some $r < \omega$ such that $B \cap (W_\ell^\beta \backslash W_{\ell+1}^\beta) \subseteq H_\ell^\beta$ for all $\ell \geq r$.

Case (i). Assume that $r = 0$, so that $B \cap (W_\ell^\beta \backslash W_{\ell+1}^\beta) \subseteq H_\ell^\beta$ for all ℓ (so B avoids the "danger zone"). Fix $\varepsilon > 0$. Applying $(\bigstar_\beta)$, fix $\nu < \omega$ so that $2^{-\nu} < \varepsilon$ (so that also $\mathrm{diam}(W_\nu^\beta \cap (\beta+1)) < \varepsilon$) and

$\forall K \in \Gamma(\mathcal{S},\nu)\ \exists J \in \Gamma(\mathcal{S},\omega)\ [d(K,J) \le 2^{-\nu}]$. Since $B \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\beta)$ and $B \cap (W^\beta_\ell \backslash W^\beta_{\ell+1}) \subseteq H^\beta_\ell$ for all ℓ, $B \in \mathrm{cl}(\Gamma(\mathcal{S},\nu), \mathring{\mathcal{T}}_\beta)$, so fix $K \in \Gamma(\mathcal{S},\nu)$ such that $d(K,B) < \varepsilon$. Then fix $J \in \Gamma(\mathcal{S},\omega)$ such that $d(K,J) < \varepsilon$. So, $d(J,B) < 2\varepsilon$. This proves that $B \in \mathrm{cl}(\Gamma(\mathcal{S},\omega), \mathring{\mathcal{T}}_\beta)$, and it follows that $B \in \mathrm{cl}(\mathcal{S}, \widehat{\mathcal{T}})$ because the topologies $\mathring{\mathcal{T}}_\beta$ and $\widehat{\mathcal{T}}$ agree on the subspace $\{\beta\} \cup \bigcup_\ell H^\beta_\ell$. (Apply Lemma 2.7 to $X = \{\beta\} \cup \bigcup_\ell H^\beta_\ell$ and $\Gamma(\mathcal{S},\omega)$.)

Case (ii). Assume that $r > 0$ is such that $B \cap (W^\beta_\ell \backslash W^\beta_{\ell+1}) \subseteq H^\beta_\ell$ for all $\ell \ge r$. We shall reduce this to the $r = 0$ case. So, assume that $B \cap (W^\beta_\ell \backslash W^\beta_{\ell+1}) \not\subseteq H^\beta_\ell$ for some ℓ; then $r > \ell$ and $W^\beta_r \ddagger B$. Let $W = W^\beta_r$. Note that $B \cap W \cap (W^\beta_\ell \backslash W^\beta_{\ell+1}) \subseteq H^\beta_\ell$ for all ℓ.

We shall now apply Lemma 2.14, with $X = \beta + 1$ and $F = B \backslash W$. Note that the topologies $\mathring{\mathcal{T}}_\beta$ and $\widehat{\mathcal{T}}$ agree on $X \backslash W$. In this common subspace $X \backslash W$, let $\mathbb{G}$ be a local base for $F = B \backslash W$ consisting of standard form clopen sets; then, $\mathbb{G}$ is a local base for F in $(X, \mathring{\mathcal{T}}_\beta)$ and $(X, \widehat{\mathcal{T}})$. Note that $B \setminus \bigcup \mathcal{G} = B \cap W \neq \emptyset$ for all $\mathcal{G} \in \mathbb{G}$. Let $T^\sharp$ denote one of the topologies $\mathring{\mathcal{T}}_\beta$ and $\widehat{\mathcal{T}}$. Then $B \in \mathrm{cl}(\mathcal{S}, \mathcal{T}^\sharp)$ iff $B \setminus \bigcup \mathcal{G} \in \mathrm{cl}(\mathcal{S} \mathbin{\setminus\!\!\setminus} \mathcal{G}, \mathcal{T}^\sharp)$ for all $\mathcal{G} \in \mathbb{G}$.

We cannot assume that $\mathbb{G} \in M_{\beta+1}$, but we can (and *do*) assume that $\mathbb{G} \subseteq M_{\beta+1}$. To see this, note that each $\mathcal{G} \in \mathbb{G}$ can be of the form $\mathcal{N}(U_0, \ldots, U_k)$, where $U_0, \ldots, U_k$ are pairwise disjoint non-empty $\widehat{\mathcal{T}}$ (equivalently $\mathring{\mathcal{T}}_\beta$) clopen subsets of $\beta \backslash W$ and $B \backslash W \subseteq \bigcup_i U_i$ and each $B \cap U_i \neq \emptyset$ (see Lemma 2.3); the U_i can be finite boolean combinations of the various V^ξ_ℓ (with $\xi < \beta$ and $\ell < \omega$) (see Lemma 2.6), and hence $U_i \in M_{\beta+1}$. By definition, $\mathcal{S} \mathbin{\setminus\!\!\setminus} \mathcal{G} = \{K \backslash \bigcup \mathcal{G} : K \in \mathcal{S}\ \&\ \forall U \in \mathcal{G}^*\ [K \cap U \neq \emptyset]\} \setminus \{\emptyset\}$. So, $\mathcal{S} \mathbin{\setminus\!\!\setminus} \mathcal{G} \in M_{\beta+1}$ because $\mathcal{S}, \mathcal{G} \in M_{\beta+1}$.

By definition of W, *Case (i)* applies to $B \cap W = B \setminus \bigcup \mathcal{G}$ and $\mathcal{S} \mathbin{\setminus\!\!\setminus} \mathcal{G}$ for each $\mathcal{G} \in \mathbb{G}$, and hence $B \setminus \bigcup \mathcal{G} \in \mathrm{cl}(\mathcal{S} \mathbin{\setminus\!\!\setminus} \mathcal{G}, \mathring{\mathcal{T}}_\beta)$ implies $B \setminus \bigcup \mathcal{G} \in \mathrm{cl}(\mathcal{S} \mathbin{\setminus\!\!\setminus} \mathcal{G}, \widehat{\mathcal{T}})$ for each $\mathcal{G} \in \mathbb{G}$. Applying the above $\mathcal{T}^\sharp$ result again, it follows that $B \in \mathrm{cl}(\mathcal{S}, \widehat{\mathcal{T}})$. ☐

Proof of Lemma 4.13. We shall show by induction that $\forall \beta < \omega_1$:

(ind(β))

for all $B \in \mathcal{K}(\omega_1, \widehat{\mathcal{T}})$ with $\max(B) = \beta$, sentence 4.13(α) holds.

Assume that $\mathrm{ind}(\gamma)$ holds for all $\gamma < \beta$, and $\max(B) = \beta$, and $B \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\alpha)$. We shall prove that $B \in \mathrm{cl}(\mathcal{S}, \widehat{\mathcal{T}})$. The fixed α partitions our proof into three cases:

Case 1. $\beta < \alpha$. The result follows by Lemma 2.7 from the fact that the topologies $\mathring{\mathcal{T}}_\alpha$ and $\widehat{\mathcal{T}}$ agree on the set α.

Case 2. $\beta = \alpha \in B \subseteq \alpha + 1$. Apply Lemma 4.14.

Case 3. $\beta > \alpha$. Apply Lemma 4.14, the induction hypothesis, and (☹).

If $B = \{\beta\}$: For $\nu \le \beta$, $B \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\nu)$ holds iff $\forall n\ \exists K \in \mathcal{S}\ K \subseteq W_n^\beta$. So, from $B \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\alpha)$ we get $B \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\beta)$; then since $\mathcal{S} \subseteq \mathcal{P}(\alpha) \subseteq \mathcal{P}(\beta)$, Lemma 4.14 Case (i) applies: $B \cap (W_\ell^\beta \backslash W_{\ell+1}^\beta) = \emptyset$, giving $B \in \mathrm{cl}(\mathcal{S}, \widehat{\mathcal{T}})$.

Now assume that $B \supsetneqq \{\beta\}$. Apply Lemma 2.15 to $(\omega_1, \mathring{\mathcal{T}}_\alpha)$, with $\mathcal{L}^- := \{W_n^\beta : W_n^\beta \ddagger B\} = \{W_n^\beta : n \ge n_0\}$ for some $n_0 \in \omega$. For $n \ge n_0$, let $\mathcal{S}_n = \mathcal{S}_{W_n^\beta} = \{K \backslash W_n^\beta : K \in \mathcal{S}\ \&\ W_n^\beta \ddagger K\}$. Since $\alpha < \beta$ and $B \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\alpha)$ we get $\forall n \ge n_0\ B \backslash W_n^\beta \in \mathrm{cl}(\mathcal{S}_n, \mathring{\mathcal{T}}_\alpha)$.

Applying (☹), $\mathcal{S}_n \in M_{\alpha+1}$. Then the induction hypothesis for $\gamma := \max(B \backslash W_n^\beta) < \max(B)$ implies that $\forall n \ge n_0\ B \backslash W_n^\beta \in \mathrm{cl}(\mathcal{S}_n, \widehat{\mathcal{T}})$. Thus, since $\widehat{\mathcal{T}}$ is a finer topology, $\forall n \ge n_0\ B \backslash W_n^\beta \in \mathrm{cl}(\mathcal{S}_n, \mathring{\mathcal{T}}_\beta)$. By Lemma 2.15 applied to $(\omega_1, \mathring{\mathcal{T}}_\beta)$, $B \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\beta)$, and hence, by Lemma 4.14, $B \in \mathrm{cl}(\mathcal{S}, \widehat{\mathcal{T}})$. □

Remarks on reduction arguments. The proof of Lemma 4.13 has two arguments using the Reduction Lemma 2.14. Both of them replace B by a "simpler" $B \setminus \bigcup \mathcal{G}$.

For *Case* (*ii*) of Lemma 4.14, we used $B \setminus \bigcup \mathcal{G} = B \cap W_r^\alpha$, which is a "tail" of B, excluding the (finite number of) $B \cap (W_\ell^\alpha \backslash W_{\ell+1}^\alpha)$ that are not subsets of H_ℓ^α, thus reducing this case to *Case* (*i*). Here, $F = B \backslash W_r^\alpha$.

In *Case 3*, we used Lemma 2.15, which is the special case of Lemma 2.14 with $F = \{p\} = \{\beta\}$, where $\beta = \max(B)$ and $B \setminus \bigcup \mathcal{G} = B \backslash W_n^\beta$, so $\max(B \setminus \bigcup \mathcal{G}) < \max(B)$, yielding a proof of Lemma 4.13 that inducts on $\max(B)$.

We can omit the hypothesis (☹) in Lemma 4.13 when B is finite; see Lemma 4.22 below. But first, we shall give some applications of Lemma 4.13 in the case that the fundamental space is second countable. When we state this as a hypothesis, it is understood that M_1 contains a countable clopen base for the fundamental space, with all the W_n^ξ chosen from that base, so that (☹) will always hold.

Our arguments that $(\omega_1, \widehat{\mathcal{T}})$ has some nice property (some variant of HS) will use the assumption that the original $(\omega_1, \mathcal{T})$ has the same property, but they will also need $(\omega_1, \mathring{\mathcal{T}}_\gamma)$ to have the same property for all $\gamma < \omega_1$. In the case that the fundamental space is second countable, we can use the next lemma, which is essentially trivial.

Lemma 4.15. *Let $(X, \mathcal{T}) = (\omega_1, \mathcal{T})$ be a second countable fundamental space. Fix $\gamma < \omega_1$. Then $(X, \mathring{\mathcal{T}}_\gamma)$ is also second countable, so that $\mathcal{K}(X, \mathring{\mathcal{T}}_\gamma)$ is second countable, and hence HS.*

Now the preceding lemmas and $\diamondsuit$ give us a simpler argument for a stronger property ($\mathcal{K}(\omega_1, \widehat{\mathcal{T}})$ is HS) than we get from CH in Theorem 4.17.

Theorem 4.16. *Let $(\omega_1, \mathcal{T})$ be a second countable fundamental space and assume $(\bigstar)$. If $\diamondsuit$ holds, then $\mathcal{K}(\omega_1, \widehat{\mathcal{T}})$ is HS.*

Proof. Suppose that $\langle B_\xi : \xi < \omega_1 \rangle$ is left separated, with $\alpha_\xi = \max(B_\xi)$. Passing to a subsequence, we may assume $\xi < \eta \to \alpha_\xi < \alpha_\eta$. For each $\gamma < \omega_1$, let ρ_γ be the least ordinal ρ such that $\gamma < \rho < \omega_1$ and $\{B_\xi : \xi < \rho\}$ is $\mathring{\mathcal{T}}_\gamma$ dense in $\{B_\xi : \xi < \omega_1\}$; there is such a ρ because $(\mathcal{K}(\omega_1), \mathring{\mathcal{T}}_\gamma)$ is HS. Note that $\gamma < \delta \to \rho_\gamma \le \rho_\delta$ because $\mathring{\mathcal{T}}_\delta$ is finer than $\mathring{\mathcal{T}}_\gamma$. Let C be a club such that for all $\eta \in C$: $\rho_\gamma < \eta$ and $\alpha_\gamma < \eta$ for all $\gamma < \eta$.

Apply $\diamondsuit$ (and Lemma 3.1) to fix $\eta \in C$ such that $\mathcal{S} := \{B_\xi : \xi < \eta\} \in M_{\eta+1}$. Now $\mathcal{S}$ is $\mathring{\mathcal{T}}_\gamma$ dense in $\{B_\xi : \xi < \omega_1\}$ for all $\gamma < \eta$, and hence also $\mathring{\mathcal{T}}_\eta$ dense because η is a limit ordinal. Then $\mathcal{S}$ is $\widehat{\mathcal{T}}$ dense in $\{B_\xi : \xi < \omega_1\}$ by Lemma 4.13, contradicting "left separated". □

If we replace $\diamondsuit$ by CH, we get a weaker result. In the above proof, for all η in the club C, $\mathcal{S}$ is a countable subset of $\mathcal{K}(\eta, \mathring{\mathcal{T}}_\eta)$, and $\diamondsuit$ allows us to get $\eta \in C$ with $\mathcal{S} \in M_{\eta+1}$. The following result employs the club C, but CH is not enough to guarantee $\mathcal{S} \in M_{\eta+1}$. Instead, we use CH to get a nice $\alpha > \eta$ with $\mathcal{S} \in M_{\alpha+1}$ so that Lemma 4.13 applies again.

Theorem 4.17. *Let $(\omega_1, \mathcal{T})$ be a second countable fundamental space and assume $(\bigstar)$. Assuming CH, there is no left separated sequence $\langle B_\xi : \xi < \omega_1 \rangle$ in $\mathcal{K}(\omega_1, \widehat{\mathcal{T}})$ such that for some fixed $\gamma < \omega_1$, the sets $B_\xi \backslash \gamma$ are pairwise disjoint.*

Proof. Repeat the first paragraph of the proof of Theorem 4.16 to get the club C, making the additional assumption that for some fixed $\gamma < \omega_1$ the sets $B_\xi \backslash \gamma$ are pairwise disjoint.

Choose $\eta \in C$ with $\eta > \gamma$. As before, $\mathcal{S}$ is $\mathring{\mathcal{T}}_\eta$ dense in $\{B_\xi : \xi < \omega_1\}$ because $\rho_\delta < \eta$ for all $\delta < \eta$, and η is a limit ordinal. By CH, fix α, ξ so that $\eta < \alpha < \alpha + 1 < \xi < \omega_1$ and $\mathcal{S} \in M_{\alpha+1}$; since the $B_\xi \backslash \gamma$ are pairwise disjoint, we can choose our ξ so that $B_\xi \cap [\eta, \alpha] = \emptyset$. Then $B_\xi \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\alpha)$ follows from $B_\xi \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\eta)$ by applying Lemma 2.7. Finally, by Lemma 4.13, $B_\xi \in \mathrm{cl}(\mathcal{S}, \widehat{\mathcal{T}})$, contradicting "left separated". □

In the case that all the B_ξ are finite, a delta system gives the following:

Corollary 4.18. *Let $(\omega_1, \mathcal{T})$ be a second countable fundamental space and assume $(\bigstar)$. Assuming CH, $(\omega_1, \widehat{\mathcal{T}})$ is a strong S-space.*

Proof. Fix $n \in \omega \backslash \{0\}$, and assume CH. Then by Theorem 4.17, $([\omega_1]^n, \widehat{\mathcal{T}})$ is HS. To finish, use Lemma 2.9. □

We do not have a proof that $[X]^{\omega+1}$ is HS; sets in $[X]^{\omega+1}$ (i.e., *homeomorphic to* $\omega+1$) may have arbitrarily large *order type*. But under CH sequences in $\mathcal{K}(X)$ of bounded order type are HS, by a delta-system-like argument:

Corollary 4.19. *Let* $(\omega_1, \mathcal{T})$ *be a second countable fundamental space and assume* ($\bigstar$). *Assuming* CH, *there is no left separated sequence* $\langle B_\xi : \xi < \omega_1\rangle$ *in* $\mathcal{K}(\omega_1, \widehat{\mathcal{T}})$ *such that* $\sup\{\mathrm{type}(B_\xi) : \xi < \omega_1\} < \omega_1$.

Proof. Suppose that $\langle B_\xi : \xi < \omega_1\rangle$ is left separated with $\sup\{\mathrm{type}(B_\xi) : \xi < \omega_1\} < \omega_1$. Thinning the sequence, we may assume there is a $\tau < \omega_1$ so that $\mathrm{type}(B_\xi) = \tau + 1$ for all ξ. Write $B_\xi = \{\beta^\nu_\xi : \nu \le \tau\}$ in increasing order. Since each $\mathcal{K}(\omega_1, \mathring{\mathcal{T}}_\alpha)$ is HS, there is a least ordinal $\sigma \le \tau$ such that $\sup\{\beta^\sigma_\xi : \xi < \omega_1\} = \omega_1$. Let $\gamma = \sup\{\beta^\nu_\xi : \xi < \omega_1 \;\&\; \nu < \sigma\} < \omega_1$. By transfinite recursion on $\xi < \omega_1$, we may thin again, so that each $\beta^\sigma_\xi > \gamma$ and β^σ_ξ is above all elements of B_η for $\eta < \xi$, whereupon Theorem 4.17 applies. $\square$

We now turn to the situation where the fundamental space need not be second countable, in which case our positive results focus mainly on $[X]^Q$ for finite Q. We begin with Lemma 4.21, which we shall use in place of Lemma 4.15. First, a preliminary:

Lemma 4.20. *For any* T_3 *space* X *and countable* $G \subset X$*:* $[X]^n$ *is HS* $\leftrightarrow$ $[X \backslash G]^n$ *is HS.*

Proof. For the $\leftarrow$ direction: Assume that $\langle B_\alpha : \alpha < \omega_1\rangle$ is left separated in $[X]^n$. We shall prove that $[X\backslash G]^m$ is not HS for some $m \le n$. Then we are done by (1) $\to$ (2) of Lemma 2.9.

Since G is countable, we may thin the sequence so that all $B_\alpha \cap G = A \in [G]^k$ for some fixed A. Any constant sequence is not left separated; so $k \ne n$. By assumption, $[X\backslash G]^n$ is HS; so $k \ne 0$. So $0 < k < n$. Let $m = n - k$.

Let $B_\alpha = \{b^\alpha_0, \ldots, b^\alpha_{k-1}, b^\alpha_k, \ldots, b^\alpha_{n-1}\}$, where $A = \{b^\alpha_0, \ldots, b^\alpha_{k-1}\} = \{b_0, \ldots, b_{k-1}\}$. Let $D_\alpha = \{b^\alpha_k, \ldots, b^\alpha_{n-1}\} \in [X\backslash G]^m$.

Let the left separating neighborhoods be $\mathcal{N}_\alpha$; so $B_\alpha \in \mathcal{N}_\alpha$ and $\alpha < \beta \to B_\alpha \notin \mathcal{N}_\beta$. Since $|B_\alpha| = n$, we may choose N_α so that $\mathcal{N}_\alpha = \mathcal{N}(V^\alpha_0, \ldots, V^\alpha_{n-1})$ (see Definition 2.2), where the V^α_i are open and pairwise disjoint and each $b^\alpha_i \in V^\alpha_i$.

Fix $\alpha < \beta$. Then $B_\alpha \notin \mathcal{N}_\beta$, which implies that ① or ② holds:

①: Some $b^\alpha_i \notin \bigcup_j V^\beta_j$. Then $i \ge k$ since $i < k$ would imply that $b^\alpha_i = b^\beta_i \in \bigcup_j V^\beta_j$. We also (trivially) have $b^\alpha_i \notin \bigcup_{k \le j < n} V^\beta_j$.

②: $B_\alpha \cap V^\beta_i = \emptyset$ for some i. Then $i \ge k$ since $i < k$ implies that $b^\alpha_i = b^\beta_i \in B_\alpha \cap V^\beta_i$. We also (trivially) have $D_\alpha \cap V^\beta_i = \emptyset$.

It follows that $\langle D_\alpha : \alpha < \omega_1 \rangle$ is left separated in $[X \backslash G]^m$, using the separating neighborhoods $\mathcal{N}(V_k^\alpha, \ldots, V_{n-1}^\alpha)$. □

Lemma 4.21. *For $n \in \omega$ and $\gamma < \omega_1$: If $([\omega_1]^n, \mathcal{T})$ is HS then $([\omega_1]^n, \mathring{\mathcal{T}}_\gamma)$ is HS.*

Proof. In fact we have:
$([\omega_1]^n, \mathcal{T})$ is HS $\leftrightarrow$ $([\omega_1 \backslash \gamma]^n, \mathcal{T})$ is HS $\leftrightarrow$ $([\omega_1 \backslash \gamma]^n, \mathring{\mathcal{T}}_\gamma)$ is HS $\leftrightarrow$ $([\omega_1]^n, \mathring{\mathcal{T}}_\gamma)$ is HS.
The center $\leftrightarrow$ holds because the two topologies agree on $\omega_1 \backslash \gamma$. The other two arrows hold by applying Lemma 4.20 with $G = \gamma$. □

Lemma 4.22. *Assume all the hypotheses of Lemma 4.13 except for (☹), and assume that $|B| = k < \omega$ and that $([\omega_1]^{k-1}, \mathcal{T})$ is HS. Then* 4.13(α) *holds.*

When $|B| = 1$, the assumption that $([\omega_1]^{k-1}, \mathcal{T})$ is HS is vacuous.

Proof. We induct on k to show that for all $\alpha < \omega_1$ and for all B with $|B| = k$, statement 4.13(α) holds.

Fix $\alpha < \omega_1$. If $B \subseteq M_{\alpha+1}$, then (☹) holds, and hence Lemma 4.13 applies. So assume $\max(B) \geq M_{\alpha+1} \cap \omega_1$. Note that $M_{\alpha+1} \cap \omega_1 > \alpha$.

Assume that $|\mathcal{S}| \leq \aleph_0$ and $\mathcal{S} \in M_{\alpha+1}$ and $\mathcal{S} \subseteq \mathcal{K}(\alpha, \widehat{\mathcal{T}} \restriction \alpha)$ and $B \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\alpha)$.

For $k = 1$: $B = \{\beta\} \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\alpha) \to \{\beta\} \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\beta) \to \{\beta\} \in \mathrm{cl}(\mathcal{S}, \widehat{\mathcal{T}})$. The first $\to$ holds because $\beta = \max(B) > \alpha$, so β has the same basic neighborhoods in $\mathring{\mathcal{T}}_\beta$ as in $\mathring{\mathcal{T}}_\alpha$ (see Lemma 2.7). The second $\to$ holds because each $W_n^\beta \in M_{\beta+1}$, so Lemma 4.13 applies.

Now assume that $|B| = k > 1$ and $([\omega_1]^{k-1}, \mathcal{T})$ is HS, and assume 4.13(γ) holds for all $\gamma < \omega_1$ and for all K with $|K| < k$. If $\alpha \notin B$, let $\hat{\alpha} = \min(B \backslash \alpha)$. Then $B \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\alpha)$ implies $B \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_{\hat{\alpha}})$ because all elements of B have the same basic neighborhoods in $\mathring{\mathcal{T}}_\alpha$ as in $\mathring{\mathcal{T}}_{\hat{\alpha}}$. Thus, if $\max(B) < M_{\hat{\alpha}+1} \cap \omega_1$, then Lemma 4.13 applies. Suppose instead $\max(B) \geq M_{\hat{\alpha}+1} \cap \omega_1$. So, we may replace α by $\hat{\alpha}$ and assume that $\alpha \in B$ and $\max(B) \geq M_{\alpha+1} \cap \omega_1$,

Let $B^- = B \cap [0, M_{\alpha+1} \cap \omega_1) = B \cap M_{\alpha+1}$ and $B^+ = B \cap [M_{\alpha+1} \cap \omega_1, \omega_1) = B \backslash M_{\alpha+1}$. Let $m = |B^-|$ and $n = |B^+|$. Then $m + n = k$ and $0 < m, n < k$. Let $\beta = \min(B^+)$; that is, the first ordinal in B but not in $M_{\alpha+1}$. Let $\xi = \max(B^-)$; that is, the largest ordinal in $B \cap M_{\alpha+1}$. Note that B^- is a finite subset of $M_{\alpha+1}$, and hence $\xi, B^- \in M_{\alpha+1}$.

Let $\mathcal{E} = \{H \in [(\xi, \omega_1)]^n : B^- \cup H \in \mathrm{cl}(\mathcal{S}, \mathring{\mathcal{T}}_\alpha)\}$. Then $B^+ \in \mathcal{E}$ and $\mathcal{E} \in M_{\alpha+1}$. Also, $([\omega_1]^n, \mathring{\mathcal{T}}_\alpha)$ is HS by Lemma 4.21. So, fix a countable $\mathcal{F} \in M_{\alpha+1}$ such that $\mathcal{F} \subseteq \mathcal{E}$ and $\mathcal{F}$ is $\mathring{\mathcal{T}}_\alpha$ dense in $\mathcal{E}$. We can choose $\mathcal{F}$ in $M_{\alpha+1}$ because $M_{\alpha+1} \prec H(\theta)$ and $\mathcal{E}, \mathring{\mathcal{T}}_\alpha \in M_{\alpha+1}$. Since $\mathcal{F}$ is countable, $\mathcal{F} \subset M_{\alpha+1}$ and $\sup(\bigcup \mathcal{F}) < M_{\alpha+1} \cap \omega_1$.

For each $H \in \mathcal{F}$, (☹) holds for $B^- \cup H$, so $B^- \cup H \in \mathrm{cl}(\mathcal{S}, \widehat{\mathcal{T}})$ by Lemma 4.13. Here Lemma 4.13 applies because $\mathcal{S} \subseteq \mathcal{K}(\alpha, \widehat{\mathcal{T}})$; and (☹) holds because $B^- \cup H \subseteq M_{\alpha+1}$.

Also, $B^+ \in \mathrm{cl}(\mathcal{F}, \mathring{\mathcal{T}}_\alpha) \to B^+ \in \mathrm{cl}(\mathcal{F}, \mathring{\mathcal{T}}_\beta) \to B^+ \in \mathrm{cl}(\mathcal{F}, \widehat{\mathcal{T}})$. To justify this: $B^+ \in \mathrm{cl}(\mathcal{F}, \mathring{\mathcal{T}}_\alpha)$ because $B^+ \in \mathcal{E}$. Then the first $\to$ holds because $\beta = \min(B^+)$, so all points of B^+ have the same basic neighborhoods in $\mathring{\mathcal{T}}_\beta$ as in $\mathring{\mathcal{T}}_\alpha$ (see Lemma 2.7). The second $\to$ holds by the inductive assumption, with $\gamma = \beta$ and $K = B^+$. The hypotheses $\mathcal{F} \in M_{\beta+1}$ & $\mathcal{F} \subseteq \mathcal{K}(\beta, \widehat{\mathcal{T}}\restriction\beta)$ of 4.13(β) are true because $\mathcal{F} \in M_{\alpha+1}$ and β is larger than all the countable ordinals in $M_{\alpha+1}$.

Now, working just with $\widehat{\mathcal{T}}$: $\forall H \in \mathcal{F}\ [B^- \cup H \in \mathrm{cl}(\mathcal{S}, \widehat{\mathcal{T}})]$ and $B^+ \in \mathrm{cl}(\mathcal{F}, \widehat{\mathcal{T}})$, so $B = B^- \cup B^+ \in \mathrm{cl}(\mathcal{S}, \widehat{\mathcal{T}})$. To justify this last "so": Let $\mathcal{N}(V_1, \ldots, V_m, V_{m+1}, \ldots, V_{m+n})$ be any standard form neighborhood of $B^- \cup B^+$ such that $|V_i \cap B^-| = |V_j \cap B^+| = 1$ whenever $1 \le i \le m < j \le m+n$. Then $\mathcal{N}(V_{m+1}, \ldots, V_{m+n})$ is a neighborhood of B^+ so it contains some H in $\mathcal{F}$ because $B^+ \in \mathrm{cl}(\mathcal{F}, \widehat{\mathcal{T}})$. But then $\mathcal{N}(V_1, \ldots, V_m, V_{m+1}, \ldots, V_{m+n})$ is also a neighborhood of $B^- \cup H$, so it contains some element of $\mathcal{S}$ because $B^- \cup H \in \mathrm{cl}(\mathcal{S}, \widehat{\mathcal{T}})$. □

It is not clear how to make the proof of Lemma 4.22 work for infinite B. The proof quotes the lemma inductively on a tail B^+ of B. But if B is infinite, then B^+ may be homeomorphic to B, so it is not clear what to induct on. But this question is not relevant for our main results here. To produce ultra strong S-spaces (with $\mathcal{K}(X)$ HS), we simply started with a second countable fundamental space. But Lemma 4.22 will be useful when we produce a strong S-space that is not ultra strong in Section 7. We shall start with $(X, \mathcal{T})$ strongly HS but $([X]^{\omega+1}, \mathcal{T})$ not HS (so then $(X, \mathcal{T})$ is *not* second countable), and then use Lemma 4.22 to prove that $(X, \widehat{\mathcal{T}})$ is strongly HS; but $([X]^{\omega+1}, \widehat{\mathcal{T}})$ will still not be HS. At the end of this section, we shall describe briefly some $(X, \mathcal{T})$ to which Lemma 4.22 applies.

Using Lemma 4.21, we get the following improvement on Corollary 4.18 with essentially the same proof, but now using Lemma 4.22:

Corollary 4.23. *Fix $n \in \omega\backslash\{0\}$. Let $(\omega_1, \mathcal{T})$ be any fundamental space and assume (★) and assume* CH *and assume that $([\omega_1]^n, \mathcal{T})$ is HS. Then $([\omega_1]^n, \widehat{\mathcal{T}})$ is HS.*

If $([X]^n, \mathcal{T})$ is not HS, then $([X]^n, \widehat{\mathcal{T}})$ is obviously not HS either, since $\widehat{\mathcal{T}}$ is a finer topology. The analogous statement for $[X]^{\omega+1}$ is not so clear because some elements of $([X]^{\omega+1}, \mathcal{T})$ may fail to be *in* $([X]^{\omega+1}, \widehat{\mathcal{T}})$.

The next lemma is a partial result in this direction. It will be used in Section 7 (Theorem 7.2) to construct a locally compact locally countable strong S-space X for which $[X]^{\omega+1}$ is not HS.

Lemma 4.24. *Assume that in the fundamental space $(X, \mathcal{T})$, there is a left separated sequence $\langle K_\xi : \xi < \omega_1 \rangle$ in $([X]^{\omega+1}, \mathcal{T})$. Assume that each K_ξ is listed as $\{t^\xi_\nu : \nu \le \omega\}$, with limit point t^ξ_ω, and that all the t^ξ_ω are different points. Then we can order X in type ω_1 and then construct $\widehat{\mathcal{T}}$ as in Lemma 4.12 and obtain ($\bigstar$), so that the set $I := \{\xi : (K_\xi, \widehat{\mathcal{T}}) \cong \omega+1\}$ will be uncountable, and hence $\langle K_\xi : \xi \in I \rangle$ will be an uncountable left separated sequence in $([X]^{\omega+1}, \widehat{\mathcal{T}})$.*

Proof. Since the t^ξ_ω are all different, we may replace $\langle K_\xi : \xi < \omega_1 \rangle$ by a subsequence thereof, and then choose an enumeration of X as $\{x_\alpha : \alpha < \omega_1\}$ so that for each ξ, t^ξ_ω is listed after all the t^ξ_ν for $\nu < \omega$. Then, working with $(\omega_1, \mathcal{T})$, we are in the following situation: We are given $J \subseteq \omega_1 \backslash \omega$, along with, for each $\alpha \in J$, a set $K_\alpha = \{\alpha\} \cup \{\eta^\alpha_n : n \in \omega\}$, where all $\eta^\alpha_n < \alpha$, and, in $\mathcal{T}$, the set $\{\eta^\alpha_n : n \in \omega\}$ is discrete and $K_\alpha \cong \omega + 1$. Then, when we do the construction from Lemma 4.12, just make sure that each $H^\alpha_\ell \supseteq K_\alpha \cap (W^\alpha_\ell \backslash W^\alpha_{\ell+1})$. This causes no problem because $K_\alpha \cap (W^\alpha_\ell \backslash W^\alpha_{\ell+1})$ is finite (because $(K_\alpha, \mathcal{T}) \cong \omega + 1$), and ensures that $(K_\alpha, \widehat{\mathcal{T}}) \cong \omega + 1$ as well. $\square$

Note that we cannot claim that *every* copy of $\omega+1$ in $\mathcal{T}$ remains homeomorphic to $\omega+1$ in $\widehat{\mathcal{T}}$, since if we choose ordinals $\xi_n \in \alpha \cap (W^\alpha_n \backslash W^\alpha_{n+1}) \backslash H^\alpha_n$, then $\{\alpha\} \cup \{\xi_n : n \in \omega\}$ is discrete in $\widehat{\mathcal{T}}$ but homeomorphic to $\omega+1$ in $\mathcal{T}$.

We conclude this section by describing the Sorgenfrey line, which yields a specific example of Corollary 4.23. This example will re-appear in Section 7. In this paper, our basic open Sorgenfrey intervals flip the usual ones (see [5]).

Definition 4.25. The *Sorgenfrey topology* on $\mathbb{R}$ is obtained by using all intervals of the form $(x, y]$ as a base.

This topology is first countable but not second countable. Under this topology: $\mathbb{R}$ is well-known to be HS and HL, but $\mathbb{R}^2$ is neither, since $\{(x, -x) : x \in \mathbb{R}\}$ is discrete; but there can be uncountable subsets of $\mathbb{R}$ that are strongly HS. This is true for *increasing* sets. This "increasing" is a weakening of the property "entangled". See [2, 1]. We give the relevant definition, since the terminology in the literature is not completely uniform:

Definition 4.26. For $n \in \omega \backslash \{0\}$ and $X \subseteq \mathbb{R}$: X is *n-increasing* iff given $\vec{x}_\alpha \in X^n$ for $\alpha < \omega_1$, there exist $\alpha < \beta < \omega_1$ such that $\forall i < n\, [x^i_\alpha \le x^i_\beta]$.

The following is easily seen from the definitions:

Lemma 4.27. *If X is n-increasing, then X^n is HS in the Sorgenfrey topology.*

Note that by Lemma 2.9, $[X]^n$ is HS iff X^n is HS in this topology. If we rephrased Definition 4.26 by replacing "$x^i_\alpha \le x^i_\beta$" by "$x^i_\alpha \ge x^i_\beta$", then we would replace "HS" by "HL" in Lemma 4.27. Both notions of n-increasing follow from X being n-entangled.

Although PFA implies that no uncountable subset of $\mathbb{R}$ can be 2-increasing, CH implies that there is an uncountable $X \subset \mathbb{R}$ that is n-increasing (and even n-entangled) for all n. Then, using this X as our fundamental space provides (by Corollary 4.23) an example of a strong S-space constructed from a fundamental space that is not second countable. A modification of this example will be used in Section 7 to construct a locally compact locally countable strong S-space X for which $[X]^{\omega+1}$ is not HS.

5. Comparing Functions with Sets

Here we relate properties of X^Q to properties of $\mathcal{K}(X)$ and its subspaces, such as $[X]^Q$ and $[X]^{\le Q}$ (see Definitions 2.1 and 2.8). We also prove Lemma 5.3, which is a version of Lemma 2.9 with $[X]^Q$ and X^Q replacing $[X]^n$ and X^n. (For X^Q see Definition 1.1.)

The space $\mathcal{K}(X)$ is somewhat more tractable than the function space X^Q. For X compact, $\mathcal{K}(X)$ is compact (Lemma 2.16) but X^Q often is not. For example, $\{0,1\}^{\omega+1}$ (that is, $C(\omega+1,\{0,1\})$ with the compact-open topology) is discrete and countably infinite, and hence not compact. Moreover, for X second countable and compact, $\mathcal{K}(X)$ is compact metric, and hence totally bounded. For example, the proof of Lemma 4.12 uses the fact that $\mathcal{K}(\bigcup_{\ell<\nu} H^\alpha_\ell)$ is totally bounded.

The space $[X]^{\le Q}$ of *images* differs significantly from the space X^Q of functions; see Section 9. Nevertheless, results in this section derive facts about X^Q from facts about $[X]^{\le Q}$ or $\mathcal{K}(X)$.

Lemma 5.1. *Suppose that X is locally compact, first countable, HS, submetrizable, and zero-dimensional, and Q is compact, second countable, zero-dimensional, and non-empty. If $\mathcal{K}(X)$ is HS, then X^Q is HS.*

We shall prove this after proving Lemma 5.3.

Lemma 5.2. *If X is a non-empty space and Q is a non-empty compact space, then the map $\Phi : X^Q \to \mathcal{K}(X)$ defined by $\Phi(f) = f(Q)$ is continuous.*

Proof. We need to show that $\Phi^{-1}(\mathcal{W})$ is open in X^Q whenever $\mathcal{W}$ is a subbasic open set in $\mathcal{K}(X)$.

If $\mathcal{W} = \{B \in \mathcal{K}(X) : B \subseteq V\}$, where V is open in X: Then $\Phi^{-1}(\mathcal{W}) = \{f \in C(Q, X) : f(Q) \subseteq V\}$, which is open because Q is compact.

If $\mathcal{W} = \{B \in \mathcal{K}(X) : B \cap V \neq \emptyset\}$, where V is open in X: Then $\Phi^{-1}(\mathcal{W}) = \{f \in C(Q, X) : f(Q) \cap V \neq \emptyset\} = \bigcup_{q \in Q}\{f \in C(Q, X) : f(\{q\}) \subseteq V\}$, which is a union of subbasic open sets in X^Q. □

Note that $[X]^{\leq Q} = \Phi(X^Q)$.

The following is our analog of Lemma 2.9, with $[X]^Q, X^Q$ replacing $[X]^n, X^n$.

Lemma 5.3. *Suppose that X is locally compact, first countable, HS, and submetrizable, and Q is compact, second countable, zero-dimensional, and non-empty. Then in the following (A①) ↔ (A②) and (B①) ↔ (B②) and (C②) → (C①):*

A.① X^Q is HS.
A.② X^P is HS for all compact P that embed into Q.
B.① $[X]^{\leq Q}$ is HS.
B.② $[X]^{\leq P}$ is HS for all compact P that embed into Q.
C.① $[X]^Q$ is HS.
C.② $[X]^P$ is HS for all compact P that embed into Q.

Also, (A) → (B) → (C②) → (C①), and, when Q is countable, all six are equivalent.

Note that (C①) → (C②) might be false because $[X]^Q$ might be empty. For example, if X is scattered and Q is the Cantor set, then $[X]^Q = \emptyset$, which is trivially HS, while if $|P| = 1$ then $X^P \cong X$, which need not be HS.

Remarks. This lemma applies to the S-spaces constructed in this paper. In particular, our S-space examples X are zero-dimensional, so it is enough to consider zero-dimensional Q. If Q is compact, and $\widetilde{Q}$ denotes the quotient space we get by collapsing each quasi-component (=connected component because Q is compact) of Q to a point, then $\widetilde{Q}$ is compact and zero-dimensional. Moreover, X^Q is HS whenever $X^{\widetilde{Q}}$ is.

Possibly the hypotheses could be weakened, but unlike in Lemma 2.9, we need some assumptions on our space X to guarantee that $[X]^Q$ is non-trivial when Q is infinite. For example, if X is extremally disconnected then $[X]^Q = \emptyset$, so (C①) is trivially true, and we can get the others all to be false in two ways: First, deleting the assumption that X is HS, X can be discrete and uncountable. Second, deleting the "locally compact and first countable", and assuming CH, X can be the space R described in Lemma 5.8.

We shall prove Lemma 5.3 after a sequence of sublemmas.

Lemma 5.4. *Each countable compact $Q \neq \emptyset$ is homeomorphic to some $\gamma + 1 < \omega_1$.*

Proof. This is a classical result of Mazurkewicz and Sierpiński [10], proved by induction on the Cantor-Bendixson rank. If $0 < n = |Q^{(\beta)}| < \aleph_0 = |Q|$ then $Q \cong (\omega^\beta \cdot n) + 1$. $\square$

Lemma 5.5. *Assume that X is first countable and HS and uncountable. Then for all $\gamma < \omega_1$, X contains a closed copy of $\gamma + 1$.*

Proof. Consider the Cantor-Bendixson sequence. Each $X^{(\alpha)} \backslash X^{(\alpha+1)}$ is countable (and possibly empty) because X is HS, so $X^{(\alpha)} \neq \emptyset$ for all $\alpha < \omega_1$. Then, it is sufficient to prove by induction on $\alpha < \omega_1$ that for all $x \in X^{(\alpha)}$, there is a continuous injection $\varphi : \omega^\alpha + 1 \to X$ such that $\varphi(\omega^\alpha) = x$. This is trivial for $\alpha = 0$. Now assume that $0 < \alpha < \omega_1$ and the result holds for all $\xi < \alpha$. There are two cases:

If $\alpha = \xi + 1$, then let the sequence $\langle y_n : n \in \omega \rangle$ converge to x with each $y_n \in X^{(\xi)}$. Let U_n be an open neighborhood of y_n, with the $\overline{U}_n$ pairwise disjoint and the sequence $\langle \overline{U}_n : n \in \omega \rangle$ converging to $\{x\}$. Then, let $\varphi_n : \omega^\xi + 1 \to X$ be a continuous injection with $\varphi_n(\omega^\xi) = x_n$. Since every clopen neighborhood of the point ω^ξ in the space $\omega^\xi + 1$ is homeomorphic to $\omega^\xi + 1$, we may assume that each $\varphi_n : \omega^\xi + 1 \to U_n$. Now define a continuous injection $\tilde{\varphi} : (\omega \times \omega^\xi) \cup \{\infty\} \to X$ by $\tilde{\varphi}(n, \mu) = \varphi_n(\mu)$ and $\tilde{\varphi}(\infty) = x$. We now obtain the desired φ using the natural homeomorphism between $(\omega \times \omega^\xi) \cup \{\infty\}$ and $\omega^\alpha + 1$.

If α is a limit ordinal, let $\xi_n \nearrow \alpha$ and let the sequence $\langle y_n : n \in \omega \rangle$ converge to x with each $y_n \in X^{(\xi_n)}$. Then use the obvious modification of the above argument, now using the natural homeomorphism between $\bigcup_n (\{n\} \times \omega^{\xi_n}) \cup \{\infty\}$ and $\omega^\alpha + 1$. $\square$

We shall show below that none of the three hypotheses, "first countable" and "HS" and "uncountable" can be omitted in Lemma 5.5.

The proof of Lemma 5.3 will make use of a convenient base for the compact open topology of X^Q. For this part of argument, we just assume that Q is compact and scattered and that X is locally compact. The obvious base is given by the sets of the form $\mathcal{N}(H_0/U_0, \ldots, H_{m-1}/U_{m-1})$, which denotes $\{g \in C(Q, X) : \forall i < m \; g(H_i) \subseteq U_i\}$, where each U_i is open in X and each H_i is closed in Q (and hence compact).

Definition 5.6. Call the basic set $\mathcal{N}(H_0/U_0, \ldots, H_{m-1}/U_{m-1})$ *tubular* iff all the H_i and U_i are non-empty, and the H_i are clopen and pairwise disjoint and $\bigcup_i H_i = Q$, and the U_i are open, and the $\overline{U}_i$ are compact and pairwise disjoint.

Lemma 5.7. *If Q is compact and zero-dimensional, and X is locally compact, and all sets in $[X]^{\leq Q}$ are zero-dimensional, then the tubular sets form a base for X^Q.*

Proof. It is sufficient to prove two things:

A. The intersection of two tubular sets is tubular or empty: Say

$$\mathcal{N} = \mathcal{N}(H_0/U_0, \ldots, H_{m-1}/U_{m-1}) \cap \mathcal{N}(K_0/V_0, \ldots, K_{n-1}/V_{n-1}) \quad .$$

Let $I = \{(i,j) : H_i \cap K_j \neq \emptyset\}$. If $U_i \cap V_j = \emptyset$ for some $(i,j) \in I$, then $\mathcal{N} = \emptyset$. Otherwise, $\mathcal{N} = \mathcal{N}(\ldots, (H_i \cap K_j)/(U_i \cap V_j), \ldots)_{(i,j)\in I}$.

B. If $g \in \mathcal{N}(F/W)$, where $\mathcal{N}(F/W)$ is a subbasic open set (so F is compact and non-empty, and W is open), then there is a tubular $\mathcal{N}$ such that $g \in \mathcal{N} \subseteq \mathcal{N}(F/W)$ (so the tubular sets form a local base at g):

If $g(Q) \subseteq W$, then, since X is locally compact and $g(Q)$ is compact, we may find an open U such that $g(Q) \subseteq U \subseteq \overline{U} \subseteq W$ and $\overline{U}$ is compact. Then $\mathcal{N}(Q/U)$ is tubular and $g \in \mathcal{N}(Q/U) \subseteq \mathcal{N}(F/W)$. From now on assume that $g(Q) \not\subseteq W$.

Consider the space $g(Q) \subseteq X$. This $g(Q)$ is compact and zero-dimensional. We have now $g(F) \subseteq (W \cap g(Q)) \subseteq g(Q)$, with $g(F)$ compact and $W \cap g(Q)$ (relatively) open in $g(Q)$. We may now partition $g(Q)$ as $g(Q) = A \mathbin{\dot\cup} B$, with both A, B relatively clopen and $g(F) \subseteq A \subseteq W \cap g(Q)$; note that $A \neq \emptyset$ and $B \neq \emptyset$. We may then get open subsets of X, $\tilde{W}, \tilde{V}$ satisfying: $\mathrm{cl}(\tilde{W}) \cap \mathrm{cl}(\tilde{V}) = \emptyset$ (in X) and $\mathrm{cl}(\tilde{W}), \mathrm{cl}(\tilde{V})$ both compact and $\tilde{W} \cap g(Q) = A$ and $\tilde{V} \cap g(Q) = B$ and $\tilde{W} \subseteq W$. Then Q is partitioned into clopen sets as $Q = g^{-1}(A) \mathbin{\dot\cup} g^{-1}(B)$ and $g \in \mathcal{N} := \mathcal{N}(f^{-1}(A)/\tilde{W}, f^{-1}(B)/\tilde{V}) \subseteq \mathcal{N}(F/W)$, and $\mathcal{N}$ is tubular.

To get the sets $\tilde{W}, \tilde{V}$: First get an open $\tilde{W}$ with $A \subseteq \tilde{W} \subseteq \mathrm{cl}(\tilde{W}) \subseteq W \backslash B$ and $\mathrm{cl}(\tilde{W})$ compact. Then get an open $\tilde{V}$ with $B \subseteq \tilde{V} \subseteq \mathrm{cl}(\tilde{V}) \subseteq X \backslash \mathrm{cl}(\tilde{W})$ and $\mathrm{cl}(\tilde{V})$ compact. $\square$

Two situations in which all sets in $[X]^{\leq Q}$ are zero-dimensional appear naturally: One is that Q is scattered (for countable Q of Lemma 5.3), and another is that X is zero-dimensional (as in Lemmas 5.1 and 5.16).

Proof of Lemma 5.3. For each of $(A), (B), (C)$, the statement ② $\to$ ① is trivial.

For $(A①) \to (A②)$, use the fact that P is a retract of Q when $P \subseteq Q$. To see this, note that Q is metric and zero-dimensional and P is a closed subset of Q. Then, if $\rho : Q \twoheadrightarrow P$ is the retraction, X^P embeds into X^Q via the map $f \mapsto f \circ \rho$.

For $(B①) \to (B②)$, observe that when P embeds into Q, $[X]^{\leq P}$ is a subspace of $[X]^{\leq Q}$ (because P is a continuous image of Q).

$(A) \to (B)$ holds because $[X]^{\leq Q}$ is the continuous image of X^Q under the map Φ of Lemma 5.2.

$(B②) \to (C②)$ holds because $[X]^P \subseteq [X]^{\leq P}$.

Now suppose that Q is countable. Then we need to prove that $(C①) \to (A)$. To do this, we shall prove $(C①) \to (C②)$ and $(C②) \to (B)$ and $(B) \to (A)$.

First, $(C②) \to (B)$ holds because $(C②) \to (B①)$ holds, because $[X]^{\leq Q}$ is the *countable* union of the $[X]^P$ such that P embeds into Q.

Second, we prove that $(B) \to (A)$ by showing that $(B②) \to (A①)$. Note that, as in Lemma 5.4, the countable compact Q is scattered.

From now on, use $\widehat{\mathcal{T}}$ for the topology on X and $\mathcal{T}$ for the coarser metric topology, which is separable because $(X, \widehat{\mathcal{T}})$ is HS. Note that if $F \subseteq X$ is $\widehat{\mathcal{T}}$ compact, then F is also $\mathcal{T}$ compact and $(F, \widehat{\mathcal{T}}) \cong (F, \mathcal{T})$. Let $\mathcal{B}$ be a countable open base for $(X, \mathcal{T})$, closed under finite unions.

Assume that $(A①)$ is false, so assume that $C(Q, X, \widehat{\mathcal{T}})$ is not HS, with left separated sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$. We shall prove that $[X]^P$ is not HS for some P that embeds into Q.

Let $\mathcal{N}_\alpha$ be the left separating neighborhoods, which by Lemma 5.7 we may assume are *tubular* basic clopen sets in $C(Q, X, \widehat{\mathcal{T}})$. So, $f_\alpha \in \mathcal{N}_\alpha$, and $\alpha < \beta \to f_\alpha \notin \mathcal{N}_\beta$.

Thinning the sequence, we may assume that $\mathcal{N}_\alpha = \mathcal{N}(H_0/U_0^\alpha, \ldots, H_{m-1}/U_{m-1}^\alpha)$, where m is independent of α, and the H_i form a clopen partition of Q (independently of α, since there are only $\aleph_0$ such partitions), and, in $(X, \widehat{\mathcal{T}})$, the U_i^α are open and non-empty, and the $\overline{U}_i^\alpha$ are compact and pairwise disjoint.

Then the $\overline{U}_i^\alpha$ are also compact in the coarser topology $(X, \mathcal{T})$, so, since $\mathcal{B}$ is closed under finite unions, we can find $G_0, \ldots, G_{m-1} \in \mathcal{B}$ such that in $(X, \mathcal{T})$: The sets $\overline{G}_0, \ldots, \overline{G}_{m-1}$ are pairwise disjoint and each $U_i^\alpha \subseteq G_i$. Since $|\mathcal{B}| = \aleph_0$, thinning the sequence again, we may assume that the G_i are independent of α.

To get P, recall that Q is homeomorphic to $\gamma + 1$ for some $\gamma < \omega_1$, by Lemma 5.4. Similarly, each $f_\alpha(Q)$ is homeomorphic to a successor ordinal. By Lemma 9.1, each $P_\alpha = f_\alpha(Q)$ embeds into $\gamma + 1$. There are only $\aleph_0$ such P_α, so thinning the sequence again, P is independent of α. Now $f_\alpha(Q) \in [X]^P$. Let $U^\alpha = \bigcup_i U_i^\alpha$.

Since $f_\alpha \in \mathcal{N}_\alpha = \mathcal{N}(H_0/U_0^\alpha, \ldots, H_{m-1}/U_{m-1}^\alpha)$, each $f_\alpha(H_i) \subseteq U_i^\alpha$, so $f_\alpha(Q) \subseteq U^\alpha$. Also, if $\alpha < \beta$ then $f_\alpha \notin \mathcal{N}(H_0/U_0^\beta, \ldots, H_{m-1}/U_{m-1}^\beta)$, so some $f_\alpha(H_i) \not\subseteq U_i^\beta$. Because of the separating sets $\overline{G}_0, \ldots, \overline{G}_{m-1}$, this implies that $f_\alpha(Q) \not\subseteq U^\beta$.

So, $[X]^P$ is not HS because the sequence $\langle f_\alpha(Q) : \alpha < \omega_1 \rangle$ is left separated by the Vietoris neighborhoods $\{B : B \subseteq U^\alpha\}$.

Third, we prove that $(C①) \to (C②)$. This is like the proof of $(1) \to (2)$ of Lemma 2.9. Let $\mathcal{T}$, $\widehat{\mathcal{T}}$, and $\mathcal{B}$ be as above. Assume that $(C②)$ is false, and let $\langle K_\alpha : \alpha < \omega_1 \rangle$ be a left separated sequence in $[X]^P$, where P embeds into Q. Then P is homeomorphic to $(K_\alpha, \widehat{\mathcal{T}})$ and to $(K_\alpha, \mathcal{T})$. For each α, we may choose an uncountable $U_\alpha \in \mathcal{B}$ such that $\mathrm{cl}(U_\alpha, \mathcal{T}) \cap K_\alpha = \emptyset$. Then, since $\mathcal{B}$ is countable, WLOG $U_\alpha = U$ for all α. Observe that since P and Q are homeomorphic to successor ordinals, P is homeomorphic to a clopen subset of Q. Then, using Lemma 5.5, fix $H \subset U$ such that $(H, \widehat{\mathcal{T}})$ and $(H, \mathcal{T})$ are homeomorphic and Q is homeomorphic to the disjoint union of P and H. Then $\langle K_\alpha \cup H : \alpha < \omega_1 \rangle$ is a left separated sequence in $[X]^Q$. □

Proof of Lemma 5.1. This is like the proof of $(B) \to (A)$, or $\neg(A) \to \neg(B)$. Here, Q may be uncountable, but we do not need to "thin" our left separated sequence down to one in some fixed $[X]^P$. □

Remark. Assume that X is as in Lemma 5.3 and $\mathcal{K}(X)$ is HS. Then $\mathcal{K}(X)$ is strongly HS, by an argument similar to the proof of Theorem 5.17 below.

The next lemma helps obtain counter-examples to strengthenings of Lemmas 5.3 and 5.5.

Lemma 5.8. *Assuming* CH*, there is a Luzin set R that is submetrizable and extremally disconnected and HS and HL.*

Proof. Assume CH, and let X be compact, ccc, and crowded, with $w(X) \leq \aleph_1$. Let $E(X) = \mathrm{st}(\mathrm{ro}(X))$ be the *absolute* or *projective cover* of X (see Engelking [5], Problems 6.3.19 and 6.3.20). Then $w(E(X)) = |\mathrm{ro}(X)| = \aleph_1$, and $\aleph_1 \leq |E(X)| \leq 2^{\aleph_1}$, and $E(X)$ is compact and extremally disconnected.

Let $\varphi : E(X) \to X$ be the usual projection, so $\{\varphi(y)\} = \bigcap\{\overline{U} : U \in y\}$; here, y is an ultrafilter on $\mathrm{ro}(X)$. So, φ is a continuous irreducible surjection.

Let $L \subset X$ be a Luzin set and $\aleph_1$-dense in X. Such an L exists because X is ccc, so that there are only $\aleph_1$ closed nowhere dense sets.

Then choose $R \subset E(X)$ such that $\varphi \restriction R$ maps R one-to-one onto L. Then R is dense in $E(X)$ because φ is irreducible and $\varphi(\overline{R}) = X$. So, R is also extremally disconnected. Also, R is a Luzin set because φ is irreducible, so that every nowhere dense subset of R maps to a nowhere dense subset of X.

This R is HL because it is a Luzin space. But also, *if* X is HS *then* R is HS: Suppose that $\langle y_\alpha : \alpha < \omega_1 \rangle$ were left separated in R. Let $x_\alpha = \varphi(y_\alpha)$. Since L is Luzin, we may thin the sequence and assume WLOG that there is an open $U \subseteq X$ such that all $x_\alpha \in U$ and $\{x_\alpha : \alpha < \omega_1\}$ is $\aleph_1$-dense in U. Then since X is HS, fix $\delta < \omega_1$ such that $\{x_\alpha : \alpha < \delta\}$ is dense in U.

Since φ is irreducible, $\{y_\alpha : \alpha < \delta\}$ is dense in $\varphi^{-1}(U)$ and hence dense in $\{y_\alpha : \alpha < \omega_1\}$, contradicting "left separated". To get the desired example R that is HS and HL, let X be $[0, 1]$. □

Now, we see that in Lemma 5.5, none of the three hypotheses "first countable" and "HS" and "uncountable" can be eliminated. For "first countable" use R ; for "HS" use a discrete space of size $\aleph_1$; for "uncountable" use a discrete space of size $\aleph_0$.

For Lemma 5.3, consider the following example:

Example 5.9. Assuming CH, there is a space R that is submetrizable and extremally disconnected and HS and HL, such that $R \times R$ and $[R]^2$ are neither HS nor HL.

Thus, if Q from Lemma 5.3 is infinite, then (C①) holds because $[R]^Q = \emptyset$ is HS, but (C②) fails because $2 = \{0, 1\}$ embeds into Q.

Proof. Let X be the double arrow space formed from $[-1, 1]$ by replacing each $t \in [-1, 1]$ by the pair $\{t^-, t^+\}$. Then let $\tilde{L} \subset (-1, 1)$ be an $\aleph_1$-dense Luzin set, using the standard real topology on $(-1, 1)$, with $t \in \tilde{L} \leftrightarrow -t \in \tilde{L}$ for all $t \in (-1, 1)$. Then, let $L = \{t^- : t \in \tilde{L}\} \subset X$, and choose R as in the proof of Lemma 5.8. Here, R is submetrizable because it maps homeomorphically onto L, which is a subspace of the Sorgenfrey line $\{t^- : -1 < t < 1\}$.

In $L \times L$, there is an uncountable discrete set $D = \{(t^-, (-t)^-) : 0 < t < 1 \;\&\; t \in L\}$. Then $(R \times R) \cap (\varphi \times \varphi)^{-1}(D)$ is discrete. Likewise, $[R]^2$ has an uncountable discrete set obtained by replacing D by $\{\{t^-, (-t)^-\} : 0 < t < 1 \;\&\; t \in L\}$. □

Section 4 described a general method for constructing S-spaces X that are locally countable and locally compact. Furthermore, we showed that we can build X so that even $\mathcal{K}(X)$ is HS, which implies that X is strongly HS. Section 7 will show that such an X can be strongly HS *without* $\mathcal{K}(X)$ being HS.

We can also ask whether we can get our X with the still stronger property that $\mathcal{K}(\mathcal{K}(X))$ is HS. But this turns out not to be stronger. Theorem 5.17 will show that if $\mathcal{K}(X)$ is HS, then $\mathcal{K}(\mathcal{K}(X))$ is HS, as is $\mathcal{K}(\mathcal{K}(\mathcal{K}(X)))$, $\mathcal{K}(\mathcal{K}(\mathcal{K}(\mathcal{K}(X))))$, etc.

We begin by discussing some properties that pass from X to $\mathcal{K}(X)$. We may think of $\mathcal{K}(X)$ as an extension of X, since the map $x \mapsto \{x\}$ embeds X into $\mathcal{K}(X)$.

Proposition 5.10. (1) *For infinite X, $w(\mathcal{K}(X)) = w(X)$.*

(2) *If X is locally compact, then $\mathcal{K}(X)$ is locally compact.*

(3) *If X is locally second countable, then $\mathcal{K}(X)$ is locally second countable.*

Proof. (1) is clear from Lemma 2.3.

For (2): Fix $K \in \mathcal{K}(X)$. Then, since K is compact, there is an open $U \supseteq K$ with $\overline{U}$ compact. Then, in $\mathcal{K}(X)$, $\mathcal{N}(U)$ is an open neighborhood of K and $\overline{\mathcal{N}(U)} \subseteq \mathcal{K}(\overline{U})$, which is compact.

The proof of (3) is exactly the same as that for (2), replacing "compact" by second countable. $\square$

On the other hand, "locally countable" does not extend upward from X to $\mathcal{K}(X)$. Our S-space is locally countable, but if X contains a closed copy of $\omega + 1$ (as does our X), then $\mathcal{K}(X)$ will contain a Cantor set, and hence fail to be locally countable.

Definition 5.11. Let "SM" abbreviate "submetrizable", and then say that X is ωSM iff there is a coarser topology $\mathcal{T}$ on X such that $(X, \mathcal{T})$ is a *separable* metric space.

Thus X is ωSM iff there is a countable point-separating subfamily of $C(X, \mathbb{R})$.

Proposition 5.12. *If X is SM then $\mathcal{K}(X)$ is SM; if X is ωSM then $\mathcal{K}(X)$ is ωSM.*

Proof. Assume that X is SM, and let $\widehat{\mathcal{T}}$ denote the original topology on X and let ρ be a metric such that (X, ρ) is coarser than $(X, \widehat{\mathcal{T}})$. Note that $\mathcal{K}(X, \widehat{\mathcal{T}}) \subseteq \mathcal{K}(X, \rho)$. Let $\rho^H : \mathcal{K}(X, \rho) \times \mathcal{K}(X, \rho) \to \mathbb{R}$ be the Hausdorff metric on $\mathcal{K}(X, \rho)$. Let $\widehat{\mathcal{T}}^V$ be the Vietoris topology on $\mathcal{K}(X, \widehat{\mathcal{T}})$. Then $(\mathcal{K}(X, \widehat{\mathcal{T}}), \rho^H)$ is coarser than $(\mathcal{K}(X, \widehat{\mathcal{T}}), \widehat{\mathcal{T}}^V)$, so $(\mathcal{K}(X, \widehat{\mathcal{T}}), \widehat{\mathcal{T}}^V)$ is SM. Furthermore, if (X, ρ) is separable then $(\mathcal{K}(X, \rho), \rho^H)$ is separable, so that $(\mathcal{K}(X, \widehat{\mathcal{T}}), \widehat{\mathcal{T}}^V)$ is ωSM. $\square$

We now relate $\mathcal{K}(X)$ to $\mathcal{K}(\mathcal{K}(X))$.

Lemma 5.13. *If $\mathcal{E} \in \mathcal{K}(\mathcal{K}(X))$ then $\bigcup \mathcal{E} \in \mathcal{K}(X)$.*

Proof. Suppose that $\mathcal{E} \subseteq \mathcal{K}(X)$ and $\mathcal{E}$ is compact in the Vietoris topology. Then $\bigcup \mathcal{E} \subseteq X$, and we must show that $\bigcup \mathcal{E}$ is compact. So, let $\mathcal{U} \subseteq \mathcal{P}(X)$ be an open cover of $\bigcup \mathcal{E}$ such that $\mathcal{U}$ is closed under finite unions. We must produce a $U \in \mathcal{U}$ with $\bigcup \mathcal{E} \subseteq U$.

In $\mathcal{K}(X)$, we have, for each $U \in \mathcal{U}$, the open set $\mathcal{N}(U) = \{H \in K(X) : H \subseteq U\}$.

For each $E \in \mathcal{E}$: $E \subseteq \bigcup \mathcal{E} \subseteq \bigcup \mathcal{U}$, and $\mathcal{U}$ is closed under finite unions, so there is some $U_E \in \mathcal{U}$ such that $E \subseteq U_E$, and hence $E \in \mathcal{N}(U_E)$. So, $\mathcal{E} \subseteq \bigcup\{\mathcal{N}(U) : U \in \mathcal{U}\}$. Since $\mathcal{E}$ is compact, fix $n < \omega$ and $U_i \in \mathcal{U}$ for $i < n$ such that $\mathcal{E} \subseteq \bigcup\{\mathcal{N}(U_i) : i < n\}$. Let $U = \bigcup\{U_i : i < n\}$. Then $U \in \mathcal{U}$ and $\mathcal{E} \subseteq \mathcal{N}(U)$, which implies that $\bigcup \mathcal{E} \subseteq U$. □

So, $\bigcup$ maps $\mathcal{K}(\mathcal{K}(X))$ onto $\mathcal{K}(X)$; it is onto because $\bigcup\{E\} = E$. It is not one-to-one if $|X| \geq 2$, since $\bigcup^{-1}(\{x, y\})$ is the five-element set $\big\{\{\{x, y\}\}, \{\{x\}, \{y\}\}, \{\{x\}, \{x, y\}\}, \{\{y\}, \{x, y\}\}, \{\{x\}, \{y\}, \{x, y\}\}\big\}$.

Lemma 5.14. *The map $\bigcup : \mathcal{K}(\mathcal{K}(X)) \twoheadrightarrow \mathcal{K}(X)$ is continuous, and its point inverses are compact.*

Proof. For continuity, it is sufficient to show that $\mathcal{R} := \{\mathcal{E} \in \mathcal{K}(\mathcal{K}(X)) : \bigcup \mathcal{E} \in \mathcal{W}\}$ is open whenever $\mathcal{W}$ is a subbasic open subset of $\mathcal{K}(X)$. There are two cases for each open $U \subseteq X$:

$\mathcal{W} = \{H : H \subseteq U\}$: $\mathcal{R} = \{\mathcal{E} \in \mathcal{K}(\mathcal{K}(X)) : \mathcal{E} \subseteq \{F \in \mathcal{K}(X) : F \subseteq U\}\}$, which is a subbasic open subset of $\mathcal{K}(\mathcal{K}(X))$.

$\mathcal{W} = \{H : H \cap U \neq \emptyset\}$: $\mathcal{R} = \{\mathcal{E} \in \mathcal{K}(\mathcal{K}(X)) : \mathcal{E} \cap \{F \in \mathcal{K}(X) : F \cap U \neq \emptyset\} \neq \emptyset\}$, which is a subbasic open subset of $\mathcal{K}(\mathcal{K}(X))$.

For each $H \in \mathcal{K}(X)$, its preimage $\{\mathcal{E} \in \mathcal{K}(\mathcal{K}(X)) : \bigcup \mathcal{E} = H\}$ is a closed subset of the compact $\{\mathcal{E} \in \mathcal{K}(\mathcal{K}(X)) : \bigcup \mathcal{E} \subseteq H\} \cong \mathcal{K}(\mathcal{K}(H))$. □

The proofs of Proposition 5.15 and Lemma 5.16 use that fact that every non-empty second countable compact space is a continuous image of the Cantor set (see Engelking [5], Problem 4.5.9(b)).

Proposition 5.15. *If X is an S-space and X^Q is HS, where Q is the Cantor set, then X is ultra strong.*

Proof. This is like the proof of $(A①) \to (A②)$ in Lemma 5.3. If P is any second countable compact space and $\varphi : Q \twoheadrightarrow P$ is a continuous surjection, then X^P embeds into X^Q via the map $f \mapsto f \circ \varphi$, so X^P is HS. □

Lemma 5.16 is a variation on Lemma 5.3, which compares spaces of images of and functions on compact P that embed into compact Q.

Lemma 5.16. *Assume that X is SM, locally compact, and zero-dimensional. Let Q be the Cantor set. Then X^Q is HS iff $\mathcal{K}(X)$ is HS.*

Proof. We may assume that X is HS, since both X^Q and $\mathcal{K}(X)$ contain a copy of X. Then, observe that X is ωSM (since it is SM and HS) and locally second countable (since it is ωSM and locally compact).

For $\to$: Since X is locally second countable, all compact subsets of X are second countable, and hence $[X]^{\leq Q} = \mathcal{K}(X)$. But $[X]^{\leq Q} = \Phi(X^Q)$, where $\Phi : X^Q \to \mathcal{K}(X)$ is the continuous map of Lemma 5.2. Thus, if X^Q is HS, then its image $\mathcal{K}(X)$ is HS.

For $\leftarrow$: From now on, use $\widehat{\mathcal{T}}$ for the topology on X and $\mathcal{T}$ for the coarser separable metric topology; so our X^Q becomes $C(Q, X, \widehat{\mathcal{T}})$. Note that if $F \subseteq X$ is $\widehat{\mathcal{T}}$ compact, then F is also $\mathcal{T}$ compact and $(F, \widehat{\mathcal{T}}) \cong (F, \mathcal{T})$. Let $\mathcal{B}$ be a countable open base for $(X, \mathcal{T})$, closed under finite unions.

Assume that $C(Q, X, \widehat{\mathcal{T}})$ is not HS, with left separated sequence $\langle f_\alpha : \alpha < \omega_1\rangle$. We shall find a left separated sequence in $\mathcal{K}(X, \widehat{\mathcal{T}})$.

Let $\mathcal{N}_\alpha$ be the left separating neighborhoods; so, $f_\alpha \in \mathcal{N}_\alpha$, and $\alpha < \beta \to f_\alpha \notin \mathcal{N}_\beta$. By Lemma 5.7, we may assume that the $\mathcal{N}_\alpha$ are *tubular* basic clopen sets in $C(Q, X, \widehat{\mathcal{T}})$.

Thinning the sequence, we may assume that $\mathcal{N}_\alpha = \mathcal{N}(H_0/U_0^\alpha, \ldots, H_{m-1}/U_{m-1}^\alpha)$, where m is independent of α, and the H_i form a clopen partition of Q (independently of α, since there are only $\aleph_0$ such partitions). Also, since X is zero-dimensional and locally compact, we may assume that the U_i^α are compact and clopen in $(X, \widehat{\mathcal{T}})$.

Then the U_i^α are also compact in the coarser topology $(X, \mathcal{T})$, so, since $\mathcal{B}$ is closed under finite unions, we can find $G_0, \ldots, G_{m-1} \in \mathcal{B}$ such that in $(X, \mathcal{T})$: The sets $\overline{G}_0, \ldots, \overline{G}_{m-1}$ are pairwise disjoint and each $U_i^\alpha \subseteq G_i$. Since $|\mathcal{B}| = \aleph_0$, we may assume WLOG that the G_i are independent of α.

Let $U^\alpha = \bigcup_i U_i^\alpha$. Note that each $f_\alpha(Q) \in \mathcal{K}(X, \widehat{\mathcal{T}})$.

Since $f_\alpha \in \mathcal{N}_\alpha = \mathcal{N}(H_0/U_0^\alpha, \ldots, H_{m-1}/U_{m-1}^\alpha)$, each $f_\alpha(H_i) \subseteq U_i^\alpha$, so $f_\alpha(Q) \subseteq U^\alpha$. Also, if $\alpha < \beta$ then $f_\alpha \notin \mathcal{N}(H_0/U_0^\beta, \ldots, H_{m-1}/U_{m-1}^\beta)$, so some $f_\alpha(H_i) \not\subseteq U_i^\beta$. Because of the separating sets $\overline{G}_0, \ldots, \overline{G}_{m-1}$, this implies that $f_\alpha(Q) \not\subseteq U^\beta$.

So, $\mathcal{K}(X, \widehat{\mathcal{T}})$ is not HS because the sequence $\langle f_\alpha(Q) : \alpha < \omega_1\rangle$ is left separated by the Vietoris neighborhoods $\{B : B \subseteq U^\alpha\}$. $\square$

Theorem 5.17. *Assume that X is SM, zero-dimensional, and locally compact, and $\mathcal{K}(X)$ is HS. Then $\mathcal{K}(\mathcal{K}(X))$ is HS.*

Proof. Apply Lemma 5.16 to X to see that X^Q is HS, and hence $(X^Q)^Q$ is HS, since $(X^Q)^Q \cong X^{Q\times Q} \cong X^Q$. For the first "$\cong$", see Engelking [5], Theorem 3.4.7.

To see that $\mathcal{K}(\mathcal{K}(X))$ is HS, apply the same lemma to $\mathcal{K}(X)$, which is also ωSM and locally second countable and zero-dimensional and locally compact. As in the proof of Lemma 5.16, the continuous map $\Phi : X^Q \twoheadrightarrow \mathcal{K}(X)$ is surjective. Here, it induces $\Phi^* : (X^Q)^Q \twoheadrightarrow (\mathcal{K}(X))^Q$, making $(\mathcal{K}(X))^Q$ a continuous image of the HS $(X^Q)^Q$. $\square$

6. Remarks on the SCSP

Here, we relate our construction to the SCSP, defined in [6]. There, we said that a *compact* space X has the ω_1–SCSP (Strong Closed Set Property) iff there are non-empty closed $H_\alpha, K_\alpha \subseteq X$ for $\alpha < \omega_1$ with all $H_\alpha \cap K_\alpha = \emptyset$ such that $\alpha \neq \beta \ \to \ H_\alpha \cap K_\beta \neq \emptyset \ \& \ H_\beta \cap K_\alpha \neq \emptyset$.

Observe that if $Y \subseteq X$ and Y has the ω_1–SCSP then X has the ω_1–SCSP. Also, if X has the ω_1–SCSP then $w(X) \geq \aleph_1$. So, the *negation*, $\neg(\omega_1$–SCSP), might be considered to be a notion of smallness. But unlike other notions of smallness (small weight, small density, HS, HL, etc), this $\neg(\omega_1$–SCSP) is, under $\diamondsuit$, not closed under finite disjoint unions. As an example, let $X = \omega_1 \cup \{\infty\}$, where ω_1 has the topology $\widehat{\mathcal{T}}$ defined in Section 4 and X is its one-point compactification. Then $X \times \{0,1\}$ has the ω_1–SCSP but X does not.

To prove the first statement, choose a clopen compact $F_\alpha \subset \omega_1$ with $\alpha = \max F_\alpha$ and let $H_\alpha = F_\alpha \times \{0\} \cup (X \backslash F_\alpha) \times \{1\}$ and $K_\alpha = (X \times \{0,1\}) \setminus H_\alpha$.

To prove the second statement, suppose that we had closed $H_\alpha, K_\alpha \subseteq X$ as in the definition of the ω_1–SCSP. Expanding them, WLOG they are clopen, and expanding further, WLOG all $H_\alpha \cup K_\alpha = X$. Then, thinning the sequence and swapping H/K if necessary, WLOG $\infty \in K_\alpha$ for all α. Then each $H_\alpha \subset \omega_1$ and is clopen and compact and countable. Also, $\alpha \neq \beta \to H_\alpha \not\subseteq H_\beta$. Let $\mathcal{N}_\beta = \{F \in \mathcal{K}(\omega_1, \widehat{\mathcal{T}}) : F \subseteq H_\beta\}$. Then $\mathcal{N}_\beta$ is a neighborhood of H_β in $\mathcal{K}(\omega_1, \widehat{\mathcal{T}})$, and $\alpha \neq \beta \to H_\alpha \notin \mathcal{N}_\beta$, contradicting the fact that $\mathcal{K}(\omega_1, \widehat{\mathcal{T}})$ is HS.

7. Examples

Since the point of this paper was to produce ultra strong S-spaces, we need to show that "ultra strong" does not just follow from "strong". So, we shall produce (under CH) two examples of strong S-spaces X such that $[X]^{\omega+1}$ is not HS.

Trivial Example. Let $X = Y \oplus Z$, where Y is any strong S-space and Z is a countable space such that $[Z]^{\omega+1}$ is not HS. X is strongly HS because Z is countable.

Such a Z exists by:

Lemma 7.1. *There is a countable T_3 space Z such that $[Z]^{\omega+1}$ has an uncountable discrete subset and $[Z]^{\gamma+1} = \emptyset$ for all $\gamma \geq \omega + \omega$.*

Proof. Let $Z = I \cup \{\infty\}$, where I is a countably infinite set. To define the topology, start with $A_\alpha, C_\alpha \in [I]^{\aleph_0}$ for $\alpha < \omega_1$ satisfying the following:

1. $A_\alpha \cap C_\beta$ is finite for all α, β.

2. $A_\alpha \cap A_\beta$ and $C_\alpha \cap C_\beta$ are finite for all $\alpha \neq \beta$.
3. $A_\alpha \cap C_\alpha = \emptyset$ for all α.
4. $A_\alpha \cap C_\beta \neq \emptyset$ for all $\alpha \neq \beta$.

Let $\mathcal{F}$ be the filter on I generated by all cofinite sets plus $\{I \backslash A_\alpha : \alpha < \omega_1\}$. Define $U \subseteq Z$ to be open iff either $\infty \notin U$ or $U \backslash \{\infty\} \in \mathcal{F}$. Then Z is T_3 and all $z \in I$ are isolated, whereas ∞ is not isolated. Since there is only one non-isolated point, $[Z]^{\gamma+1} = \emptyset$ when $\gamma \geq \omega + \omega$.

Let $K_\alpha = C_\alpha \cup \{\infty\}$. Then $K_\alpha \cong \omega + 1$ by (1). Furthermore, by (3), $\{H \in \mathcal{K}(Z) : H \subseteq (I \backslash A_\alpha) \cup \{\infty\}\}$ is a neighborhood of K_α that, by (4), does not contain any K_β for $\beta \neq \alpha$. So, $\{K_\alpha : \alpha < \omega_1\}$ is a discrete set in $\mathcal{K}(Z)$.

Sets A_α, C_α satisfying (1)(2)(3)(4) form a type of Luzin gap. They may be constructed as follows (similarly to Todorčević [15]): Let $I = \{0,1\}^{<\omega} \backslash \{\emptyset\}$. For $\sigma \in I$, let $\mathcal{L}(\sigma) = \mathrm{lh}(\sigma) - 1$. Define $\hat{\sigma}$ by: $\mathrm{lh}(\hat{\sigma}) = \mathrm{lh}(\sigma)$ & $\hat{\sigma} \restriction \mathcal{L}(\sigma) = \sigma \restriction \mathcal{L}(\sigma)$ & $\hat{\sigma}(\mathcal{L}(\sigma)) = 1 - \sigma(\mathcal{L}(\sigma))$ (just change the last element of the sequence σ). Fix any distinct $f_\alpha \in \{0,1\}^\omega$ for $\alpha < \omega_1$. Let $A_\alpha = \{\sigma \in I : \sigma \subset f_\alpha\}$ and $C_\alpha = \{\sigma \in I : \hat{\sigma} \subset f_\alpha\}$. $\square$

Less Trivial Example. We can get X to be locally compact and locally countable (and hence first countable). Note that any Z satisfying Lemma 7.1 cannot be first countable. Also, if we let Z be uncountable, it is possible that $Y \oplus Z$ will not be strongly HS, even if both Y and Z are strongly HS. But we can still prove:

Theorem 7.2. *Assuming* CH*, there is a locally compact locally countable* strong *S-space X such that $[X]^{\omega+1}$ and $X^{\omega+1}$ are not HS and $X \cup \{\infty\}$ satisfies the ω_1-SCSP.*

In the proof, we shall focus on getting $[X]^{\omega+1}$ to be not HS; then $X^{\omega+1}$ also will not be HS by Lemma 5.3.

Our fundamental space $(X, \mathcal{T})$ will be related to the Sorgenfrey topology (Definition 4.25) on a set of real numbers. It will be strongly HS, but $([X]^{\omega+1}, \mathcal{T})$ will not be HS. Then, $(X, \widehat{\mathcal{T}})$ will also be strongly HS by Corollary 4.23, while $([X]^{\omega+1}, \widehat{\mathcal{T}})$ will not be HS by Lemma 4.24.

Now if $X \subset \mathbb{R}$ and X is n-increasing for all n (Definition 4.26) and $\mathcal{T}$ is the Sorgenfrey topology, then $(X, \mathcal{T})$ is indeed strongly HS by Lemma 4.27, but the same proof will show that $([X]^{\omega+1}, \mathcal{T})$ is HS, as is even $\mathcal{K}(X, \mathcal{T})$. So, our actual $(X, \mathcal{T})$ will be a bit more complicated.

We begin with two simple lemmas on increasing sets. The following is easily proved from the definitions:

Lemma 7.3. *Suppose that $E \subset \mathbb{R}$ is n-increasing and E is partitioned into disjoint sets E_j for $j \in \omega$. Let D_j for $j \in \omega$ be disjoint subsets* *of $\mathbb{R}$ such that each D_j is order-isomorphic to E_j. Then $\bigcup_j D_j$ is also n-increasing.*

In particular (assuming CH), if $C \subset \mathbb{R}$ is a Cantor set, then we may start with an $E \subset C$ such that E is n-increasing for all $n \in \omega$, and also E is $\aleph_1$-dense in C; then partition E into disjoint sets E_j for $j \in \omega$ so that each E_j is also $\aleph_1$-dense in C. We may now let C_j for $j \in \omega$ be disjoint Cantor sets such that every open interval contains infinitely many of them. In each C_j, we may place an isomorphic copy E_j^* of E_j. Then $\bigcup_j E_j^*$ is n-increasing for all $n \in \omega$, and also $\aleph_1$-dense in $\mathbb{R}$. Re-indexing (our C_0 becomes the C of Lemma 7.4):

Lemma 7.4. *Assuming CH: Let $C \subset \mathbb{R}$ be any Cantor set. Then there is an $E \subset \mathbb{R}$ such that E is n-increasing for all $n \in \omega$, and E is $\aleph_1$-dense in $\mathbb{R}$, and also $E \cap C$ is $\aleph_1$-dense in C.*

Proof of Theorem 7.2. Fix an $\aleph_1$-dense $E \subseteq \mathbb{R}$ with $\mathbb{Q} \subset E$. The fundamental space X will be the set $(\omega+1) \times E$ with a topology $\mathcal{T}$ that we shall now describe.

First, let $\mathcal{T}^-$ denote the Sorgenfrey topology on E. So, an open base for $\mathcal{T}^-$ consists of all $(x, y]$ such that $x < y$ and $x, y \in E$; then these sets are clopen as well.

Let $\mathcal{T}^-$ also denote the natural product topology on X, formed giving $\omega+1$ its usual topology. This has as a clopen base all sets $\{n\} \times (x, y]$ and $((\omega+1)\backslash n) \times (x, y]$, with $x < y$ and $x, y \in E$. We shall now refine $(X, \mathcal{T}^-)$ to a space $(X, \mathcal{T})$, and we shall use $(X, \mathcal{T})$ as our fundamental space.

For $\alpha < \omega_1$, choose $K_\alpha \subset X$ with the following five properties: ① Each $K_\alpha = \{(\mu, t^\alpha_\mu) : \mu \le \omega\}$ and $t^\alpha_\mu \in \mathbb{Q}$ for $\mu < \omega$, while $t^\alpha_\omega \in E$. So, K_α is actually a function from $\omega+1$ to $\mathbb{R}$; $K_\alpha(\mu) = t^\alpha_\mu$. ② The sequence $\langle t^\alpha_\mu : \mu < \omega\rangle$ is strictly increasing and converges to t^α_ω in the real numbers; that is, $t^\alpha_\omega = \sup_\mu t^\alpha_\mu$. ③ $\alpha \ne \beta \to t^\alpha_\omega \ne t^\beta_\omega$. Note that each $(K_\alpha, \mathcal{T}^-) \cong \omega+1$, and this will remain true with respect to $\mathcal{T}$ and $\widehat{\mathcal{T}}$; the set $\{K_\alpha : \alpha < \omega_1\}$ will be discrete in $(\mathcal{K}(X), \widehat{\mathcal{T}})$, proving that $([X]^{\omega+1}, \widehat{\mathcal{T}})$ is not HS. Let $K_\alpha\!\downarrow = \{(\mu, z) \in X : z \le t^\alpha_\mu\}$; so, each $K_\alpha \subset K_\alpha\!\downarrow$ and $K_\alpha\!\downarrow$ is $\mathcal{T}^-$ closed but not open. But, $K_\alpha\!\downarrow \setminus \{(\omega, t^\alpha_\omega)\}$ is $\mathcal{T}^-$ open because $\langle t^\alpha_\mu : \mu < \omega\rangle \nearrow t^\alpha_\omega$.

Now, refine $(X, \mathcal{T}^-)$ to $(X, \mathcal{T})$ by declaring each $K_\alpha\!\downarrow$ to be clopen. Note that each $(K_\alpha, \mathcal{T}) \cong \omega+1$ because $\alpha \ne \beta \to t^\alpha_\omega \ne t^\beta_\omega$. Also, note that for each $\mu \le \omega$, $\{\mu\} \times E$ has the same topology in $\mathcal{T}$ as in $\mathcal{T}^-$.

④ The set E is n-increasing for all n. It follows (see Lemma 4.27) that in $\mathcal{T}$, all finite products of these $\{\mu\} \times E$ are HS, which implies that X^n is HS for all $n \in \omega$.

⑤ The K_α are chosen so that also there exists $\mu < \omega$ $[t^\alpha_\mu > t^\beta_\mu]$ whenever $\alpha \neq \beta$. This is obvious when $t^\alpha_\omega > t^\beta_\omega$, since then $t^\alpha_\mu > t^\beta_\mu$ for all but finitely many μ. It is a bit tricky when $t^\alpha_\omega < t^\beta_\omega$, since then $t^\alpha_\mu < t^\beta_\mu$ for all but finitely many μ, but ⑤ can be done, as explained below.

Requirement ⑤ implies that $\alpha \neq \beta \to K_\alpha \not\subseteq K_\beta\!\downarrow$, so that $\langle K_\alpha : \alpha < \omega_1\rangle$ is a discrete sequence in $([X]^{\omega+1}, \mathcal{T})$.

We now refine $\mathcal{T}$ to $\widehat{\mathcal{T}}$. Here, $(X, \mathcal{T})$ is our fundamental space, which we enumerate as $\{x_\delta : \delta < \omega_1\}$ and then apply the construction in Section 4 (identifying x_δ with δ) to obtain $\widehat{\mathcal{T}}$. Choose the enumeration as given by Lemma 4.24. Then (passing to a subsequence), we can make $\langle K_\alpha : \alpha < \omega_1\rangle$ discrete in $([X]^{\omega+1}, \widehat{\mathcal{T}})$ as well.

But it follows from Corollary 4.23 that $(X, \widehat{\mathcal{T}})$ is strongly HS.

Regarding the SCSP and working in $(X, \widehat{\mathcal{T}})$: Each $K_\alpha\!\downarrow$ is clopen, so choose a compact clopen countable H_α with $K_\alpha \subseteq H_\alpha \subseteq K_\alpha\!\downarrow$. Then the H_α remain compact clopen in $X \cup \{\infty\}$, and $\alpha \neq \beta \to H_\alpha \not\subseteq H_\beta$, establishing the ω_1-SCSP for $X \cup \{\infty\}$.

We are now done if we explain how to obtain Requirement ⑤.

Apply CH plus Lemma 7.4, where the Cantor set C is the set of real numbers of the form $\sum_{j\in\omega} f(j)10^{-j-1}$, where all $f(j) \in \{5, 7\}$; these are the numbers in $(0, 1)$ whose decimal expansion contains only 5s and 7s. So, choose E so that $\mathbb{Q} \subset E \subset \mathbb{R}$ and E is $\aleph_1$-dense in $\mathbb{R}$ and $E \cap C$ is $\aleph_1$-dense in C and E is n-increasing for all $n \in \omega$.

Now, it is sufficient to show that for each $y \in C$, we can choose $t^y_n \in \mathbb{Q}$ for $n < \omega$ such that $y < z \to \exists n\ t^y_n > t^z_n$ and $\langle t^y_n : n \in \omega\rangle$ is a strictly increasing sequence converging to y. The following illustrates our choice of t^y_n:

y	t^y_0	t^y_1	t^y_2	t^y_3
$.5575\cdots$	.3	.53	.551	5573
$.5757\cdots$	.3	.51	.573	5751
$.5755\cdots$	.3	.51	.573	5753
$.7575\cdots$	.1	.73	.751	7573

More formally, define $\hat{5} = 3$ and $\hat{7} = 1$. Then, if $y = \sum_{j\in\omega} f(j)10^{-j-1}$, we let $t^y_n = \sum_{j<n} f(j)10^{-j-1} + \widehat{f(n)}10^{-n-1}$. Now suppose that $y < z$, with $y = \sum_{j\in\omega} f(j)10^{-j-1}$ and $z = \sum_{j\in\omega} g(j)10^{-j-1}$. Let n be least such that $f(n) \neq g(n)$; so $f(n) = 5$ and $g(n) = 7$. Then $\widehat{f(n)} > \widehat{g(n)}$, so $t^y_n > t^z_n$. $\square$

Another remark on strong S-spaces:

If X is an S-space and $X \cong X \times X$ then $X^n \cong X$ for all $n \geq 1$, so X is a strong S-space. This suggests a new way of building a strong S-space — except that we do not see how to build a homeomorphism from X onto

X^2 directly into our construction in Section 4. We describe below a way of getting $X \cong X^2$ *after* we have built a strong S-space.

Call X *good* iff $|X| = \aleph_1$ and X is T_3 and locally compact and locally countable (and hence also scattered and zero-dimensional). All the S-spaces constructed in this paper are good. If X is good then X is not HL, since using the Cantor-Bendixson sequence, one can list X as $\{x_\alpha : \alpha < \omega_1\}$, where $\{x_{\omega\cdot\xi+n} : n \in \omega\}$ lists $X^{(\xi)} \setminus X^{(\xi+1)}$. This listing is right separated.

Note that the product of two good spaces is good.

CH implies that there is a good S-space X such that $X \cong X \times X$. *Proof.* Let Y be any good strong S-space. Let $X = \bigoplus_{n<\omega} \omega \times Y^n$. Note that Y^0 is a one point space. Since $\omega \times \omega \cong \omega$, we have $X^2 = \bigoplus_{m,n<\omega} \omega \times Y^{m+n} \cong X$.

The X that we build in this manner may or may not have $\mathcal{K}(X)$ HS. If we start with $[Y]^{\omega+1}$ not HS (as in Theorem 7.2), then $[X]^{\omega+1}$ will also not be HS, since Y embeds into X. But if we start with $\mathcal{K}(Y)$ HS, and assume that Y is also ωSM, then $\mathcal{K}(X)$ will also be HS; this is easily proved using Lemma 5.16.

Question. Can we also get a good strong S-space X such that $X \not\cong X \times X$?

8. On Compact S-spaces

Our ultra strong S-space X was locally compact but not compact. The one-point compactification $X \cup \{\infty\}$ is compact and a strong S-space, since adding one point does not destroy the property of being strongly HS.

It is not clear whether $X \cup \{\infty\}$ is ultra strong. In fact, we do not know whether $(X \cup \{\infty\})^{\omega+1}$ is HS. But $\mathcal{K}(X \cup \{\infty\})$ is not HS. This follows easily from:

Lemma 8.1. *If Y is compact and not HL, then $\mathcal{K}(Y)$ is not HS.*

Proof. Since Y is not HL, there is a sequence of sets, $\langle F_\alpha : \alpha < \omega_1\rangle$, such that each F_α is closed (and hence compact) and $\alpha < \beta \to F_\beta \subsetneqq F_\alpha$. Choose $p_\alpha \in F_\alpha \backslash F_{\alpha+1}$. Let $\mathcal{N}_\alpha = \{K \in \mathcal{K}(Y) : p_\alpha \notin K\} = \{K \in \mathcal{K}(Y) : K \subseteq Y \backslash \{p_\alpha\}\}$. Then $\mathcal{N}_\alpha$ is open in the Vietoris topology and $\mathcal{N}_\alpha \cap \{F_\xi : \xi < \omega_1\} = \{F_\xi : \alpha < \xi < \omega_1\}$, so $\{F_\xi : \xi \leq \alpha\}$ is relatively closed in $\{F_\xi : \xi < \omega_1\}$ for each α, so $\{F_\xi : \xi < \omega_1\}$ is not separable. $\square$

9. On Iterated Exponentials

Despite results such as Lemma 5.3 which emphasize the similarities between $[X]^{\leq P}$ and X^P, there is a significant difference between them.

For example, let $P = \omega + 1$ and consider $A, B \in [\mathbb{R}]^{\omega+1} \subset \mathcal{K}(\mathbb{R})$, where

$A = \{0,1\} \cup \{10^{-5-n} : n \in \omega\}$ and $B = \{0,1\} \cup \{1 - 10^{-5-n} : n \in \omega\}$.

Then A, B are "close" as elements of $\mathcal{K}(\mathbb{R})$ since $d(A,B) = 0.00001$, where d is the Hausdorff metric. But if $A = \mathrm{ran}(f)$ and $B = \mathrm{ran}(g)$ for some $f, g \in \mathbb{R}^{\omega+1}$, then $\|f - g\| = 1$ (since $f(\omega) - g(\omega) = 1$), so A, B are not "close" if viewed as arising from functions in $\mathbb{R}^{\omega+1}$. Also, although $(X^Q)^P$ has an obvious identification with $X^{P\times Q}$, and hence with $X^{Q\times P}$ and $(X^P)^Q$, we shall see below that even when P, Q are countable compacta, there is no such identification of $[[X]^{\le Q}]^{\le P}$ with $[X]^{\le (P\times Q)}$, and the two spaces $[[X]^{\le Q}]^{\le P}$ and $[[X]^{\le P}]^{\le Q}$ can be significantly different from each other.

In the following, P, Q denote arbitrary compact spaces. Note that $[X]^{\le Q} \subseteq \mathcal{K}(X)$ and $X \subseteq Y \to [X]^{\le P} \subseteq [Y]^{\le P}$. So, $[[X]^{\le Q}]^{\le P} \subseteq [\mathcal{K}(X)]^{\le P} \subseteq \mathcal{K}(\mathcal{K}(X))$.

We may also apply the continuous map $\bigcup : \mathcal{K}(\mathcal{K}(X)) \twoheadrightarrow \mathcal{K}(X)$ (see Lemma 5.13). But now consider its restriction $\bigcup : [[X]^{\le Q}]^{\le P} \to \mathcal{K}(X)$. Note that for finite m, n, $\bigcup$ maps $[[X]^{\le n}]^{\le m}$ onto $[X]^{\le m\times n}$. But for more general P, Q, what is the range of the $\bigcup$ map on $[[X]^{\le Q}]^{\le P}$? We shall see that it need not be contained in $[X]^{\le P\times Q}$, even in the "trivial" case that $P \cong Q \cong \omega + 1$.

Another view of the $\bigcup$ map and computing $\bigcup \mathcal{E}$, where $\mathcal{E} \in [[X]^{\le Q}]^{\le P}$: Say $\mathcal{E} = \{K_p : p \in P\}$, where each $K_p \in \mathcal{K}(X)$, and each K_p is a continuous image of Q, and the map $p \mapsto K_p$ is continuous from P into $\mathcal{K}(X)$. Then we know that $\bigcup \mathcal{E} = \bigcup_p K_p$ is a compact subset of X.

Example. $\bigcup \mathcal{E} = \bigcup_p K_p$ is compact scattered if P and Q are compact scattered (see Lemma 9.4). Then, Theorem 9.5 will produce a bound on the rank of $\bigcup_p K_p$.

A related simple explicit example illustrates the difference between a continuous $f : P \to [X]^Q$, where each $f(p)$ chooses a continuous homeomorphic image of Q, and a continuous $g : P \to X^Q$, where each $g(p)$ chooses a continuous function in X^Q that produces such an image. With $P = Q = \omega + 1$, such a g yields a continuous map from $P \times Q$ into X, which, using $f(p) = g(p)(\omega + 1)$, would imply $\mathrm{rank}(\bigcup_p f(p)) \le 2$. Working in $\mathcal{K}(\mathbb{R} \times \mathbb{R})$, using the Hausdorff metric induced by the standard Euclidean metric on the plane, we shall define $f : \omega + 1 \to [\mathbb{R} \times \mathbb{R}]^{\omega+1}$ so that $\mathrm{rank}(\bigcup_p f(p)) = 3$.

Let $f(\omega) = \{0\} \times E_\omega$, where $E_\omega = \{0\} \cup \{2^{-k} : k < \omega\}$. So, $f(\omega) \cong \omega+1$, with limit point $(0,0)$. For $n < \omega$, let $f(n) = \{2^{-n}\} \times E_n$. Choose $E_n \subset [0,1]$ with the following properties:

①. $E_n \cong \omega + 1$, with limit point $y_n = 2^{-j_n}$, with $j_n \in \omega$.

②. $d(E_n, E_\omega) \le 2^{-n}$, where d is the Hausdorff metric on $\mathcal{K}(\mathbb{R})$.

③. $\forall k \in \omega\ \exists^\infty n\ [j_n = k]$.

Just using ①②, $d(f(n), f(\omega)) \le 2^{-n} + 2^{-n}$, so $f : \omega + 1 \to [\mathbb{R} \times \mathbb{R}]^{\omega+1}$ is a continuous map. Let $H = \bigcup_p f(p) = \bigcup_{\nu \le \omega} f(\nu) \subseteq \mathbb{R} \times \mathbb{R}$. Then each point $(2^{-n}, 2^{-j_n}) \in H'$, so applying ③, each $(0, 2^{-k}) \in H''$, so $(0, 0) \in H'''$; so, $\mathrm{rank}(\bigcup_p f(p)) = 3$. Therefore, no continuous choice function $g : P \to X^Q$ corresponds to this f.

Next, we list two basic lemmas that will be relevant for the rest of this section.

First, we prove the following well-known fact about maps on scattered compacta:

Lemma 9.1. *Suppose that $f \in C(X, Y)$ maps X onto Y and X is compact scattered. Then Y is compact scattered and $f(X^{(\alpha)}) \supseteq Y^{(\alpha)}$ for all α and $\mathrm{rank}(Y) \le \mathrm{rank}(X)$.*

Proof. It is enough to prove $f(X^{(\alpha)}) \supseteq Y^{(\alpha)}$. The rest follows trivially. We induct on α. The case $\alpha = 0$ is trivial.

For $\alpha = 1$: We need to prove that $f(X') \supseteq Y'$. Suppose that $y \in Y' \backslash f(X')$. Since $f(X')$ is compact, let U be an open neighborhood of y with $\overline{U} \cap f(X') = \emptyset$. Then $f^{-1}(\overline{U})$ is compact and disjoint from X', so $f^{-1}(\overline{U})$ is a finite set of isolated points of X, so that $\overline{U} = f(f^{-1}(\overline{U}))$ is finite.

For the induction, the successor steps uses the $\alpha = 1$ case. For limit γ, and assuming the result for $\alpha < \gamma$, use $f(X^{(\gamma)}) \supseteq \bigcap_{\alpha<\gamma} f(X^{(\alpha)}) \supseteq \bigcap_{\alpha<\gamma} Y^{(\alpha)} = Y^{(\gamma)}$. The first "$\supseteq$" follows by compactness and the fact that $X^{(\alpha)} \searrow_\alpha X^{(\gamma)}$. $\square$

Remark. There are no similar bounds for the ranks of individual points. That is, $\mathrm{rank}(x)$ might be $<$ or $>$ or $= \mathrm{rank}(f(x))$.

Second, we shall use the following *interpolation lemma* in the plane, with $L = \mathbb{R}^2$:

Definition 9.2. Let L be a normed linear space, and fix $A, B \in [L]^{<\aleph_0} \backslash \{\emptyset\}$, and $\theta \in [0, 1]$. Then the *interpolant* $C = \mathcal{I}(A, B, \theta)$ is defined as follows: First, let $\mathcal{G} = \{(a, b) \in A \times B : d(a, b) \le d(A, B)\}$ (the *good pairs*). For $(a, b) \in \mathcal{G}$, let $c(a, b) = (1-\theta) \cdot a + \theta \cdot b$. Then let $C = \{c(a, b) : (a, b) \in \mathcal{G}\}$.

Note that A, B need not be disjoint, so a might equal b for some $(a, b) \in \mathcal{G}$. By the definition of the Hausdorff metric, $\mathrm{dom}(\mathcal{G}) = A$ and $\mathrm{ran}(\mathcal{G}) = B$ (but $\mathcal{G}$ need not be a function); it follows that $|\mathcal{I}(A, B, \theta)| \le |A| \cdot |B|$, and $\max(|A|, |B|) \le |\mathcal{I}(A, B, \theta)|$ for $\theta \in (0, 1)$.

Lemma 9.3. *Let $L, A, B, \theta, C = \mathcal{I}(A, B, \theta)$ be as in Definition 9.2. Then $d(A, C) = \theta \cdot d(A, B)$ and $d(C, B) = (1 - \theta) \cdot d(A, B)$.*

Proof. Note that if $c = c(a,b)$, then both $d(a,c) = \theta \cdot d(a,b) \le \theta \cdot d(A,B)$ and $d(c,b) = (1-\theta) \cdot d(a,b) \le (1-\theta) \cdot d(A,B)$. Now let $C = \{c(a,b) : (a,b) \in \mathcal{G}\}$.

Then $d(A,C) \le \theta \cdot d(A,B)$. *Proof.* This follows if we can show that we have both $\forall c \in C\ \exists a \in A\ [d(a,c) \le \theta \cdot d(A,B)]$ and $\forall a \in A\ \exists c \in C\ [d(a,c) \le \theta \cdot d(A,B)]$. For the first statement, choose a such that $c = c(a,b)$ for some $b \in B$. For the second statement, choose c such that $c = c(a,b)$ for some $b \in B$.

Likewise, $d(C,B) \le (1-\theta) \cdot d(A,B)$, by essentially the same proof.

Now, by the triangle inequality, $d(A,B) \le d(A,C) + d(C,B)$, which implies that $d(A,C) = \theta \cdot d(A,B)$ and $d(C,B) = (1-\theta) \cdot d(A,B)$. $\square$

Our intended application will be in the plane, with $L = \mathbb{R} \times \mathbb{R}$ and $A \subset \{\mu\} \times \mathbb{R}$ and $B \subset \{\nu\} \times \mathbb{R}$ and $\mu < \rho < \nu$. Let $\theta = (\rho - \mu)/(\nu - \mu)$. Then $(1-\theta) \cdot \mu + \theta \cdot \nu = \theta \cdot (\nu - \mu) + \mu = \rho$, so $\mathcal{I}(A,B) \subset \{\rho\} \times \mathbb{R}$.

We now return to the following general situation, where P, Q are compact and scattered, and we are considering elements of the set $\bigcup([[X]^{\le Q}]^{\le P})$. So, we have $K_p \in \mathcal{K}(X)$ for $p \in P$ such that each K_p is a continuous image of Q and the map $p \mapsto K_p$ is continuous from P into $\mathcal{K}(X)$. We shall bound the rank of $\bigcup_p K_p$ as a function $\psi(\mathrm{rank}(P), \mathrm{rank}(Q))$. After proving an upper bound, we shall show that, at least for countable P, Q, that bound is best possible. We shall also see that our bound is sometimes strictly larger than the "obvious" $\mathrm{rank}(P \times Q) = \mathrm{rank}(Q \times P)$. In fact our $\psi(\alpha, \beta)$ is not commutative; that is, $\psi(\alpha,\beta) \ne \psi(\beta,\alpha)$ for some α, β.

Actually, we shall obtain our upper bound under the weaker assumption that the map $p \mapsto K_p$ is only *weakly* continuous. This means that for all open $U \subseteq X$, the set $\{p : K_p \subseteq U\}$ is open.

The following simple example shows that weak continuity does not imply continuity:

Fix $W \subset P$, where W is open and not closed. Fix $a, b \in X$ with $a \ne b$. Consider the map $p \mapsto K_p \in [X]^{\le 2}$, where $p \in W \to K_p = \{a\}$ and $p \notin W \to K_p = \{a, b\}$.

This map is weakly continuous, since for any $U \subseteq X$, $\{p : K_p \subseteq U\}$ is either $\emptyset$ (if $a \notin U$) or W (if $a \in U$ and $b \notin U$) or P (if $a, b \in U$).

But it is not continuous: Say $a \in U$ and $b \in V$ and U, V are open and disjoint. Then $\{p : K_p \in N(U,V)\} = P \backslash W$, which is not open.

Lemma 9.4. *Assume that the map $p \mapsto K_p$ is weakly continuous from the compact space P into X, where X is any (Hausdorff) space. Let $H = \bigcup_{p \in P} K_p$. Then H is compact. Furthermore, if P and all the K_p are scattered, then H is scattered.*

Proof. To prove that H is compact: Let $\{U_\alpha : \alpha < \kappa\}$ be a family of open subsets of X closed under finite unions such that $H \subseteq \bigcup_\alpha U_\alpha$. We shall get a finite subcover. For each α, let $W_\alpha = \{p \in P : K_p \subseteq U_\alpha\}$; this is open by weak continuity. Also $\bigcup_{\alpha<\kappa} W_\alpha = P$ because each K_p is compact and the U_α are closed under finite unions, so $\forall p\ \exists\alpha\ K_p \subseteq U_\alpha$. Now, fix $F \in [\kappa]^{<\aleph_0}$ such that $\bigcup_{\alpha\in F} W_\alpha = P$. Then $H \subseteq \bigcup_{\alpha\in F} U_\alpha$ because for each p, there is an $\alpha \in F$ such that $p \in W_\alpha$ and hence $K_p \subseteq U_\alpha$.

Now, assume also that P is scattered and all K_p are scattered. If P is countable, then H is obviously scattered because then H is a compact union of countably many compact scattered subsets, so H cannot have a perfect subset.

In the general case, let $V[G]$ be a generic extension of the universe in which P becomes countable. In $V[G]$, P and all the K_p remain compact scattered, so H becomes scattered in $V[G]$. But that implies that H is scattered in V. $\square$

We now prove our upper bound.

Theorem 9.5. *To the assumptions of Lemma 9.4, add the assumption that* $\mathrm{rank}(P) \le \alpha$ *and all* $\mathrm{rank}(K_p) \le \beta$*. Then* $\mathrm{rank}(H) \le (\beta+1)\cdot\alpha+\beta$.

Proof. WLOG, X is zero-dimensional. This will simplify the argument. In fact, WLOG $X = H$, so WLOG X is compact and scattered.

We now induct on α.

For $\alpha = 0$: P is finite, and then $\mathrm{rank}(H) \le \beta$ because H is a finite union of sets K_p of rank $\le \beta$. To handle such finite unions: Note that for compact scattered A, B: $(A\cup B)' = A' \cup B'$, and then, by induction, $(A\cup B)^{(\xi)} = A^{(\xi)} \cup B^{(\xi)}$ for all ξ, which implies that $A \cup B$ is scattered and $\mathrm{rank}(A\cup B) = \max(\mathrm{rank}(A), \mathrm{rank}(B))$.

From now on, assume that $\alpha > 0$ and that the result holds for all $\alpha' < \alpha$.

Note that $0 < |P^{(\alpha)}| < \aleph_0$. Let $F = \bigcup\{K_p : p \in P^{(\alpha)}\}$. This is a finite union, so $\mathrm{rank}(F) \le \beta$. From now on, assume that $F \subsetneqq H$, since $F = H \to \mathrm{rank}(H) \le \beta$.

Now, consider any clopen $W \subseteq X$ with $W \supseteq F$ and $W \not\supseteq H$. Let $U_W = \{p \in P : K_p \subseteq W\}$. This is open in P by weak continuity. Also $P^{(\alpha)} \subseteq U_W \subsetneqq P$ (since $W \not\supseteq H$).

Let $P_W = P \backslash U_W$ and let $\alpha' = \mathrm{rank}(P_W)$. Then $\alpha' < \alpha$ because P_W is a non-empty closed subset of P disjoint from $P^{(\alpha)}$. Also, $H\backslash W \subseteq \bigcup_{p\in P_W} K_p$.

By the inductive hypothesis, $\mathrm{rank}(H\backslash W) \le (\beta+1)\cdot\alpha' + \beta$. So, $\mathrm{rank}(H\backslash W) < (\beta+1)\cdot\alpha$. Then, $H^{((\beta+1)\cdot\alpha)}\backslash W = (H\backslash W)^{((\beta+1)\cdot\alpha)} = \emptyset$ (since W is clopen), so that $H^{((\beta+1)\cdot\alpha)} \subseteq W$. Now, W was arbitrary and

X is zero-dimensional, so, intersecting all possible such sets W, we have $H^{((\beta+1)\cdot\alpha)} \subseteq F$.

We now have $H^{((\beta+1)\cdot\alpha+\beta)} \subseteq F^{(\beta)}$, which is finite because $\operatorname{rank}(F) \leq \beta$, so that $\operatorname{rank}(H) \leq (\beta+1)\cdot\alpha+\beta$. □

We shall show below that for countable α, β, this $\operatorname{rank}(H) \leq (\beta+1)\cdot\alpha+\beta$ is best possible. That is, we shall produce an example where $\operatorname{rank}(H) = (\beta+1)\cdot\alpha+\beta$. As pointed out above, WLOG $X = H$, so if there is an example at all, there is one with X compact scattered, but it will be convenient to start out with $X = \mathbb{R}^2$.

Let $\psi(\alpha,\beta) = (\beta+1)\cdot\alpha+\beta$. Note that ψ is not commutative, although $\psi(\alpha,\beta) = \psi(\beta,\alpha) = \alpha\beta+\alpha+\beta$ for finite α, β and $\psi(\alpha,0) = \psi(0,\alpha) = \alpha$ for all α. But $\psi(\omega,1) = \omega+1 \neq \omega+\omega = \psi(1,\omega)$. So, if $P = \omega^\omega+1$ (of rank ω) and $Q = \omega+1$ (of rank 1), then one cannot identify a P-limit of Qs with a Q-limit of Ps.

We shall now show that the upper bound $(\beta+1)\cdot\alpha+\beta$ derived in Theorem 9.5 is best possible when α, β are countable. And, we get a continuous map, not just weakly continuous. *Question.* What happens with uncountable α, β?

First, a preliminary lemma, relating ranks in subsets to ranks in the whole space:

Lemma 9.6. *Assume that X, Y are compact scattered and $X \subseteq Y$ and $\operatorname{rank}(x, Y) \geq \xi$ whenever $x \in X \backslash X'$ (i.e., x is isolated in X). Then $\operatorname{rank}(x, Y) \geq \xi + \operatorname{rank}(x, X)$ for all $x \in X$.*

Proof. We have $x \in Y^{(\xi)}$ for all $x \in X\backslash X'$. Then $X \subseteq Y^{(\xi)}$ because $X\backslash X'$ is dense in X. Then, $X^{(\eta)} \subseteq Y^{(\xi+\eta)}$ holds for all η by induction on η. In particular, if $\eta = \operatorname{rank}(x, X)$ then $x \in X^{(\eta)}$, so $x \in Y^{(\xi+\eta)}$, so $\operatorname{rank}(x, Y) \geq \xi + \eta$. □

Theorem 9.7. *Fix countable compact scattered spaces P, Q, and let $\alpha = \operatorname{rank}(P)$ and $\beta = \operatorname{rank}(Q)$. Then there is a continuous one-to-one map $p \mapsto K_p$ from P into $\mathcal{K}(\mathbb{R}^2)$ such that all $K_p \cong Q$ and $\operatorname{rank}(\bigcup_p K_p) = (\beta+1)\cdot\alpha+\beta$ and $p_1 \neq p_2 \to K_{p_1} \cap K_{p_2} = \emptyset$.*

Proof. Some preliminaries: Let $H = \bigcup_p K_p$. We already know that H is compact and scattered and $\operatorname{rank}(H) \leq (\beta+1)\cdot\alpha+\beta$ by Theorem 9.5, so it will be sufficient to get $\operatorname{rank}(H) \geq (\beta+1)\cdot\alpha+\beta$. We shall always assume that $\alpha \neq 0$ and $\beta \neq 0$, since if one of α, β is 0, then we can just let $H \cong P \times Q$. Note that if $\alpha = \beta = 1$ then $\operatorname{rank}(P \times Q) = 2$, and we need $\operatorname{rank}(H) = 3$, so we cannot let $H \cong P \times Q$.

Since P and Q are homeomorphic to successor ordinals, they can be embedded into $\mathbb{R}$, so we shall always assume that P is a well ordered subset of $\mathbb{R}$ such that P is compact in the standard real topology. We shall think of P as an ordinal, and use letters like μ, ν to range over P. Each K_μ will be a subset of $\{\mu\} \times \mathbb{R}$, so $H = \bigcup_{\mu \in P} K_\mu \subset \mathbb{R}^2$.

Let $\widetilde{Q} = Q^{(\beta)}$; this is the (finite) set of elements of Q of largest rank (i.e., β). Let $\widehat{Q} = Q \backslash Q'$; this is the set of isolated points of Q; then $|\widehat{Q}| = \aleph_0$ because $\beta \neq 0$. Once we have constructed K_μ, we shall let $\widetilde{K}_\mu = (K_\mu)^{(\beta)}$ and $\widehat{K}_\mu = K_\mu \backslash (K_\mu)'$.

We let $\lhd$ be a well-order of P in type ω, where the first element is $\min(P)$ and the second element is $\max(P)$. In the following, all statements about order will refer to the real order $<$, not $\lhd$, unless otherwise mentioned. But, we shall construct the K_μ by recursion on the order $\lhd$, not $<$. When explaining the recursion, we shall sometimes write K_μ as $\{\mu\} \times E_\mu$. We start by letting $E_{\min(P)} = E_{\max(P)} \cong Q$. Of course, we cannot let all the E_μ be the same, or we would just have $H \cong P \times Q$. Along with the K_μ, we shall fix a homeomorphism φ_μ from E_μ onto Q.

The *key idea* in the proof is: For each $\mu \in P \backslash P'$, each (isolated) point $(\mu, y) \in \widehat{K}_\mu$ will be a limit of a sequence of points from $\bigcup\{\widetilde{K}_\nu : \nu \in P \cap (-\infty, \mu)\}$. This will force (μ, y) to have high rank in H, even though it has rank 0 in K_μ.

Before we start the recursion, to aid in constructing these sequences: For each $\mu \in P'$ (so $\mathrm{rank}(\mu, P) \geq 1$) and $y \in \widehat{Q}$, choose a sequence $\langle \tau^n_{\mu,y} : n \in \omega \rangle$ with the following properties. We let $T_{\mu,y} = \{\tau^n_{\mu,y} : n \in \omega\}$:

1. All $\tau^n_{\mu,y} \in P \cap (\min(P), \mu)$ and $\langle \tau^n_{\mu,y} : n \in \omega \rangle$ is strictly $\nearrow$ and converges to μ.
2. $\sup_n(\mathrm{rank}(\tau^n_{\mu,y}, P) + 1) = \mathrm{rank}(\mu, P)$.
3. If $\mathrm{rank}(\mu, P)$ is a limit, then $\langle \mathrm{rank}(\tau^n_{\mu,y}, P) : n \in \omega \rangle$ is strictly increasing.
4. If $\mathrm{rank}(\mu, P)$ is a successor, then all $\mathrm{rank}(\tau^n_{\mu,y}, P) + 1 = \mathrm{rank}(\mu, P)$.
5. $T_{\mu,y} \cap T_{\nu,z} = \emptyset$ unless both $\mu = \nu$ and $y = z$.
6. $\mu \lhd \tau^n_{\mu,y}$ for all μ, y, n.

To get these $T_{\mu,y}$: First, ignore the y, and restate $(1 \cdots 6)$ replacing the "μ, y" by just "μ". Then, obtain the re-stated (1)(2)(3)(4). Here, we ignore the $\lhd$ and we work with each μ separately. Then, note that by (1), the sets T_μ are almost disjoint, so, since P is countable, we may discard finitely many elements from each T_μ and assume that the T_μ are pairwise disjoint. Likewise, for each μ, $\mu \lhd \tau^n_\mu$ must hold for all but finitely many n because $\lhd$ has type ω, so discarding finitely many more elements from each T_μ, we may assume that $\mu \lhd \tau^n_\mu$ holds for all μ, n. Finally, re-insert the y and obtain the sets $T_{\mu,y}$ by splitting each T_μ into countably many infinite subsets.

The point of (6) is that if we define a function f on P by recursion on $\lhd$, then we shall always come to $\tau^n_{\mu,y}$ *after* we have defined $f(\mu)$.

Once we have constructed K_μ and φ_μ, we shall use $\tau^n_{\mu,y}$ for $\tau^n_{\mu,\varphi_\mu(y)}$, where $y \in \widehat{E}_\mu$.

Re-stating the *key idea*: For each (isolated) $(\mu, y) \in \widehat{K}_\mu$ (where $\mu \in P'$), we have the $\tau^n_{\mu,y} \nearrow_n \mu$, and we shall also have points $(\tau^n_{\mu,y}, z^n_{\mu,y}) \in H$ with $(\tau^n_{\mu,y}, z^n_{\mu,y}) \in \widetilde{K}_{\tau^n_{\mu,y}}$ (largest rank points) such that $(\tau^n_{\mu,y}, z^n_{\mu,y}) \to_n (\mu, y)$. Using these, we shall prove below:

$$(*) \;\; \forall y \in E_\mu \; [\mathrm{rank}((\mu, y), H) \geq (\beta + 1) \cdot \mathrm{rank}(\mu, P) + \mathrm{rank}((\mu, y), K_\mu)] \quad .$$

Once we have proved $(*)$, we are done: Fix μ with $\mathrm{rank}(\mu, P) = \alpha$ and then fix y with $\mathrm{rank}((\mu, y), K_\mu) = \beta$. By $(*)$, H has a point of rank $\geq (\beta + 1) \cdot \alpha + \beta$, so that $\mathrm{rank}(H) \geq (\beta + 1) \cdot \alpha + \beta$.

More on the *key idea*: As before, d is the Hausdorff metric on $\mathcal{K}(\mathbb{R}^2)$, induced by the standard Euclidean metric on $\mathbb{R}^2$. We shall get:

7. $\mu \neq \nu \to |\mu - \nu| \leq d(K_\mu, K_\nu) < 2|\mu - \nu|$. This implies that the map $\mu \mapsto K_\mu$ is continuous. Note that the $\leq$ is obvious, but the $<$ will require some work.
8. $d((\tau^n_{\mu,y}, z^n_{\mu,y}) \,,\, (\mu, y)) < 2|\mu - \tau^n_{\mu,y}|$. This implies that $(\tau^n_{\mu,y}, z^n_{\mu,y}) \to_n (\mu, y)$.

We shall explain below how to get (7)(8), but briefly: to ensure (8) at stage ν of the recursive construction: If $\nu = \tau^n_{\mu,y}$ for some n, μ, y, then these n, μ, y are unique, and we ensure this instance of (8). If not, then (8) says nothing for this ν.

Assuming (7)(8), we prove $(*)$ by induction on $\mathrm{rank}(\mu, P)$. If $\mathrm{rank}(\mu, P) = 0$, then μ is isolated in P, so $\mathrm{rank}((\mu, y), H) = \mathrm{rank}((\mu, y), K_\mu)$.

Now, assume that $\mathrm{rank}(\mu, P) > 0$. By Lemma 9.6, with X, Y, x, ξ replaced by $K_\mu, H, (\mu, y), (\beta + 1) \cdot \mathrm{rank}(\mu, P)$, it is sufficient to prove $(*)$ when $(\mu, y) \in \widehat{K}_\mu$ (that is, (μ, y) is isolated in K_μ). Then, we need to prove, for $(\mu, y) \in \widehat{K}_\mu$:

$$\mathrm{rank}((\mu, y), H) \geq (\beta + 1) \cdot \mathrm{rank}(\mu, P) \qquad \text{i.e.,} \qquad (\mu, y) \in H^{((\beta+1)\cdot\mathrm{rank}(\mu,P))} \quad .$$

Now we have $(\tau^n_{\mu,y}, z^n_{\mu,y}) \in H$ with $(\tau^n_{\mu,y}, z^n_{\mu,y}) \in \widetilde{K}_{\tau^n_{\mu,y}}$ (largest rank points) such that $(\tau^n_{\mu,y}, z^n_{\mu,y}) \to_n (\mu, y)$. Since $\mathrm{rank}(\tau^n_{\mu,y}, P) < \mathrm{rank}(\mu, P)$, we may inductively apply $(*)$ to $(\tau^n_{\mu,y}, z^n_{\mu,y})$. Using the fact that $\mathrm{rank}((\tau^n_{\mu,y}, z^n_{\mu,y}), K_{\tau^n_{\mu,y}}) = \beta$, $(*)$ gives us:

$$\mathrm{rank}((\tau^n_{\mu,y}, z^n_{\mu,y}), H) \geq (\beta + 1) \cdot \mathrm{rank}(\tau^n_{\mu,y}, P) + \beta \quad .$$

There are now two cases, referring to (3)(4) above:

If $\mathrm{rank}(\mu, P)$ is a limit: Then $\langle \mathrm{rank}(\tau^n_{\mu,y}, P) : n \in \omega\rangle$ is strictly increasing, converging to $\mathrm{rank}(\mu, P)$. Then, just using $\mathrm{rank}((\tau^n_{\mu,y}, z^n_{\mu,y}), H) \geq (\beta+1)\cdot \mathrm{rank}(\tau^n_{\mu,y}, P)$, we get $\mathrm{rank}((\mu, y), H) \geq (\beta+1)\cdot \mathrm{rank}(\mu, P)$.

If $\mathrm{rank}(\mu, P) = \delta + 1$: Then all $\mathrm{rank}(\tau^n_{\mu,y}, P) = \delta$, so $(\tau^n_{\mu,y}, z^n_{\mu,y}) \in H^{((\beta+1)\cdot\delta+\beta)}$ for each n. Since $(\tau^n_{\mu,y}, z^n_{\mu,y}) \to_n (\mu, y)$, we have $(\mu, y) \in H^{((\beta+1)\cdot\delta+\beta+1)}$, so that $\mathrm{rank}((\mu, y), H) \geq (\beta+1)\cdot\delta + \beta + 1 = (\beta+1)\cdot(\delta+1) = (\beta+1)\cdot \mathrm{rank}(\mu, P)$.

We are now done if we can show how to get (7)(8) to hold. To do this, we construct the $K_\mu = \{\mu\} \times E_\mu$ in ω steps by recursion on the order $\lhd$. As indicated above, we let $E_{\min(P)} = E_{\max(P)} \cong Q$. So far, with $\mu, \nu \in \{\min(P), \max(P)\}$, (7)(8) are obvious.

Now, fix $\rho \in P \setminus \{\min(P), \max(P)\}$, and assume that we have constructed K_π for all $\pi \lhd \rho$, and we describe how to construct K_ρ. Let μ be the $<$ largest element of $\{\pi : \pi \lhd \rho \ \&\ \pi < \rho\}$. Let ν be the $<$ least element of $\{\pi : \pi \lhd \rho \ \&\ \rho < \pi\}$. So, $\mu < \rho < \nu$. Now we wish to make sure that all the instances of (7)(8) that involve ρ and the points $\lhd\ \rho$ hold. Since we have just fixed μ, ν, we restate (7)(8) with ξ, η:

7. $\xi \neq \eta \to |\xi - \eta| \leq d(K_\xi, K_\eta) < 2|\xi - \eta|$.
8. $d(\,(\tau^n_{\xi,y}, z^n_{\xi,y})\,,\,(\xi, y)) < 2|\xi - \tau^n_{\xi,y}|$.

Now (7) is symmetric in ξ, η. If we let $\xi = \rho$, then note that, since we are assuming inductively that (7) holds below (w.r.t. $\lhd$) ρ, we get (7) for all values of $\eta \lhd \rho$ if we just get if for $\eta = \mu$ and $\eta = \nu$. Regarding (8), note that $\xi \lhd \tau^n_{\xi,y}$ by (6), so if we let ξ be ρ, then (8) involves a point that is not $\lhd\ \rho$. So, (8) is only relevant when $\rho = \tau^n_{\xi,y}$ for some $\xi \lhd \rho$ (so $\rho < \xi$) and $y \in \widehat{K}_\xi$ and $n \in \omega$. Then, by (5), this triple ξ, y, n is unique, so there is only one instance of (8) to consider. If there is no such triple, then (8) is vacuous here. Our requirements are now:

7. $d(K_\rho, K_\mu) < 2|\rho - \mu|$ and $d(K_\nu, K_\rho) < 2|\nu - \rho|$.
8. $\rho = \tau^n_{\xi,y} \to d(\,(\rho, z^n_{\xi,y})\,,\,(\xi, y)) < 2|\xi - \rho|$.

We shall show next that for a suitably small $\varepsilon > 0$, the following procedure will construct K_ρ satisfying (7)(8):

First, fix finite non-empty $A \subset K_\mu$ and $B \subset K_\nu$ such that $d(A, K_\mu) < \varepsilon$ and $d(B, K_\nu) < \varepsilon$. Let $\theta = (\rho - \mu)/(\nu - \mu)$; so $1 - \theta = (\nu - \rho)/(\nu - \mu)$. Then, let $C = \mathcal{I}(A, B, \theta)$ (applying Definition 9.2 with $L = \mathbb{R}^2$). Then $d(A, C) = \theta \cdot d(A, B)$ and $d(C, B) = (1 - \theta)\cdot d(A, B)$, and $C \subset \{\rho\} \times \mathbb{R}$ (see Lemma 9.3 and the following remarks).

Then, *if* $\rho = \tau^n_{\xi,y}$, then that n, ξ, y is unique; choose $z^n_{\xi,y}$ so that $(\rho, z^n_{\xi,y}) \in C$ and $d((\rho, z^n_{\xi,y})\,,\,(\xi, y))$ is the least element of $\{d((\rho, z)\,,\,(\xi, y)) : (\rho, z) \in C\}$. If ρ is not equal to any $\tau^n_{\xi,y}$, then $(\rho, z^n_{\xi,y})$ can be an arbitrary element of C.

We shall choose $K_\rho \subset \{\rho\} \times \mathbb{R}$ such that $K_\rho \cong Q$ and $K_\rho \supset C$ and $d(K_\rho, C) < \varepsilon$ and $\mathrm{rank}((\rho, z^n_{\xi,y}), K_\rho) = \mathrm{rank}(K_\rho) = \mathrm{rank}(Q) = \beta$.

Define δ by: $d(K_\mu, K_\nu) = 2(\nu - \mu) - \delta$. So, $\delta > 0$ and δ is "small". Then $d(A, B) < 2(\nu - \mu) - \delta + 2\varepsilon$. We now have:

$$\begin{aligned} d(A,C) &= \theta d(A,B) \\ &< 2(\rho-\mu) - \theta(\delta - 2\varepsilon) \\ &< 2(\rho-\mu) - \theta\delta + 2\varepsilon \\ d(C,B) &= (1-\theta)d(A,B) \\ &< 2(\nu-\rho) - (1-\theta)(\delta-2\varepsilon) \\ &< 2(\nu-\rho) - (1-\theta)\delta + 2\varepsilon \end{aligned} \quad .$$

Then, if $\varepsilon < \min(\theta\delta/4, (1-\theta)\delta/4)$, choosing K_ρ so that $d(K_\rho, C) < \varepsilon$ gives us the following, ensuring (7):

$$\begin{aligned} d(K_\mu, K_\rho) &< 2(\rho-\mu) - \theta\delta + 4\varepsilon < 2(\rho-\mu) \\ d(K_\rho, K_\nu) &< 2(\nu-\rho) - (1-\theta)\delta + 4\varepsilon < 2(\nu-\rho) \end{aligned} \quad .$$

For (8), assume that $\rho = \tau^n_{\xi,y}$; otherwise (8) is vacuous. Then $\xi \lhd \rho$ and $\nu \le \xi$ and

$$\begin{gathered} d(C, K_\xi) \le d(C,B) + d(B, K_\nu) + d(K_\nu, K_\xi) < \\ 2(\nu-\rho) - (1-\theta)\delta + 2\varepsilon + \varepsilon + 2(\xi-\nu) = 2(\xi-\rho) - (1-\theta)\delta + 3\varepsilon \end{gathered} \quad .$$

Now $(\xi, y) \in K_\xi$, so choose $z^n_{\xi,y}$ so that $(\rho, z^n_{\xi,y}) \in C$ and

$$d((\rho, z^n_{\xi,y}), (\xi, y)) < 2(\xi - \rho) - (1-\theta)\delta + 3\varepsilon \quad .$$

Finally, choose K_ρ as above so that *also* $C \subset K_\rho$ and $(\rho, z^n_{\xi,y}) \in \widetilde{K}_\rho$. This yields (8) as long as we have $3\varepsilon < (1-\theta)\delta$. $\square$

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