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DEGREES OF MAPS BETWEEN ISOTROPIC GRASSMANN MANIFOLDS

SAMIK BASU AND SWAGATA SARKAR

Abstract. Let $\widetilde{I}_{2n,k}$ denote the space of k-dimensional, oriented isotropic subspaces of $\mathbb{R}^{2n}$, called the oriented isotropic Grassmannian. Let $f\colon \widetilde{I}_{2n,k} \to \widetilde{I}_{2m,l}$ be a map between two oriented isotropic Grassmannians of the same dimension, where $k, l \geq 2$. We show that either $(n,k) = (m,l)$ or $\deg f = 0$. Let $\mathbb{R}\widetilde{G}_{m,l}$ denote the oriented real Grassmann manifold. For $k, l \geq 2$ and $\dim \widetilde{I}_{2n,k} = \dim \mathbb{R}\widetilde{G}_{m,l}$, we also show that the degree of maps $g\colon \mathbb{R}\widetilde{G}_{m,l} \to \widetilde{I}_{2n,k}$ and $h\colon \widetilde{I}_{2n,k} \to \mathbb{R}\widetilde{G}_{m,l}$ must be zero.

1. Introduction

It has been proved in [5] that maps between two different oriented real Grassmann manifolds of the same dimension cannot have non-zero degree, provided the target space is not a sphere. A similar result is obtained for complex Grassmann manifolds in [4], when the map is a morphism of projective varieties. For arbitrary maps, this result has been verified for the complex Grassmann manifolds for many cases in [5] and [6].

In this paper we consider the analogous question for the space $\widetilde{I}_{2n,k}$ of oriented k-dimensional isotropic subspaces of a symplectic vector space of dimension $2n$. The oriented isotropic Grassmannian was considered in [3] and its cohomology was computed with real coefficients. Their method involves identifying $\widetilde{I}_{2n,k}$ as a homogeneous space $\widetilde{I}_{2n,k} \simeq U(n)/(SO(k) \times U(n-k))$. One may similarly consider $I_{2n,k}$, the isotropic Grassmannian of k-dimensional isotropic subspaces of a symplectic $2n$ dimensional vector

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space, which is $\simeq U(n)/(O(k)\times U(n-k))$. It turns out that the isotropic Grassmannian is orientable if and only if k is odd ([1]). In this paper we consider maps between oriented isotropic Grassmannians of the same dimension and prove (see Theorem 3.1)

Theorem 1.1. *Let n,k,m,l be integers such that $2 \leq l \leq m$ and $\dim \widetilde{I}_{2n,k} = \dim \widetilde{I}_{2m,l}$. Let $f\colon \widetilde{I}_{2n,k} \to \widetilde{I}_{2m,l}$. Then either $(n,k)=(m,l)$ or $\deg f = 0$.*

Note that $\widetilde{I}_{2n,1} \simeq S^{2n-1}$ and so it is possible to get maps of arbitrary, non-zero degree, $\phi : \widetilde{I}_{2n,k} \to \widetilde{I}_{2m,1}$ whenever $\dim(\widetilde{I}_{2n,k}) = 2m-1$. We also prove

Theorem 1.2. *Consider maps $h\colon \widetilde{I}_{2n,k} \to \mathbb{R}\widetilde{G}_{m,l}$ and $g\colon \mathbb{R}\widetilde{G}_{m,l} \to \widetilde{I}_{2n,k}$, where $2 \leq l \leq \frac{m}{2}$, $2 \leq k \leq n$ and $\dim \widetilde{I}_{2n,k} = \dim \mathbb{R}\widetilde{G}_{m,l}$. Then $\deg g = \deg h = 0$.*

The main technique used to prove the statements above is the classical result that if $f : X \to Y$ (with $\dim X = \dim Y$) is a map of non-zero degree, then $f^* : H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ is a monomorphism. We rewrite the known cohomology ring of $\widetilde{I}_{2n,k}$ in a convenient form to deduce the above theorems.

The paper is organised as follows. In section 2 we recall the description of the isotropic Grassmannians and compute their cohomology. In section 3, we prove the main theorems.

2. Isotropic Grassmannian

In this section we set up the relevant notation and describe the isotropic Grassmannian as a homogeneous space. Most of the ideas are discussed in [2] and [3]. Let $U(n) := U(n;\mathbb{C})$ denote the group of unitary linear transformations $\mathbb{C}^n \to \mathbb{C}^n$, and $O(n) := O(n;\mathbb{R})$ denote the group of orthogonal linear transformations $\mathbb{R}^n \to \mathbb{R}^n$.

Endow $\mathbb{R}^{2n}$ with the standard symplectic form and define the isotropic Grassmannian, $I_{2n,k}$, to be the space of k-dimensional isotropic vector subspaces of $\mathbb{R}^{2n}$. Then one may identify this as the homogeneous space $U(n)/(O(k)\times U(n-k))$ as in [1]. We recall this description briefly. The group $U(n)$ acts on $\mathbb{R}^{2n}$ via symplectic morphisms, so this induces an $U(n)$ action on $I_{2n,k}$ which may be checked to be transitive. Let V_k denote the k-dimensional isotropic subspace generated by the basis vectors $e_1,\cdots,e_k$. Any isometry $A \in U(n)$ with $A(V_k) = V_k$, is the complexification of an orthogonal matrix on the subspace $V_k + \imath V_k$. Thus such an isometry splits as a sum of isometries $V_k \oplus \imath V_k \to V_k \oplus \imath V_k$ and $(V_k \oplus \imath V_k)^\perp \to (V_k \oplus \imath V_k)^\perp$. Therefore, the stabilizer of V_k is $O(k)\times U(n-k)$.

From the above identification of $I_{2n,k}$ as a homogeneous space, it follows that $\dim I_{2n,k} = 2k(n-k) + \frac{k(k+1)}{2}$. In fact, for $k = 1$, $I_{2n,k}$ is the real projective space, $\mathbb{RP}^{2n-1}$. One notes that $I_{2n,k}$ is orientable if and only if k is odd ([1]).

We consider $\widetilde{I}_{2n,k}$, the space of k-dimensional, oriented, isotropic subspaces of $\mathbb{R}^{2n}$, called the oriented isotropic Grassmannian. We may analogously deduce that $\widetilde{I}_{2n,k} \cong U(n)/(SO(k) \times U(n-k))$ and $\dim \widetilde{I}_{2n,k} = 2k(n-k) + \frac{k(k+1)}{2}$.

Next we turn to the cohomology of the oriented isotropic Grassmannian. The cohomology with $\mathbb{R}$ coefficients was computed in [3] using formulas for the real cohomology of homogeneous spaces. We compute the same algebraically, fixing appropriate notation along the way. We use the Serre spectral sequence and the following fibrations:

$$\widetilde{I}_{2n,k} \longrightarrow BSO(k) \times BU(n-k) \longrightarrow BU(n)$$

$$U(n) \longrightarrow \widetilde{I}_{2n,k} \longrightarrow BSO(k) \times BU(n-k)$$

One notes that the cohomology expressions (Proposition 2.2) may be deduced directly from the computations in [3]. This approach identifies a subring of the cohomology as the cohomology of certain real and complex Grassmannians. This fact is exploited again in Section 4.

The first fibration induces the Serre spectral sequence with $E_2^{p,q}$ term given by

$$E_2^{p,q} = H^p(BU(n);\mathbb{Q}) \otimes H^q(\widetilde{I}_{2n,k};\mathbb{Q}) \tag{2.1}$$

which converges to $H^{p+q}(BSO(k) \times BU(n-k);\mathbb{Q})$.

The second fibration induces the Serre spectral sequence with $E_2^{p,q}$ term given by

$$E_2^{p,q} = H^q(U(n);\mathbb{Q}) \otimes H^p(BSO(k) \otimes BU(n-k);\mathbb{Q}) \tag{2.2}$$

which converges to $H^{p+q}(\widetilde{I}_{2n,k};\mathbb{Q})$.

The following expressions are well-known ([2])

$$H^*(BU(n-k);\mathbb{Q}) \cong \mathbb{Q}[c_1, c_2, \cdots, c_{n-k}],$$

$$H^*(U(n);\mathbb{Q}) \cong \wedge_{\mathbb{Q}}[x_1, x_3, \cdots, x_{2n-1}],$$

$$H^*(BSO(2m+1);\mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \cdots, p_m]$$

$$H^*(BSO(2m);\mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \cdots, p_{m-1}, e_{2m}]$$

where $c_i \in H^{2i}BU(n)$ stand for universal Chern classes of the canonical bundle γ_n, $p_i \in H^{4i}BSO(n)$ for the Pontrjagin classes of the canonical bundle ξ_k, and $e_k \in H^k(BSO(k);\mathbb{Q})$ for the Euler class. The classes x_i lie in grading $2i-1$.

Note that the inclusion $SO(k)\times U(n-k)\subset U(n)$ is induces on classifying spaces $BSO(k)\times BU(n-k)\to BU(n)$ which classifies the complex bundle $\xi_k\otimes\mathbb{C}\oplus\gamma_{n-k}$. This leads to a commutative diagram of fibrations

$$\begin{tikzcd} U(n) \arrow[r] \arrow[d, equal] & \widetilde{I}_{2n,k} \arrow[r] \arrow[d] & BSO(k)\times BU(n-k) \arrow[d, "(\xi_k\otimes\mathbb{C})\oplus\gamma_{n-k}"] \\ U(n) \arrow[r] & EU(n) \arrow[r] & BU(n) \end{tikzcd}$$

which gives a diagram of spectral sequences. Let λ^* denote the homomorphism from the spectral sequence $\wedge_{\mathbb{Q}}(x_1,x_3\cdots)\otimes\mathbb{Q}[c_1,c_2,\cdots] \implies H^*(pt;\mathbb{Q})$ to 2.2. As the classes x_{2i-1} are transgressive with $d_{2i}(x_{2i-1}) = c_i$, so are the classes $\lambda^*(x_{2i-1})$. Therefore we obtain

$$\begin{aligned} d(x_{2i-1}) &= \lambda^*(c_i) \\ &= c_i((\xi_k\otimes\mathbb{C})\oplus\gamma_{n-k}) \\ &= \textstyle\sum_{j=0}^{\infty} c_j(\xi_k\otimes\mathbb{C})c_{i-j}(\gamma_{n-k}) \\ &= \textstyle\sum_{j=0}^{[i/2]} p_j c_{i-2j} \\ &= c_i + \textstyle\sum_{j=1}^{[i/2]} p_j c_{i-2j} \end{aligned} \tag{2.3}$$

Proposition 2.1. *Let $2 < k \le n$. If $k < n$, the cohomology groups $H^i(\widetilde{I}_{2n,k})$ are 0 if $i \le 3$ and $H^4(\widetilde{I}_{2n,k})$ is generated by p_1. In the case $k = n$, the cohomology group $H^1(\widetilde{I}_{2n,n})$ is isomorphic to $\mathbb{Z}$ and $H^4(\widetilde{I}_{2n,n})$ is zero.*

Proof. For $k = n$, the space $\widetilde{I}_{2n,k}\cong U(n)/SO(n)$, thus the fundamental group and hence H^1 is $\cong\mathbb{Z}$. Otherwise in the spectral sequence 2.2 one has a class c_1 in $E_2^{2,0}$. In degrees ≤ 3 the spectral sequence 2.2 is $\cong \wedge(x_1,x_3)\otimes\mathbb{Q}[c_1]$ and from 2.3 we get that $d_2(x_1) = c_1$. Hence the only possible class in $H^{*\le 3}$ is x_3.

Note that 2.3 also gives $d_4(x_3) = c_2 + p_1$ if $k \ge 2$ and $d_4(x_3) = c_2$ if $k = 1$. Thus we conclude that $H^{*\le 3}$ is 0 if $k < n$, and if in addition $n > k \ge 2$ then $H^4(\widetilde{I}_{2n,k})(\cong\mathbb{Q})$ generated by p_1. In the case $k = n$, $d_4(x_3) = p_1$, and hence $H^4(\widetilde{I}_{2n,k})$ becomes zero. □

We may compute further in the spectral sequence 2.2. Notice that the formula 2.3 is of the form $d(x_{2i-1}) = c_i + \cdots$ and so the class c_i is not zero if $i \le n-k$. Thus the elements $d(x_1), d(x_3)\dots d(x_{2(n-k)-1})$ form a regular sequence in $E_2^{*,0}$. It follows that no multiple of x_{2j-1}, for $j \le n-k$, can be a permanent cycle. Therefore any positive degree classes surviving to the E_∞-page must have degree $> 2(n-k)+1$. In fact we have the Proposition:

Proposition 2.2. *Suppose $2 \leq k < n$. The cohomology algebra $H^*(\widetilde{I}_{2n,k};\mathbb{Q})$ has algebra generators $p_1, \cdots, p_m$ in degrees $4, 8, \cdots$ when $k = 2m+1$ is odd. If $k = 2m$ there is an additional generator e_m in degree $2m$ that satisfies $e_m^2 = p_m$. Other algebra generators are in degrees $\geq 2(n-k)+1$.*

Proof. In view of the discussion above it suffices to prove the first two statements for the horizontal 0-line $E_\infty^{*,0}$. Note that for j odd, the equation 2.3 gives

$$d(x_{2j-1}) = c_j + \sum_{l=1}^{[j/2]} p_l c_{j-2l}$$

Inductively we conclude that $d_{2j}(x_{2j-1}) = c_j$. We have $d_2(x_1) = c_1$ and thus c_1 is 0 in the E_3-page. Inductively c_{j-1} is 0 in the group $E_{2j-3}^{2j-4,0}$. Hence, the equation above implies $d_{2j}(x_{2j-1}) = c_j$ as the other $c_{odd \leq j-1}$ are 0 in E_{2j}. It follows that c_j is 0 in E_{2j+1}.

The remaining classes in the horizontal 0-line are p_i , c_{2j} and e_m if k is even. The remaining differentials are generated by

$$d(x_{4j-1}) = \sum_{l=0}^{j} p_l c_{2j-2l}$$

Hence the horizontal 0-line is the graded algebra $A(n,k)$ below. The Proposition now follows from Lemma 2.3 and [5]. $\square$

Define the graded algebras $A(n,k)$ for $n > k$ as (with notations as above)

$$A(n,k) = \begin{cases} \frac{\mathbb{Q}[p_1,\cdots,p_m,c_2,c_4,\cdots,c_{n-2m-2}]}{(d(x_3),\ldots,d(x_{2n-1}))} & \text{if } n \text{ is even, } k = 2m+1 \\ \frac{\mathbb{Q}[p_1,\cdots,p_m,c_2,c_4,\cdots,c_{n-2m-1}]}{(d(x_3),\ldots,d(x_{2n-3}))} & \text{if } n \text{ is odd, } k = 2m+1 \\ \frac{\mathbb{Q}[p_1,\cdots,p_{m-1},e_m,c_2,c_4,\cdots,c_{n-2m}]}{(d(x_3),\ldots,d(x_{2n-1}))} & \text{if } n \text{ is even, } k = 2m \\ \frac{\mathbb{Q}[p_1,\cdots,p_{m-1},e_m,c_2,c_4,\cdots,c_{n-2m-2}]}{(d(x_3),\ldots,d(x_{2n-3}))} & \text{if } n \text{ is odd, } k = 2m \end{cases}$$

Let $\mathbb{R}\widetilde{G}_{n,k}$ denote the oriented real Grassmannian of all k-dimensional oriented subspaces of $\mathbb{R}^n$. As a space this is $\simeq SO(n)/SO(k) \times SO(n-k)$. Let $\mathbb{C}G_{n,k}$ denote the Grassmannian of k-planes in $\mathbb{C}^n$. We readily observe the following from known expressions about the cohomology of Grassmannians.

Lemma 2.3. *There are isomorphisms of graded algebras*
a) $A(2s, 2m+1) \cong H^{*/2}(\mathbb{C}G_{s-1,m};\mathbb{Q})$
b) $A(2s+1, 2m+1) \cong H^{*/2}(\mathbb{C}G_{s,m};\mathbb{Q})$
c) $A(2s, 2m) \cong H^*(\mathbb{R}\widetilde{G}_{2s+1,2m};\mathbb{Q})$
d) $A(2s+1, 2m) \cong H^*(\mathbb{R}\widetilde{G}_{2s+1,2m};\mathbb{Q})$

Remark 2.4. One may compare this to the expression obtained in [3]. Observe that the ring of characteristic classes $\mathcal{A}$ of the principal bundle $U(n) \to \widetilde{I}_{2n,k}$ matches the graded algebra $A(n,k)$ above. Note that the cohomology of $\widetilde{I}_{2n,k}$ is $\mathcal{A} \otimes \Lambda$ where Λ is an exterior algebra on classes in degrees $d \in S_{n,k}$ ([3], Theorem 1.7) with

$$S_{n,k} = \begin{cases} \{4[\frac{n-k+1}{2}]+1, 4[\frac{n-k+1}{2}]+3, \cdots, 2n-3\} & \text{if both } n \text{ and } k \text{ are even} \\ \{4[\frac{n-k+1}{2}]+1, 4[\frac{n-k+1}{2}]+3, \cdots, 2n-1\} & \text{otherwise} \end{cases}$$

It follows that the Poincaré polynomial of $\widetilde{I}_{2n,k}$ is given by

$$p_{\widetilde{I}_{2n,k}}(x) = p_{A(n,k)}(x)\Pi_{d \in S_{n,k}}(1+x^d)$$

which may be computed from the known formulas for complex and real Grassmannians.

Recall that height of a nilpotent element, x, in an algebra, is defined to be the least positive integer n, such that $x^n \neq 0$ but $x^{n+1} = 0$.

We have identified that $A(n,k)$ is a subalgebra of $H^*(\widetilde{I}_{2n,k})$, and the Lemma 2.3 implies that $A(n,k)$ is sometimes the cohomology of complex Grassmannians and sometimes of real Grassmannians. The cohomology ring of Grassmannians are well studied objects and the literature contains a lot of results about them. We may use this to deduce the height of the elements p_1 in $A(n,k)$. In the cases where we encounter the complex Grassmannian, this simply follows from the fact that c_1 is the cohomology class of a Kahler form and so its height is half the dimension. For the other cases we refer to the Proof of Theorem 1 of [5]. Thus we obtain the following result.

Proposition 2.5. *The height of the element, p_1 in $H^*(\widetilde{I}_{2n,k})$ is $t(s-t)$ for $(n,k) \in \{(2s+2, 2t+1), (2s+1, 2t+1), (2s, 2t), (2s+1, 2t)\}$.*

3. Main results

In this section we consider the question of possible Brouwer degrees of maps $f : \widetilde{I}_{2n,k} \to \widetilde{I}_{2m,l}$, where $\widetilde{I}_{2n,k}$ and $\widetilde{I}_{2m,l}$ are oriented isotropic Grassmannians, such that $\dim \widetilde{I}_{2n,k} = \dim \widetilde{I}_{2m,l}$.

Note that when $l = 1$, the space $\widetilde{I}_{2m,l} \simeq U(m)/U(m-1) \simeq S^{2m-1}$. Also note that $\dim \widetilde{I}_{2n,k}$ $(= 2k(n-k) + \frac{k(k+1)}{2})$ is odd if and only if $k \equiv 1, 2\ (mod\ 4)$. In these cases ($\dim \widetilde{I}_{2n,k} = 2m-1$) given any $\lambda \in \mathbb{Z}$, there exists a map $f_\lambda : \widetilde{I}_{2n,k} \to S^{2m-1}$ with $\deg f_\lambda = \lambda$. We prove that these are the only possible cases of non zero degree.

Theorem 3.1. *Let n, k, m, l be integers such that $2 \leq k \leq n$ and $2 \leq l \leq m$ and $\dim \widetilde{I}_{2n,k} = \dim \widetilde{I}_{2m,l}$. Let $f : \widetilde{I}_{2n,k} \to \widetilde{I}_{2m,l}$. Then either $(n,k) = (m,l)$ or $\deg f = 0$.*

Proof. Suppose $n = k$ and $m = l$. Then $\dim \widetilde{I}_{2n,k} = \dim \widetilde{I}_{2m,l}$ implies $n(n+1)/2 = m(m+1)/2$ and it follows that $n = m$. Note that the space $\widetilde{I}_{4,2} \simeq U(2)/SO(2)$ is an oriented manifold of dimension 3. Observe that $k \geq 2$ and $n \neq k$ implies $\dim \widetilde{I}_{2n,k} \geq 4$. Hence $\dim \widetilde{I}_{2n,k} = \dim \widetilde{I}_{2m,l}$ implies $(n,k) = (m,l)$ if $n = k, m = l$ or one of (n,k) or (m,l) equals $(2,2)$.

Consider the case where $n = k$, $n > 2$ and $m \neq l$. We use the fact that if $\deg f \neq 0$ then f^* is a monomorphism on cohomology with rational coefficients. By Proposition 2.1, $H^4(\widetilde{I}_{2n,n}) = 0$ and $H^4(\widetilde{I}_{2m,l}) \neq 0$. Hence $\deg f = 0$. In the case $n \neq k$, $m = l$ and $m > 2$, we have, again by Proposition 2.1, that $H^1(\widetilde{I}_{2m,m}) \cong \mathbb{Z}$ and $H^1(\widetilde{I}_{2n,k}) = 0$. Therefore $\deg f = 0$.

Now we proceed to the more general case $2 \leq k < n$ and $2 \leq l < m$. Consider $f^*\colon H^*(\widetilde{I}_{2m,l};\mathbb{Q}) \to H^*(\widetilde{I}_{2n,k};\mathbb{Q})$. Since p_1 is the generator of $H^4(\widetilde{I}_{2m,l};\mathbb{Q})$ we must have (denote by $p_1(m,l)$ the class $p_1 \in H^4(\widetilde{I}_{2m,l};\mathbb{Q})$)

$$f^*p_1(m,l) = \lambda p_1(n,k)$$

By Proposition 2.5 the height of $p_1(n,k)$ is $[k/2][(n-k)/2]$ and the height of $p_1(m,l)$ is $[l/2][(m-l)/2]$.[1] (Here, $[t]$ denotes the integral part of t.)

If $\deg f \neq 0$ then f^* is a monomorphism and so $\lambda \neq 0$. Moreover $p_1(m,l)^{[\frac{l}{2}][\frac{m-l}{2}]} \neq 0$ implies

$$f^*p_1(m,l)^{[\frac{l}{2}][\frac{m-l}{2}]} = \lambda^{[\frac{l}{2}][\frac{m-l}{2}]} p_1(n,k)^{[\frac{l}{2}][\frac{m-l}{2}]} \neq 0$$
$$\Longrightarrow \left[\frac{l}{2}\right]\left[\frac{m-l}{2}\right] \leq \left[\frac{k}{2}\right]\left[\frac{n-k}{2}\right].$$

Since f^* is a ring homomorphism, we have

$$0 = f^*p_1(m,l)^{[\frac{l}{2}][\frac{m-l}{2}]+1} = \lambda^{[\frac{l}{2}][\frac{m-l}{2}]+1} p_1(n,k)^{[\frac{l}{2}][\frac{m-l}{2}]+1}$$
$$\Longrightarrow \left[\frac{k}{2}\right]\left[\frac{n-k}{2}\right] \leq \left[\frac{l}{2}\right]\left[\frac{m-l}{2}\right]$$

Therefore $[\frac{k}{2}][\frac{n-k}{2}] = [\frac{l}{2}][\frac{m-l}{2}]$. Together with the equation $2k(n-k) + \frac{k(k+1)}{2} = 2l(m-l) + \frac{l(l+1)}{2}$ we prove that it leads to a contradiction. Assume that $k \leq l$ (there is no loss of generality in doing this, as k and l are symmetric in the two equations above.) The first equality implies $l(m-l) - 4 \leq k(n-k) \leq l(m-l) + 4$. Rearranging terms we obtain $-16 \leq (l-k)(k+l+1) \leq 16$.

---

[1]This argument about the height of elements has been used in [5] while analyzing the analogous Theorem for Grassmannians.

As both $k \geq 2$ and $l \geq 2$ we have $k+l+1 \geq 5$ and so the above inequality can hold only when $l-k = 0,1,2$. Observe that $k = l$ implies $n = m$ so that $(n,k) = (m,l)$. Note also that $(l-k)(k+l+1)$ must also be divisible by 4 being equal to $4(l(m-l)-k(n-k))$ in the second equation above. If $l = k+2$, we have $(l-k)(k+l+1) = 2(2k+3)$ is not divisible by 4. If $l = k+1$ we have $(l-k)(k+l+1) = 2k+2$ which is divisible by 4 only when k is odd. Together with the inequation $2k+2 = k+l+1 \leq 16$ we observe that the allowed values of k are $3, 5, 7$.

Case $k = 3$: We have $l = 4$ and the equation $8(m-4)+10 = 6(n-3)+6$ which implies $4m = 3n+5$. This implies $n = 4s+1, m = 3s+2$ for some positive integer s. The equation $[\frac{l}{2}][\frac{m-l}{2}] = [\frac{k}{2}][\frac{n-k}{2}]$ implies $2s-1 = 2[\frac{3s-2}{2}]$. But the LHS is bigger for $s = 1$ and the RHS is always bigger for $s > 1$.

Case $k = 5$: We have $l = 6$ and the equation $12(m-6)+21 = 10(n-5)+15$ which implies $6m = 5n+8$ which has the only solution $n = 6s+2, m = 5s+3$. The equation $[\frac{k}{2}][\frac{n-k}{2}] = [\frac{l}{2}][\frac{m-l}{2}]$ implies $3[\frac{5s-3}{2}] = 2(3s-2)$. The LHS is always bigger for $s > 0$.

Case $k = 7$: We have $l = 8$ and the equation $16(m-8)+36 = 14(n-7)+28$ which implies $8m = 7n+11$ which has the only solution $n = 8s+3, m = 7s+4$. The equation $[\frac{k}{2}][\frac{n-k}{2}] = [\frac{l}{2}][\frac{m-l}{2}]$ implies $4[\frac{7s-4}{2}] = 3(4s-2)$. For $s = 1$, the LHS is 4 and the RHS is 6. For $s > 1$ the LHS is bigger. $\square$

The arguments in the above case can be extended to prove the following:

Theorem 3.2. *Consider maps $h\colon \widetilde{I}_{2n,k} \to \mathbb{R}\widetilde{G}_{m,l}$ and $g\colon \mathbb{R}\widetilde{G}_{m,l} \to \widetilde{I}_{2n,k}$, where $2 \leq l \leq \frac{m}{2}$, $2 \leq k \leq n$ and $\dim \widetilde{I}_{2n,k} = \dim \mathbb{R}\widetilde{G}_{m,l}$. Then $\deg g = \deg h = 0$.*

Proof. Note that when $l = 1$, $\mathbb{R}\widetilde{G}_{m,l} \simeq S^{m-1}$. Hence there exists a map $h_\lambda\colon \widetilde{I}_{2n,k} \to \mathbb{R}\widetilde{G}_{m,1}$ of any degree $\lambda \in \mathbb{Z}$ whenever $\dim(\widetilde{I}_{2n,k}) = m-1$. Similarly, we have a map $g_\lambda\colon \mathbb{R}\widetilde{G}_{m,l} \to \widetilde{I}_{2n,1}$ of any specified degree λ whenever $\dim(\mathbb{R}\widetilde{G}_{m,l}) = 2n-1$.

If $n = k = 2$, $\dim \widetilde{I}_{2n,k} = \dim \mathbb{R}\widetilde{G}_{m,l} = 3$ implies either $l = 1$ or $m - l = 1$. Since $\mathbb{R}\widetilde{G}_{m,l}$ is diffeomorphic to $\mathbb{R}\widetilde{G}_{m,m-l}$, both these cases reduce to the cases discussed in the previous paragraph.

Now consider the case where $n = k > 2$. Then, by Proposition 2.1, we have $H^4(\widetilde{I}_{2n,n}) = 0$ and $H^1(\widetilde{I}_{2n,n}) \neq 0$, which respectively imply $\deg h = 0$ and $\deg g = 0$.

Henceforth we restrict ourselves to the cases $2 \leq k < n$. Consider $h^*\colon H^*(\mathbb{R}\widetilde{G}_{m,l};\mathbb{Q}) \to H^*(\widetilde{I}_{2n,k};\mathbb{Q})$. Recall that $H^4(\mathbb{R}\widetilde{G}_{m,l};\mathbb{Q})$ is generated by p_1 which has order $[l/2][(m-l)/2]$. By Proposition 2.5, order of $p_1 \in H^4(\widetilde{I}_{2n,k};\mathbb{Q})$ is $[k/2][(n-k)/2]$. And, h^* takes $p_1 \in H^4(\mathbb{R}\widetilde{G}_{m,l};\mathbb{Q})$ to some multiple of p_1 in $H^4(\widetilde{I}_{2n,k};\mathbb{Q})$.

Therefore, as in the proof of Theorem 3.1, we have that if $\deg h \neq 0$, $l(m-l)-4 \leq k(n-k) \leq l(m-l)+4$. Observe that $\dim \widetilde{I}_{2n,k} = \dim \mathbb{R}\widetilde{G}_{m,l}$ implies $2k(n-k) + k(k+1)/2 = l(m-l)$. Hence the bound gives us $k(n-k) + k(k+1)/2 \leq 4$ which is not possible if $k \geq 2$ and $n > k$. Therefore we have $\deg h = 0$. The proof that $\deg g = 0$ is similar. □

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