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GENERAL PROPERTIES OF THE HYPERSPACE OF CONVERGENT SEQUENCES

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ABSTRACT. Given a Hausdorff space X, the symbol $\mathcal{S}_c(X)$ denotes the topological space which results of endowing the set of all infinite convergent sequences in X with the Vietoris topology. This hyperspace was introduced in [5].

In this paper we present answers to some questions posed in that article, namely, we show that if X is either metrizable or second countable, then X is pathwise connected as long as $\mathcal{S}_c(X)$ is so, and we exhibit a dendroid X for which $\mathcal{S}_c(X)$ is not pathwise connected. Continuing with negative examples, we present a normal (resp. Fréchet-Urysohn) space whose hyperspace of converging sequences is not normal (resp. Fréchet-Urysohn).

By proving that the hypothesis X *is connected* implies that $\mathcal{S}_c(X)$ is connected we generalize one of the results from the article mentioned above. Moreover, it is proved here that the reverse implication holds whenever $\mathcal{S}_c(X) \neq \emptyset$ and similiar results are obtained when we replace *connected* with *locally connected*.

A section is included where the weight, the character and the density of $\mathcal{S}_c(X)$ are compared with the corresponding cardinal functions of X. Then we turn our attention to the study of the topological dimension of the hyperspace of convergent sequences of compact metrizable spaces. Finally, we characterize the continuous functions from $\mathcal{S}_c(X)$ to $\mathcal{S}_c(Y)$ which are inducible.

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1. Introduction

The hyperspace of all nonempty closed subsets of a space X and the hyperspace of all nonempty connected closed subsets of X are examples of certain valuable spaces which have been found as topological invariants of X. The study of hyperspaces can provide information about the topological behavior of the original space and vice versa. It is the purpose of this paper to study a particular invariant space associated with a Hausdorff space X, namely $\mathcal{S}_c(X)$, which consists of all nontrivial convergent sequences in X and was introduced for metric spaces in [5]. Our main interests are the following: first, to establish relations between connectedness, local connectedness and path connectedness of a Hausdorff space X and the corresponding properties for $\mathcal{S}_c(X)$; second, to study the interrelation between the main cardinal functions of $\mathcal{S}_c(X)$ and those of X, and, third, to find necessary and sufficient conditions to obtain that a continuous function between the hyperspaces $\mathcal{S}_c(X)$ and $\mathcal{S}_c(Y)$ is inducible.

After Introduction and Preliminaries, in Section 3 of this paper, some helpful topological properties are obtained; we also present examples to show that the normality and the property of being Fréchet-Urysohn are not preserved by the hyperspace operation $\mathcal{S}_c(\cdot)$.

In Section 4 we answer questions 2.14 and 2.15 of [5, p. 802] in the affirmative for both the class of metrizable spaces and the class of second countable spaces. A positive answer for [5, Question 2.16, p. 802] is given as well. We provide an example which shows that questions 2.9 and 2.10 have negative answers. We finish this section by generalizing [5, Theorem 2.12, p. 801] and exploring the connection between the local connectedness of the ground space and the corresponding property of its hyperspace of convergent sequences.

The interrelations between weight, character and density of $\mathcal{S}_c(X)$ and those of the space X are analysed in Section 5. A brief section about dimension is presented next.

Finally, in Section 7, we give a characterization of the continuous functions g from $\mathcal{S}_c(X)$ to $\mathcal{S}_c(Y)$ for which there exists a continuous function f from X to Y satisfying $g(S) = f[S]$ for every $S \in \mathcal{S}_c(X)$ (this is done when X is a crowded sequential Hausdorff space and Y is a Hausdorff space).

2. Preliminaries

All topological notions and all set-theoretic notions whose definition is not included here should be understood as in [4] and [9], respectively.

The symbol ω denotes both, the first infinite ordinal and the first infinite cardinal. In particular, we consider all nonnegative integers as ordinals too; thus, $n \in \omega$ implies that $n = \{0, \ldots, n-1\}$ and $\omega \setminus n = \{k \in \omega : k \geq n\}$. The successor of ω is the ordinal $\omega + 1 = \omega \cup \{\omega\}$ and so the symbols $i \in \omega + 1$, $i < \omega + 1$ and $i \leq \omega$ all represent the same. The set $\omega \setminus \{0\}$ is denoted by $\mathbb{N}$. As usual, $\mathfrak{c}$ will be used to represent the cardinality of the real line, $\mathbb{R}$.

If X is a set and κ is a cardinal, $|X|$ will represent the cardinality of X and $[X]^{<\kappa}$ denotes the family of all subsets of X whose cardinality is $< \kappa$. In particular, $[X]^{<\omega}$ is the collection of all finite subsets of X and $[X]^{<n+1}$ is the collection of all subsets of X having at most n elements, whenever $n \in \mathbb{N}$.

For a function f, $\operatorname{ran}(f)$ will denote its range, and given a subset A of the domain of f, the set $\{f(x) : x \in A\}$ is denoted by $f[A]$.

The *cartesian product* of a family $\{X_\alpha : \alpha \in I\}$ of sets, i.e., the set of all functions f from I into $\bigcup_{\alpha \in I} X_\alpha$ such that $f(\alpha) \in X_\alpha$ for every $\alpha \in I$, is denoted by $\prod_{\alpha \in I} X_\alpha$.

Given a family $\mathcal{A}$ of nonempty sets, we will say that c is *a choice function for* $\mathcal{A}$ if $c : \mathcal{A} \to \bigcup \mathcal{A}$ and $c(A) \in A$, for each $A \in \mathcal{A}$.

In this paper, *space* means Hausdorff space. For a topological space X, the symbol τ_X will denote the collection of all open subsets of X. Also, for a set $A \subseteq X$, we will use $\operatorname{int}_X A$ and $\operatorname{cl}_X A$ (or, if there is no risk of confusion, $\overline{A}$) to represent its interior in X and its closure in X, respectively.

The topological product of a family of topological spaces $\{X_\alpha : \alpha \in I\}$ is the topological space which results from endowing the cartesian product $\prod_{\alpha \in I} X_\alpha$ with the product topology.

A *convergent sequence* in a topological space X is a function f from ω into X for which there is $x \in X$ in such a way that for each $U \in \tau_X$ with $x \in U$ there exists $n \in \omega$ with $f[\omega \setminus n] \subseteq U$. In this case, we will say that either f converges to x or x is the limit of f, and this fact will be denoted by either $\lim\limits_{n \to \infty} f(n) = x$ or $f(n) \to x$. We shall write $(f(n))_{n \in \omega}$ to refer to f. If $|\operatorname{ran}(f)| = \omega$, we say that f is *nontrivial*. In connection with this concept, in this paper, a subset S of a space X will be called a *nontrivial convergent sequence in* X if S is countably infinite and there is $x \in S$ in such a way that $S \setminus U \in [X]^{<\omega}$ for each $U \in \tau_X$ with $x \in U$. When this happens, the point x is called *the limit point of* S and we will say that S *converges to* x and write either $S \to x$ or $\lim S = x$. Throughout this paper, the reader will be able to identify from the context what is the intended meaning of *nontrivial convergent sequence* in the discussion.

In [5, p. 796], additionally, a nontrivial convergent sequence S satisfies that $S \setminus \{\lim S\}$ is discrete. Notice that a nontrivial convergent sequence, as defined in this paper in a Hausdorff space, has this property. Thus, our definition and that of [5, p. 796] are equivalent.

For a space X, let

$$\begin{aligned}\mathcal{CL}(X) &= \{A \subseteq X : A \text{ is closed in } X \text{ and } A \neq \emptyset\},\\ \mathcal{K}(X) &= \{A \in \mathcal{CL}(X) : A \text{ is compact}\},\\ \mathcal{S}_c(X) &= \{S \in \mathcal{K}(X) : S \text{ is a nontrivial convergent sequence in } X\} \text{ and}\\ \mathcal{C}(X) &= \{A \in \mathcal{K}(X) : A \text{ is connected}\}.\end{aligned}$$

Given a family $\mathcal{U}$ of subsets of X, we define

$$\langle \mathcal{U} \rangle = \left\{ A \in \mathcal{CL}(X) : A \subseteq \bigcup \mathcal{U} \ \wedge \ \forall\, U \in \mathcal{U}\ (A \cap U \neq \emptyset) \right\}.$$

The *Vietoris topology* is the topology on $\mathcal{CL}(X)$ generated by the base consisting of all sets of the form $\langle \mathcal{U} \rangle$, where $\mathcal{U} \in [\tau_X]^{<\omega}$ (see [10, Proposition 2.1, p. 155]). The hyperspaces $\mathcal{C}(X)$, $\mathcal{S}_c(X)$ and $\mathcal{K}(X)$ will be considered as subspaces of $\mathcal{CL}(X)$. In particular, a base for the topology of $\mathcal{S}_c(X)$ consists of all sets of the form $\langle \mathcal{U} \rangle_c = \langle \mathcal{U} \rangle \cap \mathcal{S}_c(X)$, where $\mathcal{U} \in [\tau_X]^{<\omega}$. For each $n \in \mathbb{N}$, we will denote by $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ the subspaces $[X]^{<n+1} \setminus \{\emptyset\}$ and $[X]^{<\omega} \setminus \{\emptyset\}$, respectively, of $\mathcal{CL}(X)$.

For a subset U of a space X, let $U^+ = \langle \{U\} \rangle$, $U_c^+ = \langle \{U\} \rangle_c$, $U^- = \langle \{X, U\} \rangle$ and $U_c^- = \langle \{X, U\} \rangle_c$. Thus, when V is open (closed, resp.) in X, then V^+ and V^- are open (closed, resp.) in $\mathcal{CL}(X)$, and hence, V_c^+ and V_c^- are open (closed, resp.) in $\mathcal{S}_c(X)$.

A *cellular family* in a topological space X is a pairwise disjoint family of nonempty open subsets of X. The collection of all finite cellular families of X is denoted by $\mathfrak{C}(X)$.

A topological space having no isolated points will be called *crowded*. We will say that a topological space X is *abundant in sequences* if the set consisting of all points which are the limit of a member of $\mathcal{S}_c(X)$ is dense in X. Equivalently, a space X is abundant in sequences if and only if $\mathcal{S}_c(X)$ is dense in $\mathcal{CL}(X)$.

Let us go over an example of a space which is abundant in sequences. Assume that κ is an infinite cardinal and $\{X_\alpha : \alpha < \kappa\}$ is a family of spaces satisfying $|X_\alpha| \geq 2$, for each $\alpha < \kappa$. Hence, each nonempty canonical basic open set in the topological product $\prod_{\alpha<\kappa} X_\alpha$ contains a copy of the cube 2^κ, where 2 is endowed with the discrete topology, and since $\kappa \geq \omega$, this means that 2^ω, the Cantor set, embeds into any nonempty open subset of $\prod_{\alpha<\kappa} X_\alpha$, i.e., this product is abundant in sequences.

A topological space X is called *sequential* if for each $A \subseteq X$ with $A \neq \overline{A}$, there exist $(x_n)_{n\in\omega}$, a sequence in A, and $x \in X \setminus A$ such that $\lim\limits_{n\to\infty} x_n = x$. Recall that a topological space X is *Fréchet-Urysohn* if for all $B \subseteq X$ and for each $x \in \overline{B}$ there is a sequence $(x_n)_{n\in\omega}$ contained in B whose limit is x. Every Fréchet-Urysohn space is a sequential space (see [4, Theorem 1.6.14, p. 53]).

Next, we discuss the interrelations between the properties of being abundant in sequences, being Fréchet-Urysohn and being sequential in the class of crowded spaces.

Let X be a crowded sequential space. Since X is crowded, for each $x \in X$ we get $x \in \overline{X \setminus \{x\}}$ and so there is a sequence $(x_n)_{n\in\omega}$ in $X \setminus \{x\}$ whose limit is x. Thus, $S = \{x\} \cup \{x_n : n \in \omega\} \in \mathcal{S}_c(X)$ and $\lim S = x$. In other words, all crowded sequential spaces are abundant in sequences, and therefore, all crowded Fréchet-Urysohn spaces are abundant in sequences.

The long segment (see [4, 3.12.19, p. 237]) is an example of a crowded space which is abundant in sequences but fails to be sequential and therefore it is not Fréchet-Urysohn.

Given a family of topological spaces $\{X_\alpha : \alpha \in I\}$, the symbol $\bigoplus_{\alpha\in I} X_\alpha$ denotes the topological sum of $\{X_\alpha : \alpha \in I\}$ (see [4, p. 74]).

Lemma 2.1. *Let X be a space. Then, the function $\varphi : \bigoplus_{m\in\mathbb{N}} \mathcal{K}(X)^m \to \mathcal{K}(X)$ defined by $\varphi(\mathbf{t}) = \bigcup \operatorname{ran}(\mathbf{t})$ is continuous.*

Proof. It follows from [10, 2.4.3 of Proposition 2.4, p. 156] and [10, 5.7.2 of Theorem 5.7, p. 168]. $\square$

3. General properties of $\mathcal{S}_c(X)$

In this section we present some topological properties of the hyperspace $\mathcal{S}_c(X)$ which will be used constantly in this paper.

Theorem 3.1. *Let X and Y be spaces. If X and Y are homeomorphic, then $\mathcal{S}_c(X)$ and $\mathcal{S}_c(Y)$ are homeomorphic as well.*

Proof. Let $h : X \to Y$ be a homeomorphism. In the proof of [8, Theorem 1.3, p. 5] it is shown that the function $h^* : \mathcal{CL}(X) \to \mathcal{CL}(Y)$ defined by $h^*(A) = h[A]$ is a homeomorphism. Notice that if $S \in \mathcal{S}_c(Y)$, then $T = h^{-1}[S] \in \mathcal{S}_c(X)$ and $h[T] = S$, so $h^*[\mathcal{S}_c(X)] = \mathcal{S}_c(Y)$. $\square$

Proposition 3.2. *For an arbitrary space X, $\{\langle \mathcal{U} \rangle_c : \mathcal{U} \in \mathfrak{C}(X)\}$ is a base for $\mathcal{S}_c(X)$.*

Proof. Start by fixing $\mathcal{V} \in [\tau_X]^{<\omega}$ and $S \in \mathcal{S}_c(X)$ in such a way that $S \in \langle \mathcal{V} \rangle_c$. Moreover, assume that $S \to a$.

For each $x \in S$, define $\mathcal{V}_x = \{V \in \mathcal{V} : x \in V\}$ and set $V_x = \bigcap \mathcal{V}_x$. Clearly, $F = \{a\} \cup (S \setminus V_a)$ is a finite set. Now, observe that by letting $\mathcal{V}^* = \bigcup\{\mathcal{V}_x : x \in F\}$ one gets that $W \in \mathcal{V} \setminus \mathcal{V}^*$ implies that $W \cap S \neq \emptyset$ and $W \cap F = \emptyset$; in other words, $W \cap S \cap V_a \neq \emptyset$. This remark guarantees the existence of c, a choice function for $\{W \cap S \cap V_a : W \in \mathcal{V} \setminus \mathcal{V}^*\}$ (when $\mathcal{V} \setminus \mathcal{V}^* = \emptyset$, c is the empty function). In particular, $(S \setminus V_a) \cap \operatorname{ran}(c) = \emptyset$ and $S \cap V_a \setminus \operatorname{ran}(c)$ is a compact subset of X which is disjoint from the finite set $(S \setminus V_a) \cup \operatorname{ran}(c)$.

From the previous paragraph we deduce that there is a family of open subsets of X, $\mathcal{U} = \{U_x : x \in F \cup \operatorname{ran}(c)\}$, satisfying

(1) $S \cap V_a \setminus \operatorname{ran}(c) \subseteq U_a \subseteq V_a$,
(2) if $x \in F \cup \operatorname{ran}(c)$, then $x \in U_x \subseteq V_x$, and
(3) for all $x, y \in F \cup \operatorname{ran}(c)$, $x \neq y$ implies that $U_x \cap U_y = \emptyset$.

Thus, $\mathcal{U} \in \mathfrak{C}(X)$ and each member of $\mathcal{U}$ hits S. Also, as a consequence of the equality $S = (S \cap V_a \setminus \operatorname{ran}(c)) \cup \operatorname{ran}(c) \cup (S \setminus V_a)$, we obtain that $\mathcal{U}$ covers S. Hence, $S \in \langle \mathcal{U} \rangle_c$.

To prove that $\langle \mathcal{U} \rangle_c \subseteq \langle \mathcal{V} \rangle_c$ we will show that each member of $\mathcal{V}$ contains an element of $\mathcal{U}$ and invoke [10, 2.3.1 of Lemma 2.3, p. 156]. So, let $W \in \mathcal{V}$ be arbitrary. When $W \in \mathcal{V}^*$, there is $x \in F$ with $W \in \mathcal{V}_x$ and therefore, $U_x \subseteq V_x \subseteq W$. On the other hand, $W \notin \mathcal{V}^*$ ensures that $c(W) \in W \cap S \cap V_a$ and, according to condition (2) above, $c(W) \in U_{c(W)} \subseteq V_{c(W)}$; thus, $W \in \mathcal{V}_{c(W)}$ and $U_{c(W)} \subseteq V_{c(W)} \subseteq W$. □

Lemma 3.3. *Let X be a space. If $\mathsf{A} \subseteq \mathcal{S}_c(X)$ and $S \in \operatorname{cl}_{\mathcal{S}_c(X)} \mathsf{A}$, then $S \subseteq \operatorname{cl}_X \left(\bigcup \mathsf{A}\right)$.*

Proof. By contrapositive, if $S \setminus \operatorname{cl}_X \left(\bigcup \mathsf{A}\right) \neq \emptyset$, then $\left(X \setminus \operatorname{cl}_X \left(\bigcup \mathsf{A}\right)\right)_c^-$ is a neighborhood of S which is disjoint from A. □

The following result generalizes [5, Lemma 1.1, p. 796].

Lemma 3.4. *Let X be a space, let $\mathcal{A} \in [\mathcal{CL}(X)]^{<\omega}$ be a pairwise disjoint family and let $n \in \mathbb{N}$. If $S \in \mathcal{S}_c(X)$ satisfies $S \cap \bigcup \mathcal{A} = \emptyset$, then*

$$\mathsf{E}_n(\mathcal{A}, S) = \left\{ S \cup \bigcup \operatorname{ran}(\mathbf{t}) : \mathbf{t} \in \prod_{A \in \mathcal{A}} \mathcal{F}_n(A) \right\}$$

is a closed nowhere dense subset of $\mathcal{S}_c(X)$ which is homeomorphic to $\prod_{A \in \mathcal{A}} \mathcal{F}_n(A)$.

Proof. To simplify notation, set $\mathsf{E} = \mathsf{E}_n(\mathcal{A}, S)$ and $\mathbf{P} = \prod_{A \in \mathcal{A}} \mathcal{F}_n(A)$. Let us start by proving that the map $h : \mathbf{P} \to \mathsf{E}$ given by $h(\mathbf{t}) = S \cup \bigcup \operatorname{ran}(\mathbf{t})$ is a homeomorphism.

Clearly, h is a bijection. The continuity of h follows from the fact that, if φ is as in Lemma 2.1, then $h(\mathbf{t}) = \varphi(\mathbf{t}, S)$ for each $\mathbf{t} \in \mathbf{P}$. Now, let us argue that h^{-1} is continuous. Fix $\mathbf{t} \in \mathbf{P}$ and suppose that $\{\mathcal{V}_A : A \in \mathcal{A}\} \subseteq [\tau_X]^{<\omega}$ is such that $\mathbf{t} \in \prod_{A\in\mathcal{A}}(\langle\mathcal{V}_A\rangle \cap \mathcal{F}_n(A))$. Since $S \cap \bigcup\mathcal{A} = \emptyset$, there exist $U \in \tau_X$ and $\{\mathcal{W}_A : A \in \mathcal{A}\} \subseteq [\tau_X]^{<\omega}$ with the following properties:

(1) $S \subseteq U \subseteq X \setminus \bigcup\mathcal{A}$,
(2) $\mathbf{t}(A) \in \langle\mathcal{W}_A\rangle \cap \mathcal{F}_n(A) \subseteq \langle\mathcal{V}_A\rangle$ for each $A \in \mathcal{A}$,
(3) $(\bigcup\mathcal{W}_A) \cap (\bigcup\mathcal{W}_B) = \emptyset$ whenever $A, B \in \mathcal{A}$ with $A \neq B$,
(4) $(\bigcup\mathcal{W}_A) \cap (\bigcup(\mathcal{A} \setminus \{A\})) = \emptyset$ for every $A \in \mathcal{A}$, and
(5) $U \cap \bigcup\{\bigcup\mathcal{W}_A : A \in \mathcal{A}\} = \emptyset$.

Set $\mathcal{U} = \{U\} \cup \bigcup_{A\in\mathcal{A}}\mathcal{W}_A$. We have that $h(\mathbf{t}) \in \langle\mathcal{U}\rangle_c$. When $\mathbf{p} \in \mathbf{P}$ is such that $h(\mathbf{p}) \in \langle\mathcal{U}\rangle_c$, conditions (1), (3), (4) and (5) give $\mathbf{p}(A) \in \langle\mathcal{W}_A\rangle$ for each $A \in \mathcal{A}$. Therefore, $h^{-1}[\langle\mathcal{U}\rangle_c \cap \mathsf{E}] \subseteq \prod_{A\in\mathcal{A}}(\langle\mathcal{V}_A\rangle \cap \mathcal{F}_n(A))$.

In order to prove that E is closed, let $Q \in \mathcal{S}_c(X) \setminus \mathsf{E}$ be arbitrary. When $Q \nsubseteq S \cup \bigcup\mathcal{A} = \overline{S \cup \bigcup\mathcal{A}} = \overline{\bigcup\mathsf{E}}$, we invoke Lemma 3.3 to conclude that $Q \notin \overline{\mathsf{E}}$. On the other hand, if there exists $z \in S \setminus Q$, then $Q \in (X \setminus \{z\})^+_c$ and $(X \setminus \{z\})^+_c \cap \mathsf{E} = \emptyset$. Hence, for the rest of the argument, let us assume that $Q \subseteq S \cup \bigcup\mathcal{A}$ and $S \subseteq Q$; in other words, $Q = S \cup (Q \cap \bigcup\mathcal{A})$. If there exists $B \in \mathcal{A}$ such that $B \cap Q = \emptyset$, then $Q \in (X \setminus B)^+_c$ and $(X \setminus B)^+_c \cap \mathsf{E} = \emptyset$. When $Q \cap A \neq \emptyset$ for each $A \in \mathcal{A}$, we deduce from $Q \notin \mathsf{E}$ that $|C \cap Q| > n$ for some $C \in \mathcal{A}$, and so there is $\mathcal{V} \in \mathfrak{C}(X)$ satisfying $|\mathcal{V}| = n + 1$, $(\bigcup\mathcal{V}) \cap (S \cup \bigcup\mathcal{A}) \setminus C = \emptyset$ and such that $V \cap C \cap Q \neq \emptyset$ for each $V \in \mathcal{V}$. Thus, $\langle\{X\} \cup \mathcal{V}\rangle_c$ is an open neighborhood of Q disjoint from E. Therefore, $\mathcal{S}_c(X) \setminus \mathsf{E}$ is an open subset of $\mathcal{S}_c(X)$.

Finally, to show that E has empty interior, suppose that $\mathcal{U} \in \mathfrak{C}(X)$ is such that $h(\mathbf{q}) \in \langle\mathcal{U}\rangle_c$ for some $\mathbf{q} \in \mathbf{P}$. Fix $U \in \mathcal{U}$ with $\lim S \in U$ and let $y \in U \cap S \setminus \{\lim S\}$ be arbitrary. Then, $(S \setminus \{y\}) \cup \bigcup \operatorname{ran}(\mathbf{q}) \in \langle\mathcal{U}\rangle_c \setminus \mathsf{E}$. $\square$

Let $\mathcal{H}(X) \in \{\mathcal{CL}(X), \mathcal{K}(X), \mathcal{S}_c(X), \mathcal{C}(X), \mathcal{F}(X)\} \cup \{\mathcal{F}_n(X) : n \in \mathbb{N}\}$. A topological property P will be called:

(a) *$\mathcal{H}$-preserved* provided that if a space X has property P, so does $\mathcal{H}(X)$, and
(b) *$\mathcal{H}$-reversible* if the condition "$\mathcal{H}(X)$ has property P" implies that X has property P, for any space X.

According to 4.9.2, 4.9.8, 4.9.10, 4.9.11 of [10, Theorem 4.9, pp. 163-164], the property of being a T_i space for $i \in \{2, 3, 3\frac{1}{2}\}$ is $\mathcal{K}$-preserved and therefore $\mathcal{H}$-preserved when $\mathcal{H}(X) \in \{\mathcal{S}_c(X), \mathcal{C}(X), \mathcal{F}(X)\} \cup \{\mathcal{F}_n(X) : n \in \mathbb{N}\}$.

Proposition 3.5. *Normality is not $\mathcal{S}_c$-preserved.*

Proof. Let X be Sorgenfrey's line. In order to prove that $\mathcal{S}_c(X)$ is not normal, by [4, (c) of 1.7.12, p. 60], it suffices to prove that there exists a separable closed subset of $\mathcal{S}_c(X)$ which contains a closed discrete subspace of cardinality $\mathfrak{c}$.

Let us start by noting that the intervals $Y_i = [-1+i, i)$, for $i \in 2$, are closed subsets of X. On the other hand, $S = \{1\} \cup \{1 + 2^{-n} : n \in \omega\} \in \mathcal{S}_c(X)$ and S is disjoint from $Y_0 \cup Y_1$. Thus, by Lemma 3.4, $\mathsf{E}_1(\{Y_0, Y_1\}, S)$ is a closed subset of $\mathcal{S}_c(X)$ which is homeomorphic to $\mathcal{F}_1(Y_0) \times \mathcal{F}_1(Y_1)$. Since each $\mathcal{F}_1(Y_i)$ is homeomorphic to Y_i, $\mathsf{E}_1(\{Y_0, Y_1\}, S)$ is homeomorphic to $Y = Y_0 \times Y_1$. We have that Y is separable and the set $\{(x, -x) : x \in (-1, 0)\}$ is a closed discrete subspace of Y of cardinality $\mathfrak{c}$. Therefore, $\mathcal{S}_c(X)$ is not normal. □

Remark 3.6. Let X, Y_0, Y_1 and Y be as in Proposition 3.5. We have that $\mathcal{F}_1(Y_0) \times \mathcal{F}_1(Y_1)$ is homeomorphic to Y and the mapping h from $\mathcal{F}_1(Y_0) \times \mathcal{F}_1(Y_1)$ into $\mathcal{F}_2(X)$ defined by $h(\{x\}, \{y\}) = \{x, y\}$ is an embedding such that $h[\mathcal{F}_1(Y_0) \times \mathcal{F}_1(Y_1)]$ is a closed subset of $\mathcal{F}_2(X)$. So, since Y is separable and contains a closed discrete subspace of cardinality $\mathfrak{c}$, we infer that normality is not $\mathcal{H}$-preserved if $\mathcal{H}(X) \in \{\mathcal{K}(X), \mathcal{F}(X)\} \cup \{\mathcal{F}_n(X) : n \geq 2\}$.

Proposition 3.7. *Being Fréchet-Urysohn is not $\mathcal{S}_c$-preserved.*

Proof. Denote the unit interval $[0, 1]$ by I. Given a cardinal κ, endow κ with the discrete topology and set $J(\kappa) = (I \times \kappa)/(\{0\} \times \kappa)$, i.e., $J(\kappa)$ is the quotient space of the topological product $I \times \kappa$ which results from collapsing the set $\{0\} \times \kappa$ to a single point. Similarly, define $S(\kappa) = ((\omega+1) \times \kappa)/(\{\omega\} \times \kappa)$, where the ordinal $\omega + 1$ is considered as a linearly ordered topological space. Usually, $J(\kappa)$ and $S(\kappa)$ are called *the hedgehog of κ spines* and *the sequential fan of κ spines*, respectively.

Let $X = J(\omega_1) \times 2$, where 2 is endowed with the discrete topology. Observe that X is Fréchet-Urysohn.

By [6, Corollary 1.7 and Example 1.8, p. 303], $S(\omega_1)^2$ is not Fréchet-Urysohn. So, to prove that $\mathcal{S}_c(X)$ is not Fréchet-Urysohn, it suffices to see that there exists a subset of $\mathcal{S}_c(X)$ which is homeomorphic to $S(\omega_1)^2$.

Fix $P, Q \in \mathcal{S}_c(I)$ in such a way that $\lim Q = 0$ and $0 \notin P$. Now, let $q : I \times \omega_1 \to J(\omega_1)$ be the natural quotient map and observe that $M = q[Q \times (\omega_1 \setminus \{0\})]$ is a closed subset of $J(\omega_1)$ homeomorphic to $S(\omega_1)$ and disjoint from the convergent sequence $R = q[P \times \{0\}]$. Set $\mathcal{A} = \{M \times \{0\}, M \times \{1\}\}$ and notice that each element of $\mathcal{A}$ is disjoint from $S = R \times \{0\} \in \mathcal{S}_c(X)$. Then, by Lemma 3.4, $\mathsf{E}_1(\mathcal{A}, S)$ is a closed subset of $\mathcal{S}_c(X)$ which is homeomorphic to $\mathcal{F}_1(M \times \{0\}) \times \mathcal{F}_1(M \times \{1\})$. Clearly, each $\mathcal{F}_1(M \times \{i\})$ is homeomorphic to M. Thus, $\mathsf{E}_1(\mathcal{A}, S)$ is homeomorphic to $S(\omega_1)^2$ and so $S(\omega_1)^2$ embeds as a subspace of $\mathcal{S}_c(X)$. □

Remark 3.8. With similar arguments as in Proposition 3.7 one can show that $\mathcal{F}_1(M\times\{0\})\times\mathcal{F}_1(M\times\{1\})$ is homeomorphic to $S(\omega_1)^2$ and it can be embedded in $\mathcal{F}_2(X)$. Thus, the property of being Fréchet-Urysohn is not $\mathcal{H}$-preserved when $\mathcal{H}(X)\in\{\mathcal{K}(X),\mathcal{F}(X)\}\cup\{\mathcal{F}_n(X):n\geq 2\}$.

Proposition 3.9. *Let X be a space and let $x\in X$. If $(A_n)_{n\in\omega}$ is a sequence in $\mathcal{K}(X)$ such that $x\in A_n$ for every $n\in\omega$ and $\lim\limits_{n\to\infty}A_n$ exists in $\mathcal{K}(X)$, then $x\in\lim\limits_{n\to\infty}A_n$.*

Proof. Let $A=\lim\limits_{n\to\infty}A_n$. Suppose that $x\notin A$. Since X is T_2, we have that $X\setminus\{x\}$ is an open subset of X such that $A\in(X\setminus\{x\})^+$. Notice that $A_n\notin(X\setminus\{x\})^+$ for every $n\in\omega$. This contradicts the fact that $(A_n)_{n\in\omega}$ converges to A in $\mathcal{K}(X)$. □

4. Connectedness, path connectedness, and local connectedness

We begin this section by proving that being connected is a property which is $\mathcal{S}_c$-preserved and $\mathcal{S}_c$-reversible (as long as the hyperspace of convergent sequences is not empty). In particular, [5, Question 2.16, p. 802] has a positive answer.

Lemma 4.1. *Let X be a connected space and assume that $P,Q\in\mathcal{S}_c(X)$ are such that $P\subseteq Q$ and $Q\setminus P$ is finite. Then, P and Q belong to the same component of $\mathcal{S}_c(X)$.*

Proof. Set $n=|Q\setminus P|$ and define $\varphi:\mathcal{F}_n(X)\to\mathcal{S}_c(X)$ by $\varphi(A)=P\cup A$ for each $A\in\mathcal{F}_n(X)$ to get a continuous map (see [10, Corollary 5.8.1, p. 169]). Since X is connected, [10, Theorem 4.10, p. 165] guarantees that $\mathcal{F}_n(X)$ is connected. To finish our argument, note that $P,Q\in\operatorname{ran}(\varphi)$. □

Theorem 4.2. *If X is a connected space, then so is $\mathcal{S}_c(X)$.*

Proof. Let $S_1,S_2\in\mathcal{S}_c(X)$ be arbitrary and denote by $\mathcal{C}$ the component of $\mathcal{S}_c(X)$ containing S_1. Assume that $\{x_i:i\leq\omega\}$ and $\{y_i:i\leq\omega\}$ are adequate enumerations of S_1 and S_2, respectively.

For each $k<\omega$ set $P_k=S_1\setminus\{x_i:i\leq k\}$ and $Q_k=P_k\cup\{y_i:i\leq k\}$. From Lemma 4.1 we deduce that P_k and Q_k belong to $\mathcal{C}$.

Now, the fact that the sequence $(Q_k)_{k<\omega}$ converges to $S_3=S_2\cup\{x_\omega\}$ implies that $S_3\in\mathcal{C}$ and so, Lemma 4.1 gives $S_2\in\mathcal{C}$. □

In [5, Theorem 2.6, p. 799] it is proved that connectedness is a $\mathcal{S}_c$-reversible property in the class of first countable spaces. We will show that the same result holds in the class of topological spaces with non-void hyperspace of convergent sequences.

Theorem 4.3. *If X is a topological space for which $\mathcal{S}_c(X)$ is non-empty and connected, then X is connected as well.*

Proof. Let X be a space such that $\mathcal{S}_c(X)$ is nonempty and connected. In order to prove connectedness of X, let $U, V \in \tau_X$ satisfying $X = U \cup V$ and $U \cap V = \emptyset$. Let $S \in \mathcal{S}_c(X)$ and set $a = \lim S$. Then, either $a \in U$ or $a \in V$. We may assume that $a \in U$. So, we shall show that $V = \emptyset$. Notice that U_c^+ and V_c^- are disjoint open subsets of $\mathcal{S}_c(X)$ whose union is $\mathcal{S}_c(X)$. By the connectedness of $\mathcal{S}_c(X)$, we obtain that either $U_c^+ = \emptyset$ or $V_c^- = \emptyset$. Observe that $S \setminus U$ is finite and so $S \cap U \in U_c^+$. Thus, we deduce that $V_c^- = \emptyset$. If there were a point x in V, we would have that $S \cup \{x\} \in V_c^-$, a contradiction. Therefore, $V = \emptyset$ and X is connected. $\square$

A discrete space having at least two points can be used to show that the condition in Theorem 4.3 on the hyperspace of nontrivial convergent sequences can not be weakened.

Recall that *a path* in a space X is a continuous map from the unit interval $I = [0, 1]$ into X. In [5, Example 2.8, p. 800], the authors show that path connectedness is not $\mathcal{S}_c$-preserved. They ask if path connectedness is a $\mathcal{S}_c$-reversible property ([5, Question 2.14, p. 802]) and if for each arbitrary path $\alpha : I \to \mathcal{S}_c(X)$, it is possible to find a path in X connecting one point of $\alpha(0)$ with one point of $\alpha(1)$ ([5, Question 2.15, p. 802]). In this section we give positive answers to these questions for the class of metrizable spaces and for the class of second countable spaces.

A topological space will be called *zero-dimensional* if it possesses a base consisting of closed sets. Given a space X, the symbol $\mathcal{K}_0(X)$ is going to represent the subspace of $\mathcal{CL}(X)$ consisting of all non-empty compact zero-dimensional subspaces of X. Note that $\mathcal{S}_c(X) \subseteq \mathcal{K}_0(X)$.

Lemma 4.4. *Assume X is either metrizable or second countable. If $\alpha : I \to \mathcal{S}_c(X)$ is a path, then for each $p \in \alpha(0)$ there is a path $\beta : I \to X$ in such a way that $\beta(t) \in \alpha(t)$, whenever $t \in I$, and $\beta(0) = p$.*

Proof. By [10, 5.6.2, p. 168], $Y = \bigcup \operatorname{ran}(\alpha)$ is a compact subspace of X and so, Y is metrizable.

Let us show that $K = \bigcup_{t \in I}(\{t\} \times \alpha(t))$ is a closed subset of the topological product $I \times Y$. If $(t, y) \in (I \times Y) \setminus K$, then $y \notin \alpha(t)$ so, there are disjoint $U, V \in \tau_X$ with $y \in U$ and $\alpha(t) \subseteq V$, i.e., $\alpha(t) \in V^+$. Now, let $G \in \tau_I$ be such that $t \in G$ and $\alpha[G] \subseteq V^+$. We will argue that $G \times U$ misses K. Indeed, for each $(s, z) \in G \times U$ we get $s \in G$ and so, $\alpha(s) \subseteq V$, which together with $z \in U$ gives $z \notin \alpha(s)$ and hence, $(s, z) \notin K$.

Denote by $\pi_I : I \times Y \to I$ and $\pi_Y : I \times Y \to Y$ the corresponding projections and set $\eta = \pi_I \restriction K$ to get a continuous map $\eta : K \to I$.

Note that if $t \in I$, then $\eta^{-1}(t) = \{t\} \times \alpha(t)$ is a compact non-empty zero-dimensional subspace of K. Thus, if we prove that η is open, we would be able to invoke [12, Theorem 2.1, p. 186] in order to obtain an embedding $h : I \to K$ satisfying $h(0) = (0, p) \in \eta^{-1}(0)$ and $h^{-1} = \eta \restriction \operatorname{ran}(h)$. This, in turn, would imply that $\beta = \pi_Y \circ h$ is a path in X (actually in Y) fulfilling all our requirements. Hence, we only need to verify that η is open.

Suppose that $G \in \tau_I$ and $U \in \tau_X$ are arbitrary. Set $V = K \cap (G \times U)$ and fix $(t, y) \in V$. Since $y \in U \cap \alpha(t)$, we have that $\alpha(t) \in U^-$ and so, there is $W \in \tau_I$ with $t \in W$ and $\alpha[W] \subseteq U^-$. For each $s \in G \cap W$, there exists $z \in \alpha(s) \cap U$ and so, $s = \eta(s, z) \in \eta[V]$. In other words, $G \cap W$ is an open neighborhood of t contained in $\eta[V]$. □

In the following result, we prove that path connectedness is $\mathcal{S}_c$-reversible in the class of metrizable spaces and in the class of second countable spaces, thus answering [5, Question 2.14, p. 802] in the affirmative.

Theorem 4.5. *Assume X is metrizable or second countable. If $\mathcal{S}_c(X)$ is nonempty and pathwise connected, then so is X.*

Proof. Fix $S_0 \in \mathcal{S}_c(X)$ and $w \in S_0 \setminus \{\lim S_0\}$. Then, there is a path $\alpha : I \to \mathcal{S}_c(X)$ satisfying $\alpha(0) = S_0$ and $\alpha(1) = S_1 = S_0 \setminus \{w\}$. Thus, by Lemma 4.4, there exists a path $\beta : I \to X$ such that $\beta(0) = w$ and $\beta(1) \in S_1$. In particular, $\beta(1) \neq w$. Hence, the path component K of w in X is nondegenerate.

Let $S_2 \in \mathcal{S}_c(K)$ and fix $p \in X$. Set $S_3 = S_2 \cup \{p\} \in \mathcal{S}_c(X)$. By assumption, there exists a path in $\mathcal{S}_c(X)$ joining S_3 and S_2. Hence, by Lemma 4.4 there exists a path $\gamma : I \to X$ such that $\gamma(0) = p$ and $\gamma(1) \in S_2 \subseteq K$. This shows that $p \in K$ and, therefore, X is pathwise connected. □

In questions 2.9 and 2.10 [5, p. 801], the authors ask if $\mathcal{S}_c(X)$ is path connected when X is either a hereditarily arcwise connected continuum (a *continuum* is a compact connected metric space) or a dendroid (a *dendroid* is an arcwise connected continuum satisfying that the intersection of two connected closed subsets is connected). Our following example answers these questions in the negative.

Example 4.6. A dendroid X such that $\mathcal{S}_c(X)$ is not pathwise connected.

For each $n \in \mathbb{N}$ let L_n be the segment in the plane that joins the points $(0, \frac{1}{n})$ and $(1, 0)$. Define $A = \bigcup_{n \in \mathbb{N}} L_n$ and $\widehat{A} = \{-z : z \in A\}$. Set $X = \operatorname{cl}_{\mathbb{R}^2}(A \cup \widehat{A})$ and observe that X is a dendroid. We will denote the point $(0, 0)$ by y.

Define $S_0 = \{(0, \frac{1}{m}) : m \in \mathbb{N}\} \cup \{(0, -\frac{1}{m}) : m \in \mathbb{N}\} \cup \{y\}$ and fix $S_1 \in A_c^+$. We will show that there is no path in $\mathcal{S}_c(X)$ joining S_0 and S_1. Suppose to the contrary that there exists a path $\alpha : I \to \mathcal{S}_c(X)$ such that $\alpha(0) = S_0$ and $\alpha(1) = S_1$. Consider the set $T = \{t \in I : |\alpha(t) \cap A| = \omega = |\alpha(t) \cap \widehat{A}|\}$ and set $s = \sup T$. Let us argue that $y \in \alpha(s)$. There exists a sequence $(t_n)_{n\in\mathbb{N}}$ of elements of T converging to s. By the continuity of α, we have that $\alpha(s) = \lim_{n\to\infty} \alpha(t_n)$. For each $n \in \mathbb{N}$, observe that $y = \lim \alpha(t_n)$, then $y \in \alpha(t_n)$. Thus, by Proposition 3.9, we infer that $y \in \alpha(s)$. Fix $\mathcal{U} \in \mathfrak{C}(X)$ such that $\alpha(s) \in \langle \mathcal{U} \rangle_c$ and, moreover, if U is the element of $\mathcal{U}$ that contains y, assume that $(-1, 0), (1, 0) \notin U$. Choose $t_1 \in T$ and $t_2 \in I \setminus T$ such that $\alpha\big[[t_1, t_2]\big] \subseteq \langle \mathcal{U} \rangle_c$. Since $t_2 \notin T$, we will assume without loss of generality that $\alpha(t_2) \cap \widehat{A}$ is finite. Furthermore, since $t_1 \in T$, then $y = \lim \alpha(t_1)$ and $\alpha(t_1) \cap \widehat{A}$ intersects infinitely many path components of U. Since $\alpha(t_2) \cap \widehat{A}$ is finite, there exists a path component K of U such that $K \cap \alpha(t_1) \cap \widehat{A} \neq \emptyset$ and $K \cap \alpha(t_2) = \emptyset$. Let $p \in K \cap \alpha(t_1) \cap \widehat{A}$. Applying Lemma 4.4 to the convergent sequences $\alpha(t_1)$ and $\alpha(t_2)$, there exists a path $\beta : [t_1, t_2] \to \bigcup \alpha\big[[t_1, t_2]\big]$ such that $\beta(t_1) = p$ and $\beta(t_2) \in \alpha(t_2)$. Since $\beta\big[[t_1, t_2]\big] \subseteq \bigcup \alpha\big[[t_1, t_2]\big] \subseteq \bigcup \mathcal{U}$ and $\mathcal{U}$ is pairwise disjoint, $\beta\big[[t_1, t_2]\big] \subseteq K$; hence $\beta(t_2) \in K \cap \alpha(t_2)$, a contradiction.

Now we turn our attention to local connectedness.

Theorem 4.7. *If X is a locally connected space, then so is $\mathcal{S}_c(X)$.*

Proof. Let $\mathcal{B}$ be a base for X consisting of open, connected sets. It is well known that $\{\langle \mathcal{U} \rangle : \mathcal{U} \in [\mathcal{B}]^{<\omega}\}$ is a base for $\mathcal{K}(X)$; hence, we only need to prove that each element of $\{\langle \mathcal{U} \rangle_c : \mathcal{U} \in [\mathcal{B}]^{<\omega}\}$ is connected.

Fix $\mathcal{U} \in [\mathcal{B}]^{<\omega}$. For each $U \in \mathcal{U}$ set $Q_U = \{S \in \langle \mathcal{U} \rangle_c : \lim S \in U\}$. Also, if $Q_U \neq \emptyset$, define $\varphi_U : \mathcal{S}_c(U) \times \prod_{V \in \mathcal{U}\setminus\{U\}} \mathcal{F}(V) \to Q_U$ by $\varphi(t) = \bigcup \operatorname{ran}(t)$. Using [10, 5.7.2, p. 168] one can show that φ_U is a surjective map. Hence, by Theorem 4.2 and [10, Theorem 4.10, p. 165], Q_U is connected.

Observe that $\langle \mathcal{U} \rangle_c = \bigcup_{U \in \mathcal{U}} Q_U$. In order to prove that $\langle \mathcal{U} \rangle_c$ is connected it suffices to show that if $U, V \in \mathcal{U}$ are such that $Q_U \neq \emptyset \neq Q_V$, then $Q_V \cap \overline{Q_U} \neq \emptyset$. Fix $S_V \in Q_V$ and $S \in \mathcal{S}_c(U)$. Let $\{x_n : n \leq \omega\}$ and $\{y_n : n \leq \omega\}$ be adequate enumerations of $S_V \cap V$ and of S, respectively. For each $k \in \omega$ define $P_k = \{y_n : n > k\} \cup (S_V \setminus V) \cup \{x_n : n \leq k\}$. Then $P_k \in Q_U$ for each $k \in \omega$. Since $\lim_{k\to\infty} P_k = S_V \cup \{y_w\}$, we conclude that $S_V \cup \{y_w\} \in Q_V \cap \overline{Q_U}$. □

Lemma 4.8. *Let X be a space. If G is an open subset of $\mathcal{S}_c(X)$, then $\bigcup \mathsf{G}$ is an open subset of X.*

Proof. Let $S \in \mathsf{G}$ be arbitrary. Since G is open in $\mathcal{S}_c(X)$, there exists $\mathcal{U} \in \mathfrak{C}(X)$ such that $S \in \langle \mathcal{U} \rangle_c \subseteq \mathsf{G}$. Note that for each $z \in \bigcup \mathcal{U}$, $S \cup \{z\} \in \langle \mathcal{U} \rangle_c \subseteq \mathsf{G}$ and so, $z \in \bigcup \mathsf{G}$. In other words, $\bigcup \mathcal{U} \in \tau_X$ and $S \subseteq \bigcup \mathcal{U} \subseteq \bigcup \mathsf{G}$. $\square$

Theorem 4.9. *Let X be a space and let $x \in X$. If $\mathcal{S}_c(X)$ is locally connected and there exists $S_0 \in \mathcal{S}_c(X)$ such that $x \neq \lim S_0$, then X is locally connected at x.*

Proof. We will proceed by contrapositive. Suppose that X is not locally connected at x. Then there exists an open set W such that $x \in W$ and no open and connected neighborhood of x is contained in W. Fix two disjoint open sets V_x and V, such that $x \in V_x \subseteq W$ and $S_0 \setminus \{x\} \subseteq V$.

Let G be an open subset of $\mathcal{S}_c(X)$ such that $S_0 \cup \{x\} \in \mathsf{G} \subseteq \langle \{V_x, V\} \rangle_c$. We will show that G is not connected. By Lemma 4.8 we know that $\bigcup \mathsf{G}$ is an open subset of X and so, $V_x \cap \bigcup \mathsf{G}$ is an open neighborhood of x which, by assumption, is not connected. Let W_1 and W_2 be two nonempty and disjoint open sets whose union is $V_x \cap \bigcup \mathsf{G}$. We will assume that $x \in W_1$. Define

$$\mathsf{O}_1 = \mathsf{G} \cap \big\langle \{V, W_1\} \big\rangle_c \qquad \text{and} \qquad \mathsf{O}_2 = \mathsf{G} \cap (W_2)_c^-,$$

to get two open subsets of G with $S_0 \cup \{x\} \in \mathsf{O}_1$. Furthermore, if $y \in W_2$, then $y \in \bigcup \mathsf{G}$, i.e., there exists $S' \in \mathsf{G}$ such that $y \in S'$. Thus, $S' \in \mathsf{O}_2$. Next, for each $S \in \mathsf{G}$ observe that $S \in \mathsf{O}_1$ if and only if $\emptyset \neq S \cap V_x \subseteq W_1$. As a consequence of these remarks we deduce that O_1 and O_2 are disjoint and their union is equal to G. In other words, G is not connected. Therefore $\mathcal{S}_c(X)$ is not locally connected at $S_0 \cup \{x\}$. $\square$

To simplify our notation, define for each space X the set $L_X = \{\lim S : S \in \mathcal{S}_c(X)\}$.

Corollary 4.10. *If X is a space, any of the following conditions implies that $\mathcal{S}_c(X)$ is locally connected if and only if X is.*

(1) $|L_X| \geq 2$.
(2) *X is a sequential space.*

Proof. Observe that Theorem 4.7 gives the reverse implication in the required equivalence for both cases. Also, when (1) holds we just need to invoke Theorem 4.9 to get the direct implication. So, let us suppose that X is sequential and that $\mathcal{S}_c(X)$ is locally connected.

If X is discrete, then X is locally connected. Hence, we will assume that X has an accumulation point z. Since $z \in \overline{X \setminus \{z\}}$, let us fix $S \in \mathcal{S}_c(X)$ with $S \setminus \{z\} \subseteq X \setminus \{z\}$ and $z = \lim S \in L_X$. We will prove that $|L_X| \geq 2$ by contradiction: suppose that $L_X = \{z\}$. By assumption there is a connected open neighborhood G of S in $\mathcal{S}_c(X)$. Let $\mathcal{U} \in \mathfrak{C}(X)$ and $U \in \mathcal{U}$

be such that $S \in \langle \mathcal{U} \rangle_c \subseteq \mathsf{G}$ and $z \in U$. Now choose $x \in S \cap U \setminus \{z\}$ and observe that $S \setminus \{x\} \in \langle \mathcal{U} \rangle_c$. If x were an accumulation point of X, there would be $S_1 \in \mathcal{S}_c(X)$ such that $x = \lim S_1 \in L_X = \{z\}$, a contradiction. Therefore, x is an isolated point of X. Define $\mathsf{O} = \mathsf{G} \cap \{x\}_c^-$ and note that O is a closed and open subset of G. Since $S \in \mathsf{O} \subseteq \mathsf{G} \setminus \{S \setminus \{x\}\}$, we infer that G is not connected, again a contradiction. This proves that $|L_X| \geq 2$ and the local connectedness of X follows. $\square$

Question 4.11. Is it true that the local connectedness of $\mathcal{S}_c(X)$ implies that of X for arbitrary spaces X with nonempty hyperspace $\mathcal{S}_c(X)$?

5. Cardinal functions

In this section we compare three main cardinal functions (weight, character and density) of a space X with those of $\mathcal{S}_c(X)$. Note that if X is the remainder of the Stone-Čech compactification of ω, then $\mathcal{S}_c(X)$ is empty (see [4, Corollary 3.6.15, p. 175]); this shows that in general the cardinal functions of X and those of $\mathcal{S}_c(X)$ do not coincide, even when X does not have isolated points. Thus, in this section we will look at spaces X that have nonempty hyperspace $\mathcal{S}_c(X)$ (this is the case, for instance, of sequential crowded spaces).

Note that, in principle, we are not assuming that the cardinal functions discussed in this section are infinite. For example, $w(X)$ is the least cardinality of a base for X, so it might be the case that $w(X) < \omega$. Of course, one should note that in this case, X would be a finite discrete space and so $\mathcal{S}_c(X)$ would be empty.

5.1. **Auxiliary results.** Using [10, Proposition 2.1 and 2.3.1 of Lemma 2.3, pp. 155-156], one can show the following result.

Lemma 5.1. *If $\mathcal{B}$ is a base for a space X, then $\mathfrak{B} = \{\langle \mathcal{V} \rangle : \mathcal{V} \in [\mathcal{B}]^{<\omega}\}$ is a base for $\mathcal{K}(X)$. In particular, if $\mathcal{B}$ is infinite, then $|\mathcal{B}| = |\mathfrak{B}|$ and, thus, $w(\mathcal{K}(X)) \leq w(X)$.*

Lemma 5.2. *Let X be a space and let $S \in \mathcal{S}_c(X)$. Assume that $\mathcal{U}, \mathcal{V} \in [\tau_X]^{<\omega}$ satisfy that $S \in \langle \mathcal{V} \rangle_c \subseteq \langle \mathcal{U} \rangle_c$. Also assume that $U \in \mathcal{U}$ is such that $S \cap U$ is finite. Then, there exists $V_0 \in \mathcal{V}$ such that $V_0 \subseteq U$; moreover, if $S \cap U = \{y\}$ for some $y \in S$, then $y \in V_0$.*

Proof. Suppose to the contrary that $\emptyset \notin \{V \setminus U : V \in \mathcal{V}\}$. Let c be a choice function for $\{V \setminus U : V \in \mathcal{V}\}$. Observe that $S \setminus U \in \mathcal{S}_c(X)$ and set $S_1 = (S \setminus U) \cup \operatorname{ran}(c)$. It follows that $S_1 \in \langle \mathcal{V} \rangle_c$. However, since $S_1 \cap U = \emptyset$, we obtain that $S_1 \in \langle \mathcal{V} \rangle_c \setminus \langle \mathcal{U} \rangle_c$, a contradiction; this proves the existence of V_0. Moreover, suppose that $S \cap U = \{y\}$. Note that $\emptyset \neq S \cap V_0 \subseteq S \cap U = \{y\}$ and therefore, $y \in V_0$. $\square$

Lemma 5.3. *Let X be a space and let $\mathcal{U}, \mathcal{V} \in [\tau_X]^{<\omega}$. If $\emptyset \neq \langle \mathcal{U} \rangle_c \subseteq \langle \mathcal{V} \rangle_c$, then $\bigcup \mathcal{U} \subseteq \bigcup \mathcal{V}$.*

Proof. Observe that if $S \in \langle \mathcal{U} \rangle_c$ and $x \in \bigcup \mathcal{U}$, then $S \cup \{x\} \in \langle \mathcal{U} \rangle_c \subseteq \langle \mathcal{V} \rangle_c$, so, $x \in \bigcup \mathcal{V}$. □

Lemma 5.4. *Let X be a space and let $A \in \mathcal{CL}(X)$. If $S \in \mathcal{S}_c(X)$ is such that $|S \cap A| \leq 1$, then A can be embedded in $\mathcal{S}_c(X)$.*

Proof. If $S \cap A = \emptyset$, the result follows from [5, Lemma 1.1, p. 796]; thus, we may assume that $S \cap A = \{p\}$ for some $p \in S$. Further, if $p \neq \lim S$, then $S \setminus \{p\} \in \mathcal{S}_c(X)$ and $(S \setminus \{p\}) \cap A = \emptyset$ so, again, the result follows from [5, Lemma 1.1, p. 796]. Hence, we will assume that $p = \lim S$.

Define $h : A \to \mathcal{S}_c(X)$ by $h(a) = S \cup \{a\}$. Since $|S \cap A| \leq 1$, it is easy to see that h is one-to-one. Let φ be as in Lemma 2.1. Since $h(a) = \varphi(S, \{a\})$ for each $a \in A$, we obtain that h is continuous. Next, fix a nonempty open subset U of A and $V \in \tau_X$ such that $V \cap A = U$. In order to prove that $h[U]$ is open in $h[A]$, fix $a \in U$. We will show that $h(a) \in \operatorname{int}_{h[A]}(h[U])$. We take two cases.

Case 1. $a \neq p$.

Let U_1 and U_2 be two disjoint open subsets of X such that $a \in U_1 \subseteq V$ and $S \subseteq U_2$. Note that $\langle \{U_1, U_2\} \rangle_c \cap h[A]$ is an open subset of $h[A]$ that contains $S \cup \{a\} = h(a)$. We will show that $\langle \{U_1, U_2\} \rangle_c \cap h[A] \subseteq h[U]$. Let $Q \in \langle \{U_1, U_2\} \rangle_c \cap h[A]$. Then $Q = h(y)$ for some $y \in A$ and, moreover, $\emptyset \neq Q \cap U_1 = (S \cup \{y\}) \cap U_1$. Thus, $y \in U_1 \cap A \subseteq V \cap A = U$ and, therefore, $Q \in h[U]$. This shows that $h(a) \in \operatorname{int}_{h[A]}(h[U])$.

Case 2. $a = p$.

Observe that $\langle \{V, X \setminus A\} \rangle_c \cap h[A]$ is an open subset of $h[A]$ that contains $S = h(p)$. We will show that $\langle \{V, X \setminus A\} \rangle_c \cap h[A] \subseteq h[U]$. Let $Q \in \langle \{V, X \setminus A\} \rangle_c \cap h[A]$. Then $Q = h(y)$ for some $y \in A$ and, moreover, $S \cup \{y\} = Q \subseteq V \cup (X \setminus A)$. Thus, $y \in V \cap A = U$ and, therefore, $Q \in h[U]$. This shows that $h(p) \in \operatorname{int}_{h[A]}(h[U])$. □

5.2. **Main Theorems.**

Theorem 5.5. *Let X be a space with more than one point. Then $w(X) = w(\mathcal{S}_c(X))$ if and only if $\mathcal{S}_c(X) \neq \emptyset$.*

Proof. The necessity is easy to see.

Next, assume that $\mathcal{S}_c(X) \neq \emptyset$. Recall that Hausdorff spaces of finite weight are finite; thus, since $\mathcal{S}_c(X) \neq \emptyset$, we infer that X has infinite weight. Therefore, by Lemma 5.1 we obtain that $w(\mathcal{S}_c(X)) \leq w(\mathcal{K}(X)) \leq w(X)$.

In order to show that $w(X) \leq w(\mathcal{S}_c(X))$, set $\kappa = w(\mathcal{S}_c(X))$. According to Proposition 3.2 and [4, Theorem 1.1.15, p. 17], for each $\alpha < \kappa$ there exists $\mathcal{V}_\alpha \in \mathfrak{C}(X)$ such that $\{\langle \mathcal{V}_\alpha \rangle_c : \alpha < \kappa\}$ is a base for $\mathcal{S}_c(X)$. Define $\mathcal{B} = \bigcup_{\alpha<\kappa} \mathcal{V}_\alpha$ and note that $\mathcal{B} \in [\tau_X]^{\leq\kappa}$. It remains to show that $\mathcal{B}$ is a base for X. To this end fix $W \in \tau_X$ and $p \in W$. By assumption we may also fix $S \in \mathcal{S}_c(X)$. We take two cases.

Case 1. $p = \lim S$.

In this case $S \cap W \in W_c^+$. Thus there exists $\alpha < \kappa$ such that $S \cap W \in \langle \mathcal{V}_\alpha \rangle_c \subseteq W_c^+$. Let $V \in \mathcal{V}_\alpha$ be such that $p \in V$. Then $V \in \mathcal{B}$ and by Lemma 5.3, $p \in V \subseteq W$.

Case 2. $p \neq \lim S$.

Define $S_1 = S \cup \{p\}$. Let U_1 and U_2 be two disjoint open subsets of X such that $p \in U_1$ and $S_1 \setminus \{p\} \subseteq U_2$. Then $S_1 \in \langle \{U_1 \cap W, U_2\} \rangle_c$ and so, there exists $\alpha < \kappa$ such that $S_1 \in \langle \mathcal{V}_\alpha \rangle_c \subseteq \langle \{U_1 \cap W, U_2\} \rangle_c$. According to Lemma 5.2 there exists $V \in \mathcal{V}_\alpha \subseteq \mathcal{B}$ such that $p \in V \subseteq U_1 \cap W$ and the result follows. □

Theorem 5.6. *If X is a space, then $\chi(\mathcal{S}_c(X)) \leq \chi(X)$.*

Proof. Let $S \in \mathcal{S}_c(X)$ and set $a = \lim S$. For each $x \in S$, fix $\mathcal{B}_x$, a local base for X at x, such that $|\mathcal{B}_x| = \chi(x, X)$. Define $\mathfrak{B}_S$ as follows: $\mathsf{B} \in \mathfrak{B}_S$ if and only if there exist $B_a \in \mathcal{B}_a$ and $B_x \in \mathcal{B}_x$, for each $x \in S \setminus B_a$, such that

$$\mathsf{B} = \langle \{B_x : x \in (S \setminus B_a) \cup \{a\}\} \rangle_c.$$

Since a is not an isolated point of X, $\chi(a, X) \geq \omega$ and so we deduce that $|\mathfrak{B}_S| \leq \sup\{\chi(x, X) : x \in S\} \leq \chi(X)$.

In order to prove that $\mathfrak{B}_S$ is a local base for $\mathcal{S}_c(X)$ at S, assume that $\mathcal{U} \in \mathfrak{C}(X)$ satisfies that $S \in \langle \mathcal{U} \rangle_c$. Start by letting $V \in \mathcal{U}$ and $B_a \in \mathcal{B}_a$ be so that $B_a \subseteq V$. Now, for each $x \in S \setminus B_a$ let $B_x \in \mathcal{B}_x$ be such that $B_x \subseteq U$, for some $U \in \mathcal{U}$. Therefore, $\mathsf{B} = \langle \{B_x : x \in (S \setminus B_a) \cup \{a\}\} \rangle_c \in \mathfrak{B}_S$ and $S \in \mathsf{B} \subseteq \langle \mathcal{U} \rangle_c$. □

Theorem 5.7. *Let X be a space and let $S \in \mathcal{S}_c(X)$. If $y \in S \setminus \{\lim S\}$, then $\chi(y, X) \leq \chi(S, \mathcal{S}_c(X))$.*

Proof. Set $\kappa = \chi(S, \mathcal{S}_c(X))$. Proposition 3.2 implies that for each $\alpha < \kappa$ there exists $\mathcal{V}_\alpha \in \mathfrak{C}(X)$ such that $\{\langle \mathcal{V}_\alpha \rangle_c : \alpha < \kappa\}$ is a local base at S. In particular, for each $\alpha < \kappa$ there exists a unique $V_\alpha \in \mathcal{V}_\alpha$ that contains y. Define $\mathcal{B} = \{V_\alpha : \alpha < \kappa\}$.

In order to prove that $\mathcal{B}$ is a local base at y, fix an open set W such that $y \in W$. Let U_1 and U_2 be disjoint open subsets of X such that $y \in U_1$ and $S \setminus \{y\} \subseteq U_2$. Observe that $S \in \langle \{U_1 \cap W, U_2\} \rangle_c$. Hence, there exists $\beta < \kappa$ such that $S \in \langle \mathcal{V}_\beta \rangle_c \subseteq \langle \{U_1 \cap W, U_2\} \rangle_c$. Thus, by Lemma 5.2 there exists $V \in \mathcal{V}_\beta$ such that $y \in V \subseteq U_1 \cap W$. Hence, $V = V_\beta$ and, therefore, $\mathcal{B}$ is a local base at y. The conclusion follows from the fact that $|\mathcal{B}| \leq \kappa$. □

In the rest of this section we will use the following notation: for a space X, we denote by L_X the set $\{\lim S : S \in \mathcal{S}_c(X)\}$.

Corollary 5.8. *If X is a space such that $|L_X| \geq 2$, then $\chi(X) = \chi(\mathcal{S}_c(X))$.*

Proof. Given $y \in X$, by assumption there exists $S_0 \in \mathcal{S}_c(X)$ such that $y \neq \lim S_0$. Set $S = S_0 \cup \{y\}$. Theorem 5.7 implies that $\chi(y, X) \leq \chi(S, \mathcal{S}_c(X))$. Therefore $\chi(X) \leq \chi(\mathcal{S}_c(X))$ and the result follows from Theorem 5.6. □

There exist many spaces X such that $|L_X| = 1$ and $\chi(X) = \chi(\mathcal{S}_c(X))$, as we now show. Let $\kappa > 0$ be a cardinal, let $S(\kappa)$ be the sequential fan of κ spines (see Proposition 3.7) and let $q : (\omega+1)\times\kappa \to S(\kappa)$ be the natural quotient map. Set $S = q\big[(\omega+1)\times 1\big]$. Note that $q\big[(\omega+1)\times(\kappa\setminus 1)\big] \cap S$ has exactly one element. Since $q\big[(\omega+1)\times(\kappa\setminus 1)\big]$ is homeomorphic to $S(\kappa)$, Lemma 5.4 and Theorem 5.6 imply that $\chi(S(\kappa)) = \chi\big(\mathcal{S}_c(S(\kappa))\big)$. However, the authors do not know the answer to the following question.

Question 5.9. Is it true that $\chi(X) = \chi(\mathcal{S}_c(X))$ for every space X such that $\mathcal{S}_c(X)$ is nonempty?

Lemma 5.10. *If X is a space such that $\mathcal{S}_c(X) \neq \emptyset$, then $d(\mathcal{S}_c(X)) \geq \omega$.*

Proof. Fix $S \in \mathcal{S}_c(X)$ and note that $\big\{S \setminus \{y\} : y \in S \setminus \{\lim S\}\big\} \subseteq \mathcal{S}_c(X)$, thus $\mathcal{S}_c(X)$ is infinite. Moreover, $\mathcal{S}_c(X)$ is Hausdorff (see [10, 4.9.8, p. 164]). The result follows from the fact that Hausdorff spaces of finite density are finite. □

Theorem 5.11. *Let X be a nonempty space. Then $\mathcal{S}_c(X) \neq \emptyset$ if and only if $d(X) \leq d(\mathcal{S}_c(X))$.*

Proof. Assume that $\mathcal{S}_c(X) \neq \emptyset$. Set $\kappa = d(\mathcal{S}_c(X))$ and let D be a dense subset of $\mathcal{S}_c(X)$ of cardinality κ. Define $D = \bigcup \mathsf{D}$ and observe that Lemma 5.10 implies that $|D| \leq \kappa$. In order to show that D is dense in X, let U be a nonempty open subset of X. By assumption there exists $S \in \mathcal{S}_c(X)$ and we may take $y \in U$. Observe that $S \cup \{y\} \in U_c^-$ (i.e., U_c^- is a nonempty open subset of $\mathcal{S}_c(X)$), thus there exists $S_1 \in \mathsf{D} \cap U_c^-$. Hence $\emptyset \neq S_1 \cap U \subseteq D \cap U$. Therefore, $d(X) \leq d(\mathcal{S}_c(X))$.

The converse is immediate. □

Lemma 5.12. *If X is a space, then $d(L_X) \leq d(\mathcal{S}_c(X))$.*

Proof. If $\mathcal{S}_c(X)$ is empty so is L_X, thus we may assume that $\mathcal{S}_c(X) \neq \emptyset$. Also, if $|L_X| < \omega$, then $d(L_X) = |L_X| < d(\mathcal{S}_c(X))$ by Lemma 5.10. Hence, let us assume that L_X is infinite (and so, $d(L_X) \geq \omega$).

Let E be a subset of $\mathcal{S}_c(X)$ such that $|\mathsf{E}| < d(L_X)$. We will show that E is not dense in $\mathcal{S}_c(X)$. Note that $|\bigcup \mathsf{E}| < d(L_X)$. Thus there exists a nonempty open subset V of L_X such that $(\bigcup \mathsf{E}) \cap V = \emptyset$. Let $U \in \tau_X$ be such that $U \cap L_X = V$. Since $U \cap L_X \neq \emptyset$, it is easy to see that U_c^+ is a nonempty open subset of $\mathcal{S}_c(X)$. On the other hand, if $S \in \mathsf{E}$, then $\lim S \in (\bigcup \mathsf{E}) \cap L_X$, which implies that $S \notin U_c^+$. Therefore, E is not dense in $\mathcal{S}_c(X)$. □

In our following example we construct a space X such that $d(X) < d(\mathcal{S}_c(X))$. To this end we consider the ordinal ω with the discrete topology, we denote by $\beta\omega$ its Stone-Čech compactification and by ω^* the remainder of such compactification. The ordinal $\omega + 1$ is considered as a linearly ordered topological space.

Example 5.13. Let $X = \big(\beta\omega \times (\omega+1)\big) \setminus (\omega \times \{\omega\})$. Recall that $\mathcal{S}_c(\beta\omega) = \emptyset$ (see [4, Corollary 3.6.15, p. 175]). Hence, $L_X = \omega^* \times \{\omega\}$, thus $d(L_X) = \mathfrak{c}$ (see [4, Corollary 3.6.12, Example 3.6.18, p. 175 and 1.7.12 (a), p. 60]). Note that $d(X) = \omega$, therefore by Lemma 5.12 we obtain that $d(X) < \mathfrak{c} \leq d(\mathcal{S}_c(X))$.

Lemma 5.14. *For any space X, $d(\mathcal{S}_c(X)) \leq d(X)d(L_X)$.*

Proof. We may assume that $\mathcal{S}_c(X) \neq \emptyset$. Recall that Hausdorff spaces of finite density are finite; thus, since $\mathcal{S}_c(X) \neq \emptyset$ we infer that $d(X) \geq \omega$.

Fix dense subsets E and P of X and of L_X, respectively, such that $|E| = d(X)$ and $|P| = d(L_X)$. For each $x \in P$ fix $S_x \in \mathcal{S}_c(X)$ such that $\lim S_x = x$. Define

$$\mathsf{G} = \{(S_x \setminus F) \cup H : (x \in P) \wedge (F \in [S_x \setminus \{x\}]^{<\omega}) \wedge (H \in [E]^{<\omega})\}.$$

Since $d(X) \geq \omega$, we deduce that $|\mathsf{G}| = |E||P| = d(X)d(L_X)$. Let us show that G is dense in $\mathcal{S}_c(X)$. Fix $\mathcal{U} \in [\tau_X]^{<\omega}$ with $\langle \mathcal{U} \rangle_c \neq \emptyset$, let $S \in \langle \mathcal{U} \rangle_c$ and fix $U \in \mathcal{U}$ such that $\lim S \in U$. Since $U \cap L_X \neq \emptyset$, we may take $x \in U \cap P$. Set $H = \operatorname{ran}(e)$, where e is a choice function for $\{V \cap E : V \in \mathcal{U}\}$. Finally, define $F = S_x \setminus U$, to obtain that $(S_x \setminus F) \cup H \in \mathsf{G} \cap \langle \mathcal{U} \rangle_c$. Therefore, $d(\mathcal{S}_c(X)) \leq d(X)d(L_X)$. □

Corollary 5.15. *Let X be a space such that $\mathcal{S}_c(X) \neq \emptyset$. Then $d(X) = d(\mathcal{S}_c(X))$ if and only if $d(L_X) \leq d(X)$.*

Proof. The necessity follows from Lemma 5.12. The fact that $d(X) \geq \omega$, Theorem 5.11 and Lemma 5.14 imply the sufficiency. □

Corollary 5.16. *If X is a crowded sequential space, then $d(X) = d(\mathcal{S}_c(X))$.*

Proof. Since X is crowded and sequential, we have that $L_X = X$; in particular $\mathcal{S}_c(X) \neq \emptyset$ and the result follows from Corollary 5.15. □

6. Dimension

It is a well-known result that the property of a space being both metrizable and separable is $\mathcal{K}$-preserved and therefore, it is also $\mathcal{S}_c$-preserved (see Theorem 5.5).

Given a metric separable space X, its *topological dimension*, $\dim(X)$, should be understood as in [7]. Note that $\dim(\mathcal{K}(X))$ and $\dim(\mathcal{S}_c(X))$ make sense by the remarks from the previous paragraph.

Lemma 6.1. *Let X be a space and assume that $\{L_n : n \in \omega\}$ is a sequence of pairwise disjoint elements of $\mathcal{K}(X)$ that converges to a singleton $\{x_\omega\}$. Then $\prod_{n\in\omega} L_n$ can be embedded in $\mathcal{S}_c(X)$.*

Proof. Set $P = \prod_{n\in\omega} L_n$ and for each $n \in \omega$ denote by $\pi_n : P \to L_n$ the n^{th} projection. Since $\{L_n : n \in \omega\}$ is pairwise disjoint, the function $g : P \to \mathcal{S}_c(X)$ given by $g(\mathbf{s}) = \mathrm{ran}(\mathbf{s}) \cup \{x_\omega\}$ is well defined and one-to-one. Next we will show that g is continuous. To this end fix $\mathbf{s} \in P$ and $\mathcal{U} \in \mathfrak{C}(X)$ such that $g(\mathbf{s}) \in \langle\mathcal{U}\rangle_c$. Take $U \in \mathcal{U}$ with $x_\omega \in U$ and define $A = \{n \in \omega : L_n \subseteq U\}$. For each $n \in \omega \setminus A$ pick $U_n \in \mathcal{U}$ such that $\mathbf{s}(n) \in U_n$. Define $V = \bigcap_{n\in\omega\setminus A} {\pi_n}^{-1}[U_n \cap L_n]$. Clearly V is an open subset of P and $\mathbf{s} \in V$. Now, if $\mathbf{t} \in V$, it follows easily that $g(\mathbf{t}) \subseteq \bigcup\mathcal{U}$; further, $\mathbf{t}(n) \in g(\mathbf{t}) \cap U_n$ for each $n \in \omega \setminus A$ and $x_\omega \in g(\mathbf{t}) \cap U$, which implies that $g(\mathbf{t}) \in \langle\mathcal{U}\rangle_c$. Therefore g is continuous and, since each L_n is compact, we conclude that g is an embedding. □

Our next result follows from [7, Remark, p. 34].

Lemma 6.2. *If $\{X_1, \ldots, X_n\}$ is a finite family of compact 1-dimensional spaces, then $\dim\left(\prod_{i=1}^n X_i\right) = n$.*

Lemma 6.3. *Each compact metric space X of finite dimension ≥ 1 contains a 1-dimensional continuum.*

Proof. We proceed by induction. If $\dim(X) = 1$, then X has a nondegenerate component (see [7, D), p. 22]) which is a 1-dimensional continuum. Next assume that the lemma holds for n-dimensional compact metric spaces and let X be a compact metric space of dimension $n + 1$. Then there exist $p \in X$ and a neighborhood V of p whose boundary has dimension n; applying the induction hypothesis to such boundary we obtain the desired conclusion. □

By an *arc* in a topological space X we mean a subspace of X which is homeomorphic to the unit interval $[0, 1]$.

Theorem 6.4. *Consider the following conditions for a space X.*

(1) $\mathcal{S}_c(X)$ contains a Hilbert cube, i.e., a copy of the topological product $[0, 1]^\omega$.

(2) *X contains an arc.*
(3) $\dim(X) \neq 0$.
(4) $\dim(\mathcal{S}_c(X)) = \infty$.
(5) $\dim(\mathcal{K}(X)) = \infty$.

Then (1) and (2) are equivalent. Moreover, if X is compact and metric (thus, metric and separable) then (3), (4) and (5) are equivalent.

Proof. Assume that $\mathcal{S}_c(X)$ contains a Hilbert cube Q and let S_1 and S_2 be two distinct elements of Q. We may assume that there exists $z \in S_1 \setminus S_2$. Choose a path $\alpha : [0,1] \to Q$ from S_1 to S_2 and use Lemma 4.4 to get a path $\gamma : [0,1] \to X$ from z to a point of S_2. Since $\operatorname{ran}(\gamma)$ has at least two points, it contains an arc (see [13, Corollary 31.6, p. 222]).

Note that if A is an arc in X, there is a sequence of pairwise disjoint arcs $\{L_n : n \in \omega\} \subseteq \mathcal{K}(A)$ which converges to a singleton and so, Lemma 6.1 guarantees that condition (1) holds.

For the rest of the proof let us assume that X is compact metric. Also, for each $A \subseteq X$, the symbol $\operatorname{diam}(A)$ will denote the diameter of A in X.

Assume (3). Since X is compact, it has a nondegenerate component Y, which is evidently a continuum. Fix $S = \{x_n : n \in \omega + 1\} \in \mathcal{S}_c(Y)$ and a cellular family $\{W_n : n \in \omega\}$ in the subspace $Y \setminus \{x_\omega\}$ with the following properties:

$$S \cap W_n = \{x_n\} \text{ for each } n \in \omega \quad \text{and} \quad \lim_{n\to\infty} \operatorname{diam}(W_n) = 0.$$

According to [11, Corollary 5.5, p. 74], for each $n \in \omega$ there exists a nondegenerate continuum K_n such that $x_n \in K_n \subseteq W_n$. Thus Lemma 6.1 implies that $\prod_{n\in\omega} K_n$ can be embedded in $\mathcal{S}_c(X)$.

If K_m is infinite-dimensional for some $m \in \omega$, condition (4) follows immediately; hence we may assume that each K_n is 1-dimensional for each $n \in \omega$ (Lemma 6.3). In this case (4) follows using Lemma 6.2. Further, since $\mathcal{S}_c(X) \subseteq \mathcal{K}(X)$ then (4) implies (5). Finally, it is known that if X is 0-dimensional so is $\mathcal{K}(X)$ ([10, 4.13.1, p. 166]); therefore (5) implies (3). $\square$

Question 6.5. Let X be a metric separable space. Is it true that either $\dim(\mathcal{S}_c(X)) \in \{-1, 0\}$ or $\dim(\mathcal{S}_c(X)) = \infty$? In other words, can the assumption on the compactness of X be removed in Theorem 6.4?

7. Inducible mappings

Let X and Y be spaces and let $f : X \to Y$ be a mapping. For a fixed hyperspace $\mathcal{H}(X) \in \{\mathcal{K}(X), \mathcal{S}_c(X), \mathcal{C}(X), \mathcal{F}(X)\} \cup \{\mathcal{F}_n(X) : n \in \mathbb{N}\}$ define the *$\mathcal{H}$-induced mapping* $\mathcal{H}(f) : \mathcal{H}(X) \to \mathcal{K}(Y)$ by

$$\mathcal{H}(f)(A) = f[A] \text{ for each } A \in \mathcal{H}(X).$$

It is known that $\mathcal{K}(f)$ is continuous (see [10, 5.10.1 of Theorem 5.10, p. 170]) and $\operatorname{ran}(\mathcal{H}(f)) \subseteq \mathcal{H}(Y)$ whenever $\mathcal{H}(X) \in \{\mathcal{K}(X), \mathcal{C}(X), \mathcal{F}(X)\} \cup \{\mathcal{F}_n(X) : n \in \mathbb{N}\}$. Now, if $\mathcal{H}(X) \in \{\mathcal{S}_c(X), \mathcal{C}(X), \mathcal{F}(X)\} \cup \{\mathcal{F}_n(X) : n \in \mathbb{N}\}$, then $\mathcal{H}(X) \subseteq \mathcal{K}(X)$ and $\mathcal{H}(f)$ is the restriction of $\mathcal{K}(f)$ to $\mathcal{H}(X)$; thus, $\mathcal{H}(f)$ is continuous.

In connection with the concept of induced mapping, it is natural to ask under what conditions an arbitrary mapping between the hyperspaces $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ is an induced one. An answer to this question is presented in this section when $\mathcal{H}(X) = \mathcal{S}_c(X)$. This question has been studied for other hyperspaces in [1], [2] and [3].

7.1. **Auxiliary results.** First, we characterize the mappings f from a sequential space into a space Y such that $\operatorname{ran}(\mathcal{S}_c(f)) \subseteq \mathcal{S}_c(Y)$.

A continuous function f between topological spaces is called a *strong light map* provided that each fiber $f^{-1}(y)$ is discrete.

Theorem 7.1. *Let X and Y be spaces and let $f : X \to Y$ be a map. If f is strong light, then $f[S] \in \mathcal{S}_c(Y)$ for every $S \in \mathcal{S}_c(X)$.*

Proof. Let $S \in \mathcal{S}_c(X)$. By the continuity of f, to prove that $f[S] \in \mathcal{S}_c(Y)$ it suffices to verify that $f[S]$ is infinite countable. First, notice that $|f[S]| \leq |S| = \omega$, so $f[S]$ is countable. Now, to check that $f[S]$ is infinite, we assume to the contrary that $f[S]$ is finite; hence there exists $y \in f[S]$ such that $Q = S \cap f^{-1}(y)$ is infinite. Since Q is closed and $Q \subseteq S$, we have that $Q \in \mathcal{S}_c(X)$. Thus, $f^{-1}(y)$ can not be discrete because $\{\lim Q\}$ is not an open subset of $f^{-1}(y)$. $\square$

Theorem 7.2. *Assume X and Y are topological spaces and let $f : X \to Y$ be a (not necessarily continuous) function. If X is sequential and*

$$\{f[S] : S \in \mathcal{S}_c(X)\} \subseteq \mathcal{S}_c(Y),$$

then f is a strong light map.

Proof. Suppose $(x_n)_{n\in\omega}$ is a sequence in X converging to some $x \in X$. We claim that $(f(x_n))_{n\in\omega}$ converges to $f(x)$ in Y (observe that this claim, together with [4, Proposition 1.6.15, p. 53], implies that f is continuous). Set $S = \{x_n : n \in \omega\} \cup \{x\}$. We have two cases.

First, if $S \notin \mathcal{S}_c(X)$, then S is finite and so there is $m \in \omega$ satisfying $\{x_n : n \in \omega \setminus m\} \cup \{x\} = \{x\}$. Clearly, this equality implies that our claim holds.

Now, if $S \in \mathcal{S}_c(X)$, our hypotheses give $f[S] \in \mathcal{S}_c(Y)$. Let z be the limit of $f[S]$. Seeking a contradiction, assume $z \neq f(x)$ and set $A = S \cap f^{-1}(z) \subseteq S \setminus \{x\}$. If A were finite, $S \setminus A \in \mathcal{S}_c(X)$, but $f[S \setminus A] = f[S] \setminus \{z\} \notin \mathcal{S}_c(Y)$. On the other hand, A being infinite, would imply $A \cup \{x\} \in \mathcal{S}_c(X)$ and $f[A \cup \{x\}] = \{z, f(x)\} \notin \mathcal{S}_c(Y)$. Therefore, $z = f(x)$ and this completes the proof of our claim.

In order to prove that all fibers of f are discrete, suppose $w \in X$ is an accumulation point of $f^{-1}(y)$, for some $y \in Y$. Then, $A = f^{-1}(y) \setminus \{w\}$ is not a closed subset of X and so, by assumption, there exists $(w_n)_{n\in\omega}$, a sequence in A which converges to some $a \in X \setminus A$. Since $\overline{A} = f^{-1}(y)$, we deduce that $a = w$, i.e., $S = \{w_n : n \in \omega\} \cup \{w\} \in \mathcal{S}_c(X)$, but $f[S] = \{y\} \notin \mathcal{S}_c(Y)$; a contradiction. $\square$

According to [4, Corollary 3.6.15, p. 175] and [4, 3.6.A, p. 179], there are spaces which are infinite and crowded, but have no non-trivial convergent sequences. We will use this information in our next example, which shows that the assumption on Theorem 7.2 of X being sequential is essential.

Example 7.3. Fix M, a non-empty crowded space with $\mathcal{S}_c(M) = \emptyset$, and set $X = M \oplus (\omega + 1)$, where the ordinal $\omega + 1$ is endowed with the order topology. By letting $f : X \to \omega + 1$ be such that $f[M] = \{f(\omega)\} = \{\omega\}$ and $f(n) = n$, for each $n \in \omega$, we obtain a continuous function. Also, $S \in \mathcal{S}_c(X)$ implies that the set $S \cap M$ is finite and so, $\{f[S] : S \in \mathcal{S}_c(X)\} \subseteq \mathcal{S}_c(\omega + 1)$, but f fails to be a strong light map.

The next result follows from theorems 7.1 and 7.2.

Corollary 7.4. *Let X be a sequential space and Y be a topological space. If $f : X \to Y$ is an arbitrary function, the following statements are equivalent.*

(1) f is a strong light map.
(2) $\{f[S] : S \in \mathcal{S}_c(X)\} \subseteq \mathcal{S}_c(Y)$.

Theorem 7.5. *Suppose f is a function from the sequential space X into the topological space Y. If $\{f[S] : S \in \mathcal{S}_c(X)\} \subseteq \mathcal{S}_c(Y)$, then f is continuous and $\mathcal{S}_c(f)$ is a continuous function from $\mathcal{S}_c(X)$ into $\mathcal{S}_c(Y)$.*

Proof. The continuity of f follows from Theorem 7.2. By [10, 5.10.1 of Theorem 5.10, p. 170], $\mathcal{K}(f)$ is continuous. So, since $\mathcal{S}_c(f)$ is the restriction of $\mathcal{K}(f)$ to $\mathcal{S}_c(X)$, we can conclude that $\mathcal{S}_c(f)$ is continuous and, by assumption, $\operatorname{ran}(\mathcal{S}_c(f)) \subseteq \mathcal{S}_c(Y)$. $\square$

Suppose Y is a strong light image of a space X (i.e., there is a strong light map from X onto Y). Since Y is, in particular, a continuous image of X, we have that $d(Y) \leq d(X)$. Thus, it is natural to ask if the extra assumption of the map being strong light would imply a fixed inequality between $\phi(X)$ and $\phi(Y)$, where ϕ is one of the cardinal functions we discussed in the previous section. Our next examples show that the answer is *no*.

Example 7.6. Let $J(\omega_1)$ be the hedgehog of ω_1 spines (see the proof of Proposition 3.7) and I be the unit interval, $[0,1]$. Denote by $q : I \times \omega_1 \to J(\omega_1)$ the natural quotient map and define $f : J(\omega_1) \to I$ by $(f \circ q)(t, \alpha) = t$, for each $(t, \alpha) \in I \times \omega_1$. Observe that f is a surjective strong light map and $J(\omega_1)$ is a crowded Fréchet-Urysohn space. Moreover, a minor modification of the argument used in [4, Example 1.4.17, p. 33] shows that $\chi(J(\omega)) > \omega$ and since $J(\omega_1)$ contains a copy of $J(\omega)$, we deduce that

$$w(I) = \chi(I) < \chi(J(\omega)) \leq \chi(J(\omega_1)) \leq w(J(\omega_1)).$$

Also, $d(J(\omega_1)) = \omega_1 > d(I)$.

Example 7.7. Let $q : I \times \omega \to J(\omega)$ be the natural quotient map. Then, q is a strong light map, $I \times \omega$ is a crowded metrizable space, and

$$w(I \times \omega) = \chi(I \times \omega) < \chi(J(\omega)) \leq w(J(\omega)).$$

Also, $d(I \times \omega) = d(J(\omega))$.

7.2. **Main Theorems.** Let X and Y be spaces and let $\mathcal{H}(X)$ be a hyperspace. A mapping $g : \mathcal{H}(X) \to \mathcal{H}(Y)$ is said to be *inducible* provided that there exists a mapping $f : X \to Y$ such that $g = \mathcal{H}(f)$. In [2, Theorem 5.2, p. 256], the author characterizes inducible mappings when X and Y are continua and $\mathcal{H}(X) \in \{\mathcal{K}(X), \mathcal{C}(X), \mathcal{F}(X)\} \cup \{\mathcal{F}_n(X) : n \in \mathbb{N}\}$ (see also [1, Theorem 2.1, p. 105] and [3, Theorem 2.2, p. 7]). In order to find necessary and sufficient conditions for a mapping between the hyperspaces $\mathcal{S}_c(X)$ and $\mathcal{S}_c(Y)$ to be an induced one, it is convenient to introduce some concepts and notation.

Throughout this section, X and Y denote a crowded sequential space and a space, respectively.

Let $g : \mathcal{S}_c(X) \to \mathcal{S}_c(Y)$ be a map. Define the relation ℓ_g as follows: $(x, y) \in \ell_g$ if and only if there exists $S \in \mathcal{S}_c(X)$ such that $\lim S = x$ and $\lim g(S) = y$. Since X is crowded and sequential, then each point of X is the limit point of an element of $\mathcal{S}_c(X)$, hence, the domain of ℓ_g is X.

Recall that $\ell_g[A]$ is the set $\{y \in Y : \exists x \in A((x, y) \in \ell_g)\}$, whenever $A \subseteq X$.

Consider the following properties:

$(1)_g$ if $S, Q \in \mathcal{S}_c(X)$ satisfy $\lim S = \lim Q$, then $\lim g(S) = \lim g(Q)$.
$(2)_g$ $g^{-1}[\{y\}_c^-] \subseteq \{S \in \mathcal{S}_c(X) : y \in \ell_g[S]\}$ for every $y \in Y$.

Clearly, ℓ_g is a function if and only if condition $(1)_g$ holds.

Lemma 7.8. *Let $g : \mathcal{S}_c(X) \to \mathcal{S}_c(Y)$ be a map. If g satisfies $(1)_g$, then $\ell_g[S] \subseteq g(S)$ for every $S \in \mathcal{S}_c(X)$.*

Proof. Let $S \in \mathcal{S}_c(X)$ and $y \in \ell_g[S]$. Then, there exists $x \in S$ such that $\ell_g(x) = y$. Let $Q \in \mathcal{S}_c(X)$ be such that $\lim Q = x$. Denote $\lim S$ by a. Assume that $Q = \{q_n : n \in \omega\} \cup \{x\}$ and $S = \{s_n : n \in \omega\} \cup \{a\}$. For each $k \in \omega$, define $S_k = \{q_n : n \in \omega \setminus k\} \cup \{s_n : n \in k\} \cup \{x, a\}$. Notice that $S_k \in \mathcal{S}_c(X)$, that $\lim S_k = x$ for every $k \in \omega$ and that $\lim\limits_{k\to\infty} S_k = S$. Thus, by the continuity of g, $\lim\limits_{k\to\infty} g(S_k) = g(S)$. Now, since $\lim g(S_k) = \ell_g(x) = y$, we obtain that $y \in g(S_k)$ for every $k \in \omega$. Hence, by Proposition 3.9, we can conclude that $y \in g(S)$. This ends the proof. □

Theorem 7.9. *Let $g : \mathcal{S}_c(X) \to \mathcal{S}_c(Y)$ be a map. Then, g is inducible if and only if $(1)_g$ and $(2)_g$ are satisfied.*

Proof. First, suppose that there exists a map $f : X \to Y$ such that $g = \mathcal{S}_c(f)$. In order to see that $\mathcal{S}_c(f)$ satisfies $(1)_{\mathcal{S}_c(f)}$, let $x \in X$ and $S \in \mathcal{S}_c(X)$ be such that $\lim S = x$. Then, $\lim \mathcal{S}_c(f)(S) = f(\lim S) = f(x)$. Thus, $\ell_{\mathcal{S}_c(f)} = f$ and $\ell_{\mathcal{S}_c(f)}$ is a function. Hence, condition $(1)_{\mathcal{S}_c(f)}$ holds. Now, take $y \in Y$. We have that

$$\{S \in \mathcal{S}_c(X) : y \in \ell_{\mathcal{S}_c(f)}[S]\} = \{S \in \mathcal{S}_c(X) : y \in f[S]\} = \mathcal{S}_c(f)^{-1}[\{y\}_c^-].$$

Therefore, $(2)_{\mathcal{S}_c(f)}$ is satisfied.

Next, we assume that $(1)_g$ and $(2)_g$ are true. Then, ℓ_g is a function. We are going to verify that $g(S) = \ell_g[S]$ for every $S \in \mathcal{S}_c(X)$. Let $S \in \mathcal{S}_c(X)$ and $y \in g(S)$. Then, $S \in g^{-1}[\{y\}_c^-]$. Since $(2)_g$ is satisfied, we obtain that $y \in \ell_g[S]$. Thus, $g(S) \subseteq \ell_g[S]$. By Lemma 7.8, we have that $\ell_g[S] \subseteq g(S)$. Therefore, $\ell_g[S] = g(S) \in \mathcal{S}_c(Y)$ for every $S \in \mathcal{S}_c(X)$. So, by Theorem 7.2, ℓ_g is a strong light map. This proves that $g = \mathcal{S}_c(\ell_g)$ and so g is inducible. □

7.3. **Examples.** The following examples show that properties $(1)_g$ and $(2)_g$ are independent in the sense that none of them is implied by the other.

Example 7.10. There exists a map $g : \mathcal{S}_c([0,1]) \to \mathcal{S}_c(\mathbb{R})$ satisfying $(1)_g$ but not $(2)_g$.

Define $g : \mathcal{S}_c([0,1]) \to \mathcal{S}_c(\mathbb{R})$ by $g(S) = S \cup \{-1\}$ for every $S \in \mathcal{S}_c([0,1])$. First, we shall prove the continuity of g. Let $\varphi : \bigoplus_{m\in\mathbb{N}} \mathcal{K}(\mathbb{R})^m \to \mathcal{K}(\mathbb{R})$ be as in Lemma 2.1. Notice that $g(S) = \varphi(S, \{-1\})$ for every $S \in \mathcal{S}_c([0,1])$. Hence, since φ is continuous, so is g. Now, from the fact that $\lim g(S) = \lim S$ for every $S \in \mathcal{S}_c([0,1])$, we have that ℓ_g is a function. So, $(1)_g$ is satisfied. Clearly, $g^{-1}[\{-1\}_c^-] = \mathcal{S}_c([0,1])$. Since $\ell_g(x) = x$ for every $x \in [0,1]$, we obtain that $\{S \in \mathcal{S}_c([0,1]) : -1 \in \ell_g[S]\} = \emptyset$. Thus, $(2)_g$ is not satisfied.

Example 7.11. Consider the following subspaces of $\mathbb{R}$: $X = (0,1) \cup (1,2)$ and $Y = (-1,0) \cup X$. We will exhibit a map $g : \mathcal{S}_c(X) \to \mathcal{S}_c(Y)$ satisfying $(2)_g$ but not $(1)_g$.

Define $f : (0,1) \to Y$ by $f(x) = -x$ and let $i : X \to Y$ be the inclusion map. For each $S \in \mathcal{S}_c(X)$, let

$$g(S) = \begin{cases} \mathcal{S}_c(f)(S), & \text{if } S \in (0,1)_c^+, \\ \mathcal{S}_c(i)(S), & \text{if } S \in (1,2)_c^-. \end{cases}$$

Clearly, f and i are strong light maps. Hence, by Corollary 7.4 and Theorem 7.5, $\mathcal{S}_c(f) : \mathcal{S}_c((0,1)) \to \mathcal{S}_c(Y)$ and $\mathcal{S}_c(i) : \mathcal{S}_c(X) \to \mathcal{S}_c(Y)$ are continuous. From the fact that $(0,1)_c^+$ and $(1,2)_c^-$ are disjoint open subsets of $\mathcal{S}_c(X)$ whose union is $\mathcal{S}_c(X)$, it follows that g is well defined and continuous.

Next, let $A \in (0,1)_c^+$ and set $B = A \cup \{\frac{3}{2}\}$. We have that $B \in (1,2)_c^-$, $\lim A = \lim B$, $\lim g(A) = -\lim A$ and $\lim g(B) = \lim A$. Hence, g does not satisfy $(1)_g$.

Now, to see that $(2)_g$ is satisfied, let $p \in Y$ and let $S \in g^{-1}[\{p\}_c^-]$. Notice that $(|p|, p) \in \ell_g$ and $|p| \in S$. Then, $p \in \ell_g[S]$. This finishes the proof.

We conclude this paper with some final remarks and questions.

Let $\mathcal{H}(X) \in \{\mathcal{K}(X), \mathcal{C}(X), \mathcal{S}_c(X), \mathcal{F}(X)\} \cup \{\mathcal{F}_n(X) : n \in \mathbb{N}\}$. Given two mappings between hyperspaces $g_1, g_2 : \mathcal{H}(X) \to \mathcal{H}(Y)$, we will write $g_1 \prec g_2$ provided that $g_1(A) \subseteq g_2(A)$ for every $A \in \mathcal{H}(X)$. In [2, Theorem 5.2, p. 256], the partial order $\prec$ on the set of all mappings between $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ when X and Y are continua and $\mathcal{H}(X) \in \{\mathcal{K}(X), \mathcal{C}(X), \mathcal{F}(X)\} \cup \{\mathcal{F}_n(X) : n \in \mathbb{N}\}$ is used to classify the inducible mappings (see also [1, Theorem 2.1, p. 105] and [3, Theorem 2.2, p. 7]). Let X be a crowded sequential space, let Y be a space and let $g : \mathcal{S}_c(X) \to \mathcal{S}_c(Y)$ be a mapping. Consider the following properties:

- $(A)_g$ The set $\{x \in X : \exists Q \in \mathcal{S}_c(X)(\lim Q = x \wedge \lim g(Q) = y)\}$ is a discrete subspace of X for every $y \in Y$.
- $(B)_g$ g is minimal with respect to $\prec$ and $(1)_g$ (i.e., if a mapping $g_0 : \mathcal{S}_c(X) \to \mathcal{S}_c(Y)$ satisfies $(1)_g$ and $g_0 \prec g$, then $g_0 = g$).

Question 7.12. Assume that g satisfies $(1)_g$, $(A)_g$ and $(B)_g$. Does g satisfy $(2)_g$?

In [5, p. 796], the authors ask if $\mathcal{S}_c(\mathbb{Q})$ and $\mathcal{S}_c(\mathbb{R} \setminus \mathbb{Q})$ are homeomorphic. In connection with that question, we think the following are interesting too.

Question 7.13. Are $\mathcal{S}_c(\mathbb{R}^2)$ and $\mathcal{S}_c(\mathbb{R}^3)$ homeomorphic?

Question 7.14. More generally: Do there exist two distinct natural numbers n and m such that $\mathcal{S}_c(\mathbb{R}^n)$ and $\mathcal{S}_c(\mathbb{R}^m)$ are homeomorphic?

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