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PERTURBATIONS OF NORMS ON C^1-FUNCTION SPACES AND ASSOCIATED ISOMETRY GROUPS

KAZUHIRO KAWAMURA

Abstract. We study some families of norms and isometry groups on $C^1([0,1])$ and $C^1(\mathbb{T})$, the spaces of all complex-valued C^1-functions on the unit interval $[0,1]$ and the unit circle $\mathbb{T}$, with the C^1-topology. The norms studied in the present paper are all equivalent, while their isometry groups are rather different. "Contiuous interpolations" among these norms are introduced and perturbations of the associated isometry groups are studied.

1. Introduction and Preliminaries

The present paper deals with isometry groups associated with norms defined on C^1- function spaces. Two equivalent norms $\|\cdot\|_0$ and $\|\cdot\|_1$ on a continuous function space $F(X)$ over a compact Hausdorff space X may have rather different isometry groups. As an attempt to understand how they are similar or different, and how they are related with each other, we consider a "continuous path" $(\|\cdot\|_t)_{0\leq t\leq 1}$ of norms on $F(X)$ and study how the isometry groups $\mathcal{U}_t$ with respect to $\|\cdot\|_t$ "vary along the path." For example we may ask the following question.

Question 1 ([6]). Let T be an isometry with respect to $\|\cdot\|_0$. Does there exists a "continuous collection" $(T_t)_{0\leq t\leq 1}$ of linear operators such that $T_0 = T$ and T_t is a $\|\cdot\|_t$-isometry for each $t \in [0,1]$?

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A closely related question is:

Question 2. Let T be an isometry with respect to $\|\cdot\|_0$ on $F(X)$ and assume that $\|\cdot\|_1$ is "close" to $\|\cdot\|_0$. Does there exists an isometry T' with respect to $\|\cdot\|_1$ so that T' is "close" to T?

In order to study these questions, we need to specify the notion of continuity as well as the function spaces under consideration. This paper focuses on the spaces $C^1([0,1])$ and $C^1(\mathbb{T})$, the spaces of all complex-valued C^1-functions on the unit interval $[0,1]$ and the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, both endowed with the C^1-topology. Norms that will be studied include the following whose isometry groups have been studied for $C^1([0,1])$ in [1], [2], [3], [8], [11], [12] etc. Let $X = [0,1]$ or $\mathbb{T}$. For a function $f \in C^1(X)$ and for a point $c \in X$, let:

$$
\begin{aligned}
&\|f\|_\Sigma = ||f||_\infty + ||f'||_\infty, \\
&\|f\|_C = \sup\nolimits_{0\le t\le 1}(|f(t)| + |f'(t)|), \\
&\|f\|_{\sigma,c} = |f(c)| + ||f'||_\infty, \\
&\|f\|_M = \max(\|f\|_\infty, \|f'\|_\infty).
\end{aligned}
\tag{1.1}
$$

It turns out that answers to the above questions are related with the topology of the homeomorphism group $\mathrm{Homeo}([0,1])$, the mapping space $\mathrm{Map}([0,1],\mathbb{T})$, and the isometry group $\mathrm{Isom}(\mathbb{T})$.

The rest of this section fixes notation and defines families of norms that are studied in what follows. Section 2 studies norms and isometry groups on $C^1([0,1])$ on the basis of some previous results ([5],[6]) which is stated as Theorem 2.1. Section 3 and Section 4 deal with the space $C^1(\mathbb{T})$. We prove in Section 3 a characterization theorem of isometries on $C^1(\mathbb{T})$ with respect to norms under consideration, a counterpart to Theorem 2.1. We then proceed in Section 4 to studying the questions above for $C^1(\mathbb{T})$.

For a compact Hausdorff space X, $C(X)$ denotes the space of all *complex-valued* continuous functions on X with the supremum norm $\|\cdot\|_\infty$. The unit interval $[0,1]$ is endowed with the standard metric. Let $\mathbb{T}$ be the unit circle on the complex plane: $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, with the metric $d\theta/2\pi$. The covering map $\pi : \mathbb{R} \to \mathbb{T}$ defined by $\pi(t) = \exp(2\pi i t),\ t \in \mathbb{R}$, is a local isometry. For a function $f : \mathbb{T} \to \mathbb{C}$, we define its derivative by

$$
f'(e^{2\pi i\theta}) = \frac{1}{2\pi}\frac{d}{dt}\Big|_{t=\theta} f(\exp(2\pi i t)). \tag{1.2}
$$

We say that the function f is a C^1-function if the above derivative exists and is continuous.

For $X = [0,1]$ or $\mathbb{T}$, $C^1(X)$ denotes the space of all *complex-valued* C^1-functions on X with the C^1-topology; for a map $f \in C^1(X)$, the collection of the sets of the form:

$$\{g \in C^1(X) \mid \|g - f\|_\infty < \epsilon,\ \|g' - f'\|_\infty < \epsilon\}$$

form a neighborhood basis of f. Following [6] and [5] we define families of norms on $C^1(X)$ ($X = [0,1]$ or $\mathbb{T}$) as follows.

(i) The i-th projection $X \times X \to X$ of the product $X \times X$ onto X is denoted by p_i, $i = 1, 2$. For a compact connected subset D of $X \times X$ with $p_2(D) = X$, we define

$$\|f\|_{<D>} = \sup_{(x,y)\in D}(|f(x)| + |f'(y)|),\ f \in C^1(X). \tag{1.3}$$

As was shown in [6, Section 1], for each pair of compact connected subsets D_1, D_2 with $p_2(D_1) = p_2(D_2) = X$, the norms $\|\cdot\|_{<D_1>}$ and $\|\cdot\|_{<D_2>}$ are equivalent and both induce the C^1-topology.

(ii) For $p \in [1, \infty]$, let

$$\|f\|_{[p]} = (\|f\|_\infty^p + \|f'\|_\infty^p)^{1/p},\ f \in C^1(X). \tag{1.4}$$

We follow the standard convention for $p = \infty$:

$$\|f\|_{[\infty]} = \max(\|f\|_\infty, \|f'\|_\infty).$$

The inequality

$$\|f\|_{[\infty]} \le \|f\|_{[p]} \le 2^{1/p}\|f\|_{[\infty]}$$

implies that the norms $\|\cdot\|_{[p]}$, $p \in [1, \infty]$, are all mutually equivalent and induce the C^1-topology.

The norms defined in (1.1) are then written as follows:

$$\begin{aligned}
&\|f\|_\Sigma = \|f\|_{<[0,1]\times[0,1]>} = \|f\|_{[1]},\\
&\|f\|_{\sigma,c} = \|f\|_{<\{c\}\times[0,1]>},\\
&\|f\|_C = \|f\|_{<\Delta>},\ \text{where } \Delta = \{(t,t) | t \in X\},\ \text{and}\\
&\|f\|_M = \|f\|_{[\infty]}.
\end{aligned} \tag{1.5}$$

Now we introduce topology on the space of norms and on the space of isometries. For $X = [0,1]$ or $\mathbb{T}$, let $\mathcal{N}(X)$ denotes the space of all norms on $C^1(X)$ which induce the C^1-topology. The space $\mathcal{N}(X)$ is endowed with the weakest topology such that the "evaluation map"

$$e_f : \mathcal{N}(X) \to \mathbb{R},\ e_f(\|\cdot\|) = \|f\|$$

is continuous for each $f \in C^1(X)$. Thus a path $(\|\cdot\|_t)_{0\le t\le 1}$ is continuous if and only if the function $t \mapsto \|f\|_t$ is continuous for each $f \in C^1(X)$. Also let $\mathrm{GL}(C^1(X))$ be the space of all $\mathbb{C}$-linear isomorphisms on $C^1(X)$.

The space $\mathrm{GL}(C^1(X))$ is endowed with the weakest topology such that the map

$$E_f : \mathrm{GL}(C^1(X)) \to C^1(X),\ E_f(T) = Tf$$

is continuous for each $f \in C^1(X)$ (recall that the space $C^1(X)$ is endowed with the C^1-topology). For a norm $\|\cdot\|$ on $C^1(X)$, $\mathcal{U}_X(\|\cdot\|)$ denotes the group of all surjective $\mathbb{C}$-linear $\|\cdot\|$-isometries endowed with the subspace topology of $\mathrm{GL}(C^1(X))$. When no confusion occurs, $\mathcal{U}_X(\|\cdot\|)$ is simply denoted by $\mathcal{U}(\|\cdot\|)$. Some examples of continuous paths in $\mathcal{N}(X)$ are given below. The first example is most common, while the second and the third are important for our purpose.

Example 1.1. Let $X = [0,1]$ or $\mathbb{T}$.

(1) Let $\|\cdot\|_i,\ i = 0,1$, be norms in $\mathcal{N}(X)$ and for $t \in [0,1]$, let $\|\cdot\|_t = (1-t)\|\cdot\|_0 + t\|\cdot\|_1$. Then the map $t \mapsto \|\cdot\|_t$ is a continuous path in $\mathcal{N}(X)$. In fact the function $t \mapsto \|f\|_t$ is continuous for each $f \in C^1(X)$.

(2) Let $(D_t)_{0\le t\le 1}$ be a collection of compact connected subsets of $X \times X$ such that
  (i) $p_2(D_t) = X$ for each t and
  (ii) the map $t \mapsto D_t$ is continuous with respect to the Hausdorff metric.

  Then the map $t \mapsto \|\cdot\|_{<D_t>}$ is continuous.

(3) Assume that the extended real line $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ has the natural topology. Then the map $p \mapsto \|\cdot\|_{[p]}$ is a continuous map $[1,\infty] \to \mathcal{N}(X)$.

2. Isometry groups on the space $C^1([0,1])$

Throughout this section, $\mathbf{1}$ denotes the constant function on $[0,1]$ which takes the value 1. Also the identity map on $[0,1]$ is denoted by id. The following theorem is a consequence of Main Theorem of [6] and Theorem 1.1 of [5]. For $0 \le a \le b \le 1$, we define a map $\varphi_\gamma^{a,b} : [a,b] \to [a,b],\ \gamma = \pm 1$, by the following:

$$\varphi_1^{a,b}(x) = x,\ \varphi_{-1}^{a,b}(x) = a + b - x,\ x \in [a,b]. \tag{2.1}$$

The maps $\varphi_1^{a,b}$ and $\varphi_{-1}^{a,b}$ are the only isometries on $[a,b]$. When $a = b$, the singleton $\{a\}$ is called a *degenerate interval.* Observe $\varphi_1^{a,a} = \varphi_{-1}^{a,a}$.

Theorem 2.1. *Let $T : C^1([0,1]) \to C^1([0,1])$ be a surjective $\mathbb{C}$-linear isomorphism.*

(1) *(*[6, Main Theorem]*) Let D be a compact connected subset of $[0,1]\times[0,1]$ such that $p_2(D) = [0,1]$ and let $p_1(D) = [a,b]$, a (possibly degenerate) subinterval of $[0,1]$. Consider the norm $\|\cdot\|_{<D>}$*

defined by (1.3) and assume that T is a $\|\cdot\|_{<D>}$-isometry. Then we have the following.

(1-1) *There exist constants $\kappa \in \mathbb{T}$, $\gamma \in \{\pm 1\}$, a continuous map $\beta : [0,1] \to \mathbb{T}$ and a homeomorphism $\psi : [0,1] \to [0,1]$ such that, for each $f \in C^1([0,1])$, we have*

$$Tf(x) = \kappa f(\varphi_\gamma^{ab}(a)) + \int_a^x \beta(y) f'(\psi(y))dy, \ x \in [0,1].$$

(1-2) *If $a < b$, then we have $\beta|[a,b] \equiv \kappa\gamma$ and $\psi|[a,b] = \varphi_\gamma^{a,b}$. If moreover $0 = a < b < 1$ or $0 < a < b = 1$, then we have $\gamma = 1$.*

(1-3) *Assume that $p_1(D) = [0,1]$ so that $a = 0, b = 1$, and let $\varphi_\gamma = \varphi_\gamma^{0,1}$. Then there exist constants $\kappa \in \mathbb{T}$ and $\gamma \in \{\pm 1\}$ such that*

$$Tf(x) = \kappa f(\varphi_\gamma(x)), \ f \in C^1([0,1]), x \in [0,1].$$

(2) *(*[5, Theorem 1.1]*) Assume that T is a $\|\cdot\|_{[p]}$-isometry for $p \in [1,\infty]$. If $p > 1$, assume further that $T\mathbf{1}$ is a constant function. Then there exist constants $\kappa \in \mathbb{T}$ and $\gamma \in \{\pm 1\}$ which represent T as in the form of (1-3).*

Remark 2.2. (1) If $a = b$ in the statement (1) above, then the isometries defined by $(\kappa, 1, \beta, \psi)$ and $(\kappa, -1, \beta, \psi)$ are identical.

(2) The statement (1-3) follows directly from (1-1) and (1-2). In fact, under the hypothesis of (1-3), we see by (1-2) the maps β and ψ of (1-1) satisfy $\beta \equiv \kappa\gamma$ and $\psi = \varphi_\gamma$. Then a simple integration in (1-1) implies (1-3).

Let $\mathcal{U}(\|\cdot\|)^{\text{const}}$ be the subgroup of $\mathcal{U}(\|\cdot\|)$ consisting of the $\mathbb{C}$-linear $\|\cdot\|$-isometries T such that $T\mathbf{1}$ is a constant function. The group of all homeomorphisms of $[0,1]$ is denoted by $\mathrm{Homeo}([0,1])$ and $\mathrm{Map}([0,1],\mathbb{T})$ denotes the group of all continuous maps $[0,1] \to \mathbb{T}$ with pointwise multiplication. Recalling (1.5) and applying Theorem 2.1, we obtain the following corollary.

Corollary 2.3. (1) $\mathcal{U}(\|\cdot\|_\Sigma) = \mathcal{U}(\|\cdot\|_C) = \mathcal{U}(\|\cdot\|_M)^{\text{const}} \cong \mathbb{T} \times \mathbb{Z}_2$.

(2) $\mathcal{U}(\|\cdot\|_{\sigma,a}) \cong \mathbb{T} \times \mathrm{Map}([0,1],\mathbb{T}) \times \mathrm{Homeo}([0,1])$.

Remark 2.4. For two isometries $T_i \in \mathcal{U}(\|\cdot\|_{\sigma,a})$, $i = 1, 2$, of the form

$$T_i f(x) = \kappa_i f(a) + \int_a^x \beta_i(y) f'(\psi_i(y))dy,$$

where $\kappa_i \in \mathbb{T}, \beta_i \in \mathrm{Map}([0,1],\mathbb{T}), \psi_i \in \mathrm{Homeo}([0,1])$, the composition $T_2 \circ T_1$ is given by the formula

$$(T_2 \circ T_1)(f)(x) = \kappa_2\kappa_1 f(a) + \int_a^x \beta_2(y)\beta_1(y) f'((\psi_1 \circ \psi_2)(y))dy$$

for $f \in C^1([0,1])$. This naturally gives us a group isomorphism between $\mathcal{U}(\|\cdot\|_{\sigma,a})$ and the direct product $\mathbb{T} \times \mathrm{Map}([0,1],\mathbb{T}) \times \mathrm{Homeo}([0,1])$. The same remark applies to (1).

We perform continuous perturbations of isometries on the basis of Theorem 2.1 and the next proposition. In order to simplify the notation, a weighted composition operator T defined by $Tf(x) = \kappa f(\varphi(x))$ $(x \in [0,1],\ f \in C^1([0,1]))$, for some scalar $\kappa \in \mathbb{C}$ and a map $\varphi : [0,1] \to [0,1]$ is denoted by

$$Tf = \kappa \cdot (f \circ \varphi),\ f \in C^1([0,1]).$$

Also we introduce a function $I(\beta,\psi,f;a)$ on $[0,1]$ defined by

$$I(\beta,\psi,f;a)(x) = \int_a^x \beta(y) f'(\psi(y))dy,\ x \in [0,1], \tag{2.2}$$

where $\beta \in \mathrm{Map}([0,1],\mathbb{T})$ and $\psi \in \mathrm{Homeo}([0,1])$. An operator T given by $Tf(x) = \kappa f(\varphi(a)) + I(\beta,\psi,f;a)(x)$ $(f \in C^1([0,1]), x \in [0,1])$ is simply denoted by

$$Tf = \kappa f(\varphi(a)) + I(\beta,\psi,f;a),\ f \in C^1([0,1]).$$

Also let $\mathcal{D}([0,1])$ be a space of compact connected subsets of $[0,1] \times [0,1]$ defined by

$$\mathcal{D}([0,1]) = \{D \subset [0,1]^2 \mid D \text{ is compact connected and } p_2(D) = [0,1]\}$$

with the Hausdorff distance. The spaces $\mathrm{Homeo}([0,1])$ and $\mathrm{Map}([0,1],\mathbb{T})$ are assumed to be equipped with the compact-open topology.

Proposition 2.5. *Let $d : (0,1] \to \mathcal{D}([0,1])$ be a continuous map and let $p_1(d(t)) = [a_t, b_t]$. Assume that $\lim_{t\to 0} a_t = a$, $\lim_{t\to 0} b_t = b$ and assume that T_t is a $\|\cdot\|_{<d(t)>}$-isometry of the form*

$$T_t f = \kappa_t f(\varphi_{\gamma_t}^{a_t,b_t}(a_t)) + I(\beta_t,\psi_t,f;a_t),\ f \in C^1([0,1]),$$

where $\kappa_t \in \mathbb{T}, \beta_t \in \mathrm{Map}([0,1],\mathbb{T})$ and $\psi_t \in \mathrm{Homeo}([0,1])$.

(1) *Assume that $\lim_{t\to 0}\gamma_t = \gamma$, $\lim_{t\to 0}\psi_t = \psi$, $\lim_{t\to 0}\beta_t = \beta$ for some $\gamma \in \{\pm 1\}$, a homeomorphism $\psi \in \mathrm{Homeo}([0,1])$, and a continuous map $\beta : [0,1] \to \mathbb{T}$. Then we have $\lim_{t\to 0} T_t = T$ where T is an operator of the form*

$$Tf = \kappa f(\varphi_\gamma^{a,b}(a)) + I(\beta,\psi,f;a),\ f \in C^1([0,1]).$$

(2) *Conversely assume that* $\lim_{t\to 0} T_t = T$ *for some* $\mathbb{C}$*-linear isomorphism* T *of the form of (1) for some* $\kappa \in \mathbb{T}, \gamma \in \{\pm 1\}$*, a continuous map* $\beta : [0,1] \to \mathbb{T}$ *and a homeomorphism* $\psi : [0,1] \to [0,1]$*. Then we have* $\lim_{t\to 0} \psi_t = \psi$*,* $\lim_{t\to 0} \beta_t = \beta$*, and* $\lim_{t\to 0} \varphi_{\gamma_t}^{a_t,b_t}(a_t) = \varphi_{\gamma}^{a,b}(a)$*. If in particular* $a < b$*, then we have* $\lim_{t\to 0} \gamma_t = \gamma$*.*

Proof. (1). For each $f \in C^1([0,1])$, we have, from the assumption, that $\lim_{t\to 0} \|T_t f - Tf\|_\infty = 0$ and $\lim_{t\to 0} \|(T_t f)' - (Tf)'\|_\infty = 0$.

(2) For each $f \in C^1([0,1])$, we have from the assumption that

$$\lim_{t\to 0} \|T_t f - Tf\|_\infty = 0,\ \lim_{t\to 0} \|(T_t f)' - (Tf)'\|_\infty = 0.$$

Applying the first equality to $f = \mathbf{1}$, we see $\lim_{t\to 0} \kappa_t = \kappa$. Next we apply the second equality to

$$f = \mathrm{id} \text{ and } f(x) = \frac{x^2}{2}$$

to see $\lim_{t\to 0} \beta_t = \beta$ and $\lim_{t\to 0} \beta_t \cdot \psi_t = \beta \cdot \psi$. These two imply $\lim_{t\to 0} \psi_t = \psi$.

Noticing that $\gamma_t = \pm 1$, we may assume that $\lim_{t\to 0} \gamma_t = \gamma_\infty$. By the above and (1), we see $\lim_{t\to 0} T_t = T_\infty$ where T_∞ is the operator defined by

$$T_\infty f = \kappa f(\varphi_{\gamma_\infty}^{a,b}(a)) + I(\beta, \psi, f; a),\ f \in C^1([0,1]).$$

Hence we have $f(\varphi_{\gamma_\infty}^{a,b}(a)) = f(\varphi_{\gamma}^{a,b}(a))$ for each $f \in C^1([0,1])$, and thus $\varphi_{\gamma_\infty}^{a,b}(a) = \varphi_{\gamma}^{a,b}(a)$. If $a < b$, then this implies $\gamma_\infty = \gamma$. If $a = b$, then we have $\lim_{t\to 0} \varphi_t^{a_t,b_t}(a_t) = \varphi_{\gamma_\infty}^{a,a}(a) = a = \varphi_{\gamma}^{a,a}(a)$. This proves the proposition. □

The next theorem Theorem 2.6 is the main result of this section. It is an extension of what was proven in [6]. Let $\mathrm{Homeo}_{+1}([0,1])$ (resp. $\mathrm{Homeo}_{-1}([0,1])$) be the set of all orientation-preserving (resp. orientation-reversing) homeomorphisms of $[0,1]$. For $\gamma = \pm 1$, let $\mathcal{WC}^\gamma$ be the set of all $\mathbb{C}$-linear isomorphisms of the form:

$$Tf = \kappa \cdot (f \circ \varphi_\gamma),\ f \in C^1([0,1]),$$

for $\kappa \in \mathbb{T}$ and $\varphi_\gamma \in \mathrm{Homeo}_\gamma([0,1])$, and let $\mathcal{WC} = \mathcal{WC}^{+1} \cup \mathcal{WC}^{-1}$. For $\gamma = \pm 1$ and a point $c \in [0,1]$, define $\mathcal{V}_c^\gamma$ be the set of $\mathbb{C}$-linear isomorphisms of the form

$$Tf = \kappa f(c) + I(\beta, \psi, f; c),\ f \in C^1([0,1]),$$

where $\beta \in \mathrm{Map}([0,1], \mathbb{T})$ with $\beta(c) = \kappa\gamma$, $\psi \in \mathrm{Homeo}_\gamma([0,1])$ with $\psi(c) = c$. Also let $\mathcal{V}_c = \mathcal{V}_c^{+1} \cup \mathcal{V}_c^{-1}$.

Theorem 2.6. *Let $c \in [0,1]$ and let $d : [0,1] \to \mathcal{D}([0,1])$ be a continuous map such that*

(a) $d(0) = \{c\} \times [0,1], p_1(d(1)) = [0,1]$, *and*
(b) *for each $t \in (0,1)$, $p_1(d(t))$ is a non-degenerate interval such that $c \in p_1(d(t)) \neq [0,1]$.*

Let $\|\cdot\|_t = \|\cdot\|_{<d(t)>}$ so that $\|\cdot\|_0 = \|\cdot\|_{\sigma,c}$. Then we have the following.

(1) *Assume that $p_1(d(t)) \cap \{0,1\} = \emptyset$ for each $t \in (0,1)$.*
  (1-1) *If $(T_t)_{0 \leq t \leq 1}$ is a continuous collection of isometries associated with d, then we have $T_0 \in \mathcal{V}_c \subset \mathcal{U}(\|\cdot\|_0)$.*
  (1-2) *Conversely for each $T \in \mathcal{V}_c \subset \mathcal{U}(\|\cdot\|_0)$, there exists a continuous collection $(T_t)_{0 \leq t \leq 1}$ of isometries associated with d such that $T_0 = T$.*
  (1-3) *Also, for each $T \in \mathcal{U}(\|\cdot\|_1)$, there exists a continuous collection $(T_t)_{0 \leq t \leq 1}$ of isometries associated with d such that $T_1 = T$.*
(2) *Assume that there is a $\tau \in (0,1)$ such that $p_1(d(\tau)) \cap \{0,1\} \neq \emptyset$.*
  (2-1) *If $(T_t)_{0 \leq t \leq 1}$ is a continuous collection of isometries associated with d, then $T_0 \in \mathcal{V}_c^{+1}$ and $T_1 \in \mathcal{WC}^{+1}$.*
  (2-2) *Conversely for each $T \in \mathcal{V}_c^{+1}$, there exists a continuous collection $(T_t)_{0 \leq t \leq 1}$ of isometries associated with d such that $T_0 = T$.*
  (2-3) *Also for each $T \in \mathcal{WC}^{+1} \subset \mathcal{U}(\|\cdot\|_1)$, there exists a continuous collection $(T_t)_{0 \leq t \leq 1}$ of isometries associated with d such that $T_1 = T$.*

First we observe some direct consequences of Theorem 2.1.

Lemma 2.7. (1) *Assume that a compact connected set $D \in \mathcal{D}([0,1])$ satisfies $p_1(D) = p_2(D) = [0,1]$. Then we have*

$$\mathcal{U}(\|\cdot\|_{<D>}) = \mathcal{WC} \cong \mathbb{T} \times \mathbb{Z}_2.$$

(2) *For each $D \in \mathcal{D}([0,1])$, we have the inclusion*

$$\mathcal{WC}^{+1} \subset \mathcal{U}(\|\cdot\|_{<D>}).$$

(3) *For each $c \in [0,1]$, we have the inclusion $\mathcal{V}_c \subset \mathcal{U}(\|\cdot\|_{\sigma,c})$.*

We also need the following lemma. For $\gamma \in \{\pm 1\}$, the isometry on $[0,1]$ defined by (2.1) for $a = 0, b = 1$ is simply denoted by $\varphi_\gamma : [0,1] \to [0,1]$.

Lemma 2.8. *Let $a, b : [0,1] \to [0,1]$ be two continuous maps such that*

(a) $a(t) \leq b(t)$ *and* $[a(t), b(t)] \neq [0,1]$ *for each* $t \in [0,1)$ *and*
(b) $a(1) = 0, b(1) = 1$.

Then we have the following.

(1) *For each homeomorphism* $\psi \in \mathrm{Homeo}([0,1])$ *such that* $\psi|[a(0),b(0)] = \varphi_\gamma^{a(0),b(0)}$*, with* $\gamma = \pm 1$*, there exists a continuous map* $(\psi_t)_{0\le t\le 1} : [0,1] \to \mathrm{Homeo}([0,1])$ *such that*

$$\psi_0 = \psi,\ \psi_1 = \varphi_\gamma,\ \psi_t|[a(t),b(t)] = \varphi_\gamma^{a(t),b(t)}$$

for each $t \in [0,1]$.

(2) *Assume that* $a(0) = b(0) := c$. *For each map* $\beta \in \mathrm{Map}([0,1],\mathbb{T})$*, there exists a continuous map* $(\beta_t)_{0\le t\le 1} : [0,1] \to \mathrm{Map}([0,1],\mathbb{T})$ *such that*

$$\beta_0 = \beta, \beta_1 \equiv \beta(c),\ \beta_t|[a(t),b(t)] \equiv \beta(c)$$

for each $t \in [0,1]$.

Proof. (1) Case 1. First we assume that the functions a and b satisfy the following conditions:

(i) if $a(0) = 0$, then $a(t) = 0$ for each $t \in [0,1]$, and
(ii) if $b(0) = 1$, then $b(t) = 1$ for each $t \in [0,1]$.

Let $\xi_t : [0,a(0)] \to [0,a(t)]$ and $\eta_t : [b(0),1] \to [b(t),1]$ be the linear maps such that $\xi_t(0) = 0, \xi_t(a) = a(t)$ and $\eta_t(b) = b(t), \eta_t(1) = 1$. If $a(0) = 0$ (resp. $b(0) = 1$), then the assumption (i) (resp. (ii)) implies the interval $[0,a(t)]$ (resp. $[b(0),1]$) reduces to a singleton $\{0\}$ (resp. $\{1\}$).

Assume that $\gamma = 1$ so that $\psi(0) = 0$ and $\psi(1) = 1$. Define the map ψ_t $(0 \le t \le 1)$ by:

$$\psi_t(x) = \begin{cases} (\xi_t \circ \psi \circ \xi_t^{-1})(x), & 0 \le x \le a(t) \\ x, & a(t) \le x \le b(t) \\ (\eta_t \circ \psi \circ \eta_t^{-1})(x), & b(t) \le x \le 1 \end{cases}$$

For $\gamma = -1$ for which we have $\psi(0) = 1, \psi(1) = 0$, we define ψ_t $(0 \le t \le 1)$ by

$$\psi_t(x) = \begin{cases} (\eta_t \circ \psi \circ \xi_t)(x), & 0 \le x \le a(t) \\ a(t) + b(t) - x, & a(t) \le x \le b(t) \\ (\xi_t^{-1} \circ \psi \circ \eta_t^{-1})(x), & b(t) \le x \le 1 \end{cases}$$

Then $(\psi_t)_{0\le t\le 1}$ is the desired map.

Case 2. Suppose that $a(0) = 0$. Then by the assumption (a) we see $b(0) < 1$. Since ψ is a homeomorphism on $[0,1]$ such that $\psi([0,b(0)]) = [0,b(0)]$, we see $\psi(0) = 0$ and $\psi(b(0)) = b(0)$. Thus we have $\gamma = 1$. Let $a' : [0,1] \to [0,1]$ be the constant function $a' \equiv 0$. Then a' clearly satisfies the condition (i) of Case 1 and also

$$[a(t),b(t)] \subset [a'(t),b(t)]. \tag{2.3}$$

Applying Case 1 to the functions a' and b, we obtain a family $(\psi_t)_{0\le t\le 1}$ such that $\psi_0 = \psi, \psi_1 = \varphi_{+1}$ and $\psi_t|[0,b(t)] = \varphi_{+1}^{0,b(t)}$ for each $t \in [0,1]$.

Due to the inclusion (2.3) and $\gamma = 1$, we see that (ψ_t) is indeed the desired family for the function a and b. The same argument applies when $b(0) = 1$.

This proves (1).

(2) Let β_t be the map defined by

$$\beta_t(x) = \begin{cases} \beta(x + c - a(t)), & 0 \le x \le a(t) \\ \beta(c), & a(t) \le x \le b(t) \\ \beta(x + c - b(t)), & b(t) \le x \le 1 \end{cases}$$

Then $(\beta_t)_{0 \le t \le 1}$ is the desired map. □

Proof. (of Theorem 2.6). Let $p_1(d(t)) = [a_t, b_t]$. By the hypotheses (a) and (b), we have $a_t \le c \le b_t$, $a_t < b_t$ and $[a_t, b_t] \ne [0,1]$ for each $t \in (0,1)$, and also $a_0 = b_0 = c, a_1 = 0, b_1 = 1$.

(1) By the assumption we have $0 < a_t < b_t < 1$ for each $t \in (0,1)$.

(1-1) By Theorem 2.1, each $T_t, t \in [0,1)$ is of the form

$$T_t f = \kappa_t f(\varphi_{\gamma_t}(a_t)) + I(\beta_t, \psi_t, f; a_t),\ f \in C^1([0,1]),$$

for some $\gamma_t = \pm 1$, $\kappa_t \in \mathbb{T}$, $\beta_t \in \mathrm{Map}([0,1], \mathbb{T})$ and $\psi_t \in \mathrm{Homeo}_{\gamma_t}([0,1])$ such that

$$\beta_t | [a_t, b_t] = \kappa_t \gamma_t,\ \psi_t | [a_t, b_t] = \varphi_{\gamma_t}^{a_t, b_t}.$$

In particular we have

$$\beta_t(c) = \kappa_t \gamma_t. \tag{2.4}$$

Since $a_t < b_t$ for each $t \in (0,1)$, we see from the continuity of d and (2) of Propsition 2.5 that the map $(0,1) \to \{\pm 1\};\ t \mapsto \gamma_t$ is locally constant, and hence is a constant function. Let $\gamma_t \equiv \gamma$. Then by (2.4) $\beta_t(c) = \kappa_t \gamma$. Also by Proposition 2.5 (2) we see

$$\lim_{t \to 0} \kappa_t = \kappa_0,\ \lim_{t \to 0} \beta_t = \beta_0,\ \lim_{t \to 0} \psi_t = \psi_0.$$

Then $\beta_0(c) = \kappa_0 \gamma$ and $\psi_0(c) = \lim_{t \to 0} \varphi_{\gamma_t}^{a_t, b_t}(c) = c$. Thus we see $T_0 \in \mathcal{V}_c$.

(1-2) Let $T \in \mathcal{V}_c$ be an isometry of the form

$$Tf = \kappa f(c) + I(\beta, \psi, f; c),\ f \in C^1([0,1]),$$

with $\beta \in \mathrm{Map}([0,1], \mathbb{T})$ with $\beta(c) = \kappa\gamma$ and $\psi \in \mathrm{Home}([0,1])$ with $\psi(c) = c$. Apply Lemma 2.8 to the maps ψ and β with the functions $a(t) := a_t, b(t) := b_t$, recalling $[a(t), b(t)] \ne [0,1]$ for each $t \in [0,1)$. Then we obtain $(\psi_t)_{0 \le t \le 1}$ and $(\beta_t)_{0 \le t \le 1}$ as in the lemma. Let T_t be the operator defined by

$$T_t f = \kappa f(\varphi_{\gamma_t}(a_t)) + I(\beta_t, \psi_t, f; a_t),\ f \in C^1([0,1]).$$

By Theorem 2.1 and Proposition 2.5, we see $T_t \in \mathcal{U}(\|\cdot\|_{<d(t)>})$ for each $t \in [0,1]$ and $(T_t)_{0\le t\le 1}$ is a continuous collection of isometries associated with d. Notice that T_1 is written as

$$T_1 f = \kappa f(\varphi_\gamma(0)) + I(\kappa\gamma, \varphi_\gamma, f; 0) = \kappa \cdot (f \circ \varphi_\gamma),$$

for $f \in C^1([0,1])$.

(1-3) Assume $T \in \mathcal{U}(\|\cdot\|_1) = \mathcal{WC}$ (see Lemma 2.7(1)). If $T \in \mathcal{WC}^{+1}$, then (2) of Lemma 2.7 implies $T \in \mathcal{U}(\|\cdot\|_t)$. Hence the conclusion follows directly.

If $T \in \mathcal{WC}^{-1}$, then take an orientation reversing homeomorphism $\psi \in \mathrm{Homeo}_{-1}([0,1])$ such that $\psi(c) = c$. It follows that $c \neq 0, 1$ because ψ is orientation-reversing. We apply Lemma 2.8(1) to find a family $(\psi_t)_{0\le t\le 1}$ of homeomorphisms. Then the operator T_t defined by

$$T_t f = \kappa f(\varphi_t(a_t)) + I(-\kappa, \psi_t, f; a_t), \ f \in C^1([0,1]),$$

forms a continuous collection of isometries so that, for each $f \in C^1([0,1])$, we have

$$T_1 f(x) = \kappa f(\varphi_{-1}(0)) + I(-\kappa, \varphi_{-1}, f; 0)(x) = \kappa f(1-x) = Tf(x), \ x \in [0,1]$$

as desired.

(2) Next we assume that $p_1(d(\tau)) \cap \{0,1\} \neq \emptyset$ for some τ.

(2-1) Assume that each $T_t, t \in (0,1)$, is of the form

$$T_t f = \kappa_t f(\varphi_{\gamma_t}(a_t)) + I(\beta_t, \psi_t, f; a_t), \ f \in C^1([0,1])$$

and let

$$\begin{aligned} I_+ &= \{t \in (0,1) \mid \psi_t \in \mathrm{Homeo}_{+1}([0,1])\}, \\ I_- &= \{t \in (0,1) \mid \psi_t \in \mathrm{Homeo}_{-1}([0,1])\}. \end{aligned}$$

Since the map $(0,1) \to \mathrm{Homeo}([0,1]),\ t \mapsto \psi_t$ is continuous, both I_+ and I_- are closed in $(0,1)$. Note that $0 = a_\tau < b_\tau < 1$ or $0 < a_\tau < b_\tau = 1$ by the assumption. By Theorem 2.1 we see $\gamma_\tau = 1$ and thus $\psi_\tau \in \mathrm{Homeo}_{+1}([0,1])$. Hence $I_+ = (0,1)$. By the continuity we obtain ψ_0 and $\psi_1 \in \mathrm{Homeo}_{+1}([0,1])$, which is to be proved.

(2-2) For each $T \in \mathcal{V}_c^{+1}$ of the form

$$Tf = \kappa f(c) + I(\beta, \psi, f; c), \ f \in C^1([0,1]),$$

with $\beta(c) = \kappa$ and $\psi(c) = c$, apply Lemma 2.8 to $a(t) = a_t, b(t) = b_t$ and obtain (ψ_t) and (β_t) satisfying the conditions of the lemma. Then the operator

$$T_t f = \kappa f(\varphi_{\gamma_t}(a_t)) + I(\beta_t, \psi_t, f; a_t), \ f \in C^1([0,1])$$

defines a continuous collection of isometries satisfying $T_0 = T$.

For $T \in \mathcal{WC}^{+1}$, we see from (2) of Lemma 2.7 that $T \in \mathcal{U}(\|\cdot\|_t)$. As in (1-3) this is applied to prove (2-3). This completes the proof of the theorem. $\square$

We apply the above theorem to obtain information on perturbations of isometry groups associated with the norms $\|\cdot\|_\Sigma, \|\cdot\|_{\sigma,c}$, and $\|\cdot\|_C$. Recall that Δ denotes the diagonal set $\{(x,x) \mid x \in [0,1]\}$.

Corollary 2.9. (1) *Let $d : [0,1] \to \mathcal{D}$ be a continuous map such that $d(0) = \Delta, d(1) = [0,1] \times [0,1]$ and $p_1(d(t)) = [0,1]$ for each $t \in [0,1]$. Then for each $t \in [0,1]$, we have the equality $\mathcal{U}(\|\cdot\|_t) = \mathcal{U}(\|\cdot\|_C) = \mathcal{WC}$ for each t.*

(2) *Let $e : [0,1] \to \mathcal{D}$ be a map such that $e(0) = \{0\} \times [0,1], e(1) = [0,1] \times [0,1]$ and $p_1(e(t))$ is not a singleton for each $t \in (0,1]$ so that $\|\cdot\|_{\sigma,0} = \|\cdot\|_{<e(0)>}$ and $\|\cdot\|_\Sigma = \|\cdot\|_{<e(1)>}$.*

(2-1) *If $T \in \mathcal{V}_0^{+1}$, then there exists a continuous collection of isometries $(T_t)_{0 \le t \le 1}$ associated with e such that $T_0 = T$ and $T_1 \in \mathcal{WC}^{+1}$. While there is no such a collection if $T \in \mathcal{V}_0^{-1}$.*

(2-2) *If $S \in \mathcal{WC}^{+1}$, then there exists a continuous collection of isometries $(S_t)_{0 \le t \le 1}$ associated with e such that $S_1 = S$ and $S_0 \in \mathcal{V}_0^{+1}$. While there is no such a collection if $S \in \mathcal{WC}^{-1}$.*

Proof. (1) follows directly from Theorem 2.1.

(2) Case 1. First assume that $\{0,1\} \cap p_1(e(\tau)) \neq \emptyset$ for some $\tau \in (0,1)$. Applying Theorem 2.6 (2) we directly obtain the conclusion in (2-1) and (2-2).

Case 2. Assume that $p_1(e(t)) \cap \{0,1\} = \emptyset$ for each $t \in (0,1)$ and let $p_1(e(t)) = [a_t, b_t]$.

(2-1) Let (T_t) be a continuous collection of isometries such that each T_t is an operator of the form

$$T_t f = \kappa_t f(a_t) + I(\beta_t, \psi_t, f; a_t), \;\; f \in C^1([0,1]),$$

where $\beta_t|[a_t, b_t] \equiv \kappa_t \gamma_t, \psi_t([a_t, b_t]) = [a_t, b_t]$. Since $\lim_{t \to 0} a_t = \lim_{t \to 0} b_t = 0$, we obtain $\psi_0(0) = 0$. Thus $\psi_0 \in \mathrm{Homeo}_{+1}([0,1])$ and we cannot have such a collection if $T_0 \in \mathcal{V}_0^{-1}$.

If $T_0 \in \mathcal{V}_0^{+1}$, then we apply (2-2) of Theorem 2.6 to obtain the desired collection. For each such collection (T_t), the condition (2-1) of Theorem 2.6 ensures that $T_1 \in \mathcal{WC}^{+1}$.

(2-2) Assume S_t is an operator of the form

$$S_t f = \kappa_t f(a_t) + I(\beta_t, \psi_t, f; a_t), \;\; f \in C^1([0,1]),$$

where $\beta_t|[a_t, b_t] \equiv \kappa_t\gamma_t, \psi_t([a_t, b_t]) = [a_t, b_t]$. The same argument as in (2-1) shows that $\psi_0 \in \mathrm{Homeo}_{+1}([0, 1])$. The two sets

$$\begin{aligned} J_+ &= \{t \in [0, 1] \mid \psi_t \in \mathrm{Homeo}_{+1}([0, 1])\}, \\ J_- &= \{t \in [0, 1] \mid \psi_t \in \mathrm{Homeo}_{-1}([0, 1])\} \end{aligned}$$

are closed by the continuity of the map $[0, 1] \ni t \mapsto \psi_t \in \mathrm{Homeo}([0, 1])$, which implies that $J_+ = [0, 1]$ and in particular ψ_1 is orientation-preserving. Thus $S_1 \in \mathcal{WC}^{+1}$ and we cannot have such a collection of isometries $(S_t)_{0\le t\le 1}$ with $S_1 \in \mathcal{WC}^{-1}$. If $S_1 \in \mathcal{WC}^{+1}$, then we apply (2-3) of Theorem 2.6 to find the desired collection. By (2-1) of Theorem 2.6, we see $S_0 \in \mathcal{V}_0^{+1}$ for each such $(S_t)_{0\le t\le 1}$.

This proves the corollary. □

Let $c : [1, \infty] \to \mathcal{N}([0, 1])$ be the map defined by

$$c(p) = \|\cdot\|_{[p]},\ p \in [1, \infty].$$

As in Example 1.1, c is a continuous interpolation between the norms $\|\cdot\|_\Sigma$ and $\|\cdot\|_M$. By Theorem 2.1 and Corollary 2.3 we have

$$\mathcal{U}(\|\cdot\|_{[p]})^{\mathrm{const}} = \mathcal{U}(\|\cdot\|_\Sigma) \cong \mathbb{T} \times \mathbb{Z}_2.$$

Hence each $T \in \mathcal{U}(\|\cdot\|_\Sigma)$ defines the trivial continuous collection of isometries associated with c. The full isometry group $\mathcal{U}(\|\cdot\|_M)$ has not been determined yet and the whole picture of the perturbation of these groups remains unknown to the author.

3. Isometries on $C^1(\mathbb{T})$

The goal of this section is to prove Theorem 3.1 and Theorem 3.2 which together form a counterpart to Theorem 2.1. Throughout this section, the constant function taking the value 1 is denoted by $\mathbf{1}$. The identity on $\mathbb{T}$ is denoted by id. No confusion of these with those of Section 2 will be caused. Also $\overline{\mathrm{id}} : \mathbb{T} \to \mathbb{T}$ stands for the map defined by $\overline{\mathrm{id}}(z) = \bar{z},\ z \in \mathbb{T}$. Notice that $(\mathrm{id})'(z) = iz$ and $(\overline{\mathrm{id}})'(z) = -i\bar{z}$.

Let D be a compact connected subset of $\mathbb{T} \times \mathbb{T}$ such that $p_2(D) = \mathbb{T}$ and let $I = p_1(D)$. We consider the norm defined in (1.3).

Theorem 3.1. *Let $T : C^1(\mathbb{T}) \to C^1(\mathbb{T})$ be a surjective $\mathbb{C}$-linear $\|\cdot\|_{<D>}$-isometry. Assume either $p_1(D) = \mathbb{T}$, or T satisfies the condition:*

$$(*)\ T(\mathrm{id}) \text{ and } T(\overline{\mathrm{id}}) \text{ are } C^3\text{-functions.}$$

(1) *Assume that $I = \{a\}$. Then there exist constants $\beta, \kappa \in \mathbb{T}$ and an isometry $\varphi : \mathbb{T} \to \mathbb{T}$ such that*

$$Tf(z) = \beta f(\varphi(z)) + (\kappa f(a) - \beta f(\varphi(a)))\,,\ z \in \mathbb{T}$$

for each $f \in C^1(\mathbb{T})$.

(2) *Assume that* I *is not a singleton. Then there exist a constant* $\kappa \in \mathbb{T}$ *and an isometry* $\varphi : \mathbb{T} \to \mathbb{T}$ *such that* $\varphi(I) = I$ *and*

$$Tf(z) = \kappa f(\varphi(z)), \ z \in \mathbb{T}$$

for each $f \in C^1(\mathbb{T})$.

Theorem 3.2. *Let* $T : C^1(\mathbb{T}) \to C^1(\mathbb{T})$ *be a surjective* $\mathbb{C}$*-linear* $\|\cdot\|_{[p]}$*-isometry. When* $p > 1$*, assume that* T *satisfies*

(const) $T\mathbf{1}$ *is a constant function.*

Then there exist a constant $\kappa \in \mathbb{T}$ *and an isometry* $\varphi : \mathbb{T} \to \mathbb{T}$ *such that*

$$Tf(z) = \kappa f(\varphi(z)), \ z \in \mathbb{T}$$

for each $f \in C^1(\mathbb{T})$.

In this section, for a norm $\|\cdot\| \in \mathcal{N}(\mathbb{T})$, $\mathcal{U}(\|\cdot\|)$ denotes the group of all surjective $\mathbb{C}$-linear $\|\cdot\|$-isometries of $C^1(\mathbb{T})$. Also let $\mathcal{U}(\|\cdot\|)^*$ and $\mathcal{U}(\|\cdot\|)^{\mathrm{const}}$ be the subgroups of $\mathcal{U}(\|\cdot\|)$ consisting of all $\|\cdot\|$-isometries satisfying the condition $(*)$ and the condition (const) respectively. The group of all isometries on $\mathbb{T}$ is denoted by $\mathrm{Isom}(\mathbb{T})$. For a subset I of $\mathbb{T}$, let $\mathrm{Isom}(\mathbb{T}; I) = \{\varphi \in \mathrm{Isom}(\mathbb{T}) \mid \varphi(I) = I\}$. In particular $\mathrm{Isom}(\mathbb{T}; \{a\})$ is simply denoted by $\mathrm{Isom}(\mathbb{T}; a)$. Every isometry $\varphi : \mathbb{T} \to \mathbb{T}$ is written, by some $\lambda \in \mathbb{T}$ and $\epsilon = \pm 1$, as follows:

$$\varphi(z) = \lambda z^{\epsilon}, \ z \in \mathbb{T}$$

where

$$z^{\epsilon} = \begin{cases} z & \text{if } \epsilon = 1, \\ \bar{z} & \text{if } \epsilon = -1. \end{cases} \tag{3.1}$$

This implies

$$\mathrm{Isom}(\mathbb{T}) \cong \mathbb{T} \times \mathbb{Z}_2$$

and

$$\mathrm{Isom}(\mathbb{T}; I) \cong \mathbb{Z}_2.$$

for each nondegenerate interval I.

Corollary 3.3. (1) $\mathcal{U}(\|\cdot\|_{\Sigma})^* = \mathcal{U}(\|\cdot\|_C)^* = \mathcal{U}(\|\cdot\|_M)^{\mathrm{const}} \cong \mathbb{T} \times \mathrm{Isom}(\mathbb{T})$.

(2) $\mathcal{U}(\|\cdot\|_{\sigma,a})^* \cong \mathbb{T} \times \mathbb{T} \times \mathrm{Isom}(\mathbb{T})$.

(3) For an interval I of $\mathbb{T}$ which is not a singleton, $\mathcal{U}(\|\cdot\|_{<I\times\mathbb{T}>})^* \cong \mathbb{T} \times \mathrm{Isom}(\mathbb{T}; I)$.

The proof is divided into several steps. We make use of the extreme point method ([1], [2], [3], [7], [9], [10], [11], [12] etc.). Following the line of [6] and [5], we express isometries in question as the form of weighted composition operators. Next we examine weights and homeomorphisms that appear in the form to obtain the desired result.

3.1. **Isometries as weighted composition operators.** As before let $\|\cdot\|_{<D>}$ be the norm defined by (1.3) for a compact connected subset D of $\mathbb{T}\times\mathbb{T}$ with $p_2(D)=\mathbb{T}$ and let $I=p_1(D)$.

Proposition 3.4. *Let T be a $\|\cdot\|_{<D>}$-isometry. Then there exist a constant $\kappa\in\mathbb{T}$, a continuous map $\beta:\mathbb{T}\to\mathbb{T}$, homeomorphisms $\varphi: I\to I$ and $\psi:\mathbb{T}\to\mathbb{T}$ such that*

$$Tf(x) = \kappa f(\varphi(x)),\ x\in I, \tag{3.2}$$

$$(Tf)'(y) = \beta(y)f'(\psi(y)),\ y\in\mathbb{T}. \tag{3.3}$$

The proof of Proposition 3.4 follows that of Main Theorem of [6] with a slight modification. In [6] only the *real*-linearity is assumed, while we assume here that all isometries involved are *complex*-linear. This additional assumption considerably simplifies the argument. For completeness we give most of the argument in our context. For convenience of reference we will keep the notation of [6] as much as possible.

As in [6], the following elementary lemma will be used in what follows.

Lemma 3.5. *([6])*

(1) *Let $p,q\in\mathbb{C}$. If $|p+\lambda q|=1$ for each $\lambda\in\mathbb{T}$, then we have $pq=0$ and $|p|^2+|q|^2=1$.*
(2) *For $a,b\in\mathbb{C}$, we have $\max_{z\in\mathbb{T}}|az+b|=|a|+|b|$. If $ab\neq 0$, then the maximum is attained uniquely at the point $z=\frac{|a|b}{a|b|}$.*

For a compact connected subset D of $\mathbb{T}\times\mathbb{T}$ such that $p_2(D)=\mathbb{T}$, let $X_D=D\times\mathbb{T}$ so that each point of X_D is written as (r,s,z), where $(r,s)\in D$ and $z\in\mathbb{T}$. For a function $f\in C^1(\mathbb{T})$, let $\tilde f\in C(X_D)$ be the function defined by

$$\tilde f_D(r,s,z)=f(r)+zf'(s),\ (r,s,z)\in X_D. \tag{3.4}$$

And let $A_D=\{\tilde f_D \mid f\in C^1(\mathbb{T})\}$. It is a $\mathbb{C}$-linear subspace of $C(X_D)$. As in [6, Section 1] we have, by Lemma 3.5,

$$\|\tilde f_D\|_\infty=\sup_{(r,s,z)\in X_D}|f(r)+zf'(s)|=\sup_{(r,s)\in D}(|f(r)|+|f'(s)|)=\|f\|_{<D>}$$

for each $f\in C^1(\mathbb{T})$. Thus the map $\Lambda:(C^1(\mathbb{T}),\|\cdot\|_{<D>})\to(A_D,\|\cdot\|_\infty)$ defined by $\Lambda(f)=\tilde f$ is a $\mathbb{C}$-linear isometry. Hence every $\mathbb{C}$-linear $\|\cdot\|_{<D>}$-isometry T induces a $\mathbb{C}$-linear $\|\cdot\|_\infty$-isometry S on A_D given by $S=\Lambda\circ T\circ\Lambda^{-1}$.

Before proceeding, let us recall some terminologies. For a $\mathbb{C}$-subspace A of $C(X)$ for a compact Hausdorff space X, the complex-dual space of A with the operator norm is denoted by A^* and the unit ball of A^* is denoted by $B(A^*)$. An *extreme point* of the ball $B(A^*)$ is a point $\xi\in B(A^*)$ with the property that the equality $\xi=\frac{\eta+\zeta}{2}$ with $\eta,\zeta\in B(A^*)$

implies $\eta = \zeta = \xi$. The set of all extreme points of $B(A^*)$ is denoted by $\mathrm{ext}(A^*)$. By repeating the argument of [6, Lemma 1.6] by simply replacing $[0,1]$ with $\mathbb{T}$ we obtain the next proposition. We omit the proof.

Proposition 3.6. *(cf.[12, Lemma 3.1 and p.189]) Let* $\Omega : X_D \times \mathbb{T} \to \mathrm{ext}(A_D^*)$ *be the map defined by*

$$\Omega((r,s,z),\lambda) = \lambda\delta_{(r,s,z)},\ (r,s,z) \in X_D,\ \lambda \in \mathbb{T}.$$

Then Ω *is a homeomorphism with respect to the weak $*$-topology on* $\mathrm{ext}(A_D^*)$.

Throughout sections 3.1 and 3.2, we fix a $\mathbb{C}$-linear $\|\cdot\|_{<D>}$-isometry $T : C^1(\mathbb{T}) \to C^1(\mathbb{T})$. First consider the induced $\|\cdot\|_\infty$-isometry $S = \Lambda \circ T \circ \Lambda^{-1} : A_D \to A_D$ and let $S^* : A_D \to A_D$ be the dual operator. It is a $\mathbb{C}$-linear isometry with respect to the operator norm on A_D^*. Thus it satisfies the equality $S^*(\mathrm{ext}(A_D^*)) = \mathrm{ext}(A_D^*)$, and by Propostion 3.6 it induces a homeomorphism $\Phi := \Omega^{-1} \circ S^* \circ \Omega : X_D \times \mathbb{T} \to X_D \times \mathbb{T}$. The restriction $\Phi| : X_D \times 1 : X_D \times 1 \to X_D \times \mathbb{T}$ is a continuous map written as

$$\Phi((r,s,z),1) = ((\varphi(r,s,z),\psi(r,s,z),w(r,s,z)),\alpha(r,s,z))\,,$$

where $(\varphi(r,s,z),\psi(r,s,z),w(r,s,z)) \in D \times \mathbb{T} = X_D$ and $\alpha(r,s,z) \in \mathbb{T}$. It is convenient to write the above as,

$$S_*(\delta_{(r,s,z)}) = \alpha(r,s,z)\delta_{(\varphi(r,s,z),\psi(r,s,z),w(r,s,z))}, \tag{3.5}$$

which is translated in terms of the original isometry T as follows:

$$\begin{aligned} &Tf(r) + z(Tf)'(s) = \\ &\qquad \alpha(r,s,z)\left\{f(\varphi(r,s,z)) + w(r,s,z)f'(\psi(r,s,z))\right\}, \\ &\qquad\qquad f \in C^1([0,1]). \end{aligned} \tag{3.6}$$

In what follows we simplify the above to obtain the desired conclusion. First we note, by using the $\mathbb{C}$-linearity of S^*, the following.

Lemma 3.7. *The maps* $\varphi : X_D \to I,\ \psi : X_D \to \mathbb{T}, w : X_D \to \mathbb{T}$ *are continuous surjections. Also the map* $\hat{\alpha} : X_D \times \mathbb{T} \to \mathbb{T}$*, defined by* $\hat{\alpha}(\lambda,(r,s,z)) := \lambda\alpha(r,s,z)$ *is a continuous surjection.*

Proof. By (3.5) and the $\mathbb{C}$-linearity of the dual map, we have $S_*(\lambda\delta_{(r,s,z)}) = \lambda\alpha(r,s,z)\delta_{(\varphi(r,s,z),\psi(r,s,z),w(r,s,z))}$. Thus the homeomorphism $\Phi : X_D \times \mathbb{T} \to X_D \times \mathbb{T}$ is written as

$$\begin{aligned} \Phi((r,s,z),\lambda) &= ((\varphi(r,s,z),\psi(r,s,z),w(r,s,z)),\lambda\alpha(r,s,z)) \\ &= ((\varphi(r,s,z),\psi(r,s,z),w(r,s,z)),\hat{\alpha}(\lambda,(r,s,z)))\,, \end{aligned}$$

which implies the conclusion. $\square$

Lemma 3.8. *For each* $(r,s,z_1),(r,s,z_2) \in X_D$*, we have* $\varphi(r,s,z_1) = \varphi(r,s,z_2)$ *and* $\psi(r,s,z_1) = \psi(r,s,z_2)$.

Proof. First we prove the following:

($\sharp$) For each triple of distinct points z_0, z_1, z_2 of $\mathbb{T}$, two of the values $\varphi(r, s, z_0), \varphi(r, s, z_1), \varphi(r, s, z_2)$ are equal.

If it is not the case, then there exists an $f \in C^1(\mathbb{T})$ such that

$$f(\varphi(r, s, z_0)) = 1, f(\varphi(r, s, z_1)) = f(\varphi(r, s, z_2)) = 0, \text{ and}$$
$$f'(\psi(r, s, z_0)) = f'(\psi(r, s, z_1)) = f'(\psi(r, s, z_2)) = 0.$$

By (3.6) we have

$$\begin{aligned} Tf(r) + z_0(Tf)'(s) &= \alpha(r, s, z_0) \in \mathbb{T}, \\ Tf(r) + z_1(Tf)'(s) &= Tf(r) + z_2(Tf)'(s) = 0. \end{aligned}$$

Since $z_1 \neq z_2$, the second equality implies $Tf(r) = (Tf)'(s) = 0$ which contradicts the first. This proves ($\sharp$).

For two distinct points $z_1, z_2 \in \mathbb{T}$ and for each $z \in \mathbb{T} \setminus \{z_1, z_2\}$, we have $\varphi(r, s, z) \in \{\varphi(r, s, z_1), \varphi(r, s, z_2)\}$ by ($\sharp$). This and the connectedness of $\mathbb{T}$ imply $\varphi(r, s, z_1) = \varphi(r, s, z_2)$, which proves the conclusion for the map φ.

A similar argument works to prove the statement on ψ. □

In view of the above lemma we write $\varphi(r, s) := \varphi(r, s, z), \psi(r, s) := \psi(r, s, z)$ in (3.6).

Lemma 3.9. *The function $T\mathbf{1}$ is a constant function. Also $\alpha(r, s, z)$ is equal to the function $T\mathbf{1}$.*

Proof. Recall $p_1(D) = I$. The proof is divided into two cases.
Case 1. I is not a singleton.

Applying (3.6) to the constant function $\mathbf{1}$, we see that, for each $(r, s, z) \in X_D$,

$$(T\mathbf{1})(r) + z(T\mathbf{1})'(s) = \alpha(r, s, z). \tag{3.7}$$

Fix an arbitrary point $(r, s) \in D$ and observe the equality $|(T\mathbf{1})(r) + z(T\mathbf{1})'(s)| = 1$ for each $z \in \mathbb{T}$. Lemma 3.5 (1) implies $(T\mathbf{1})(r) \cdot (T\mathbf{1})'(s) = 0$ and $|T\mathbf{1}(r)|^2 + |(T\mathbf{1})'(s)|^2 = 1$.

Let $G = \{(r, s) \in D \mid T\mathbf{1}(r) = 0\}$ and $H = \{(r, s) \in D \mid (T\mathbf{1})'(s) = 0\}$. Then G and H are closed in D, mutually disjoint, and $G \cup H = D$. The connectedness of D implies $H = D$ or $H = \emptyset$. If $H = \emptyset$, then $T\mathbf{1}|I \equiv 0$ and hence we have $(T\mathbf{1})'|I \equiv 0$. For a point $r \in I$ with $(r, s) \in D$, we have $T\mathbf{1}(r) = 0 = (T\mathbf{1})'(s)$, contradicting to $|T\mathbf{1}(r)|^2 + |(T\mathbf{1})'(s)|^2 = 1$. Therefore we have $H = D$ and hence $(T\mathbf{1})' \equiv 0$ and $T\mathbf{1}$ is a constant function. Applying (3.6) we see $\alpha(r, s, z) \equiv T\mathbf{1}$.

This proves the conclusion for Case 1.

Case 2. $I = \{a\}$. In this case $D = \{a\} \times [0,1]$ and we have $\varphi(a,s,z) \equiv a$ for each $(a,s,z) \in X_D$. The equation (3.6) reduces to

$$Tf(a) + z(Tf)'(s) = \alpha(a,s,z)\left\{f(a) + w(a,s,z)f'(\psi(a,s))\right\}. \tag{3.8}$$

Applying the above to $f = \mathbf{1}$ and $f = \mathrm{id}$ respectively, we obtain

$$T\mathbf{1}(a) + z(T\mathbf{1})'(s) = \alpha(a,s,z), \tag{3.9}$$

and

$$\begin{aligned}&(T(\mathrm{id}))(a) + z(T(\mathrm{id}))'(s)\\ &\qquad = \alpha(a,s,z)\{a + iw(a,s,z)\psi(a,s)\}.\end{aligned} \tag{3.10}$$

We claim $T\mathbf{1}(a) \neq 0$. Suppose that $T\mathbf{1}(a) = 0$. Then by the equality (3.9), we have

$$z(T\mathbf{1})'(s) = \alpha(a,s,z). \tag{3.11}$$

Using this to the equality (3.10), we obtain

$$i\alpha(a,s,z)w(a,s,z)\psi(a,s) = T(\mathrm{id})(a) + z(T(\mathrm{id}))'(s) - az(T\mathbf{1})'(s). \tag{3.12}$$

By (3.12) and (3.11), we may rewrite (3.8) as follows:

$$\begin{aligned}&Tf(a) + z(Tf)'(s)\\ &\quad = z(T\mathbf{1})'(s)f(a) + \alpha(a,s,z)w(a,s,z)f'(\psi(a,s))\\ &\quad = z(T\mathbf{1})'(s)f(a) +\\ &\quad +(i\psi(a,s))^{-1}f'(\psi(a,s))\left\{T(\mathrm{id})(a) + z(T(\mathrm{id}))'(s) - az(T\mathbf{1})'(s)\right\}.\end{aligned}$$

Thus we have, for each $f \in C^1([0,1])$,

$$\begin{aligned}&i\psi(a,s)Tf(a) - (T(\mathrm{id})(a))f'(\psi(a,s))\\ &= z\left\{i\psi(a,s)(T\mathbf{1})'(s)f(a) - i\psi(a,s)(Tf)'(s) + (T(\mathrm{id}))'(s) - a(T\mathbf{1})'(s)\right\}.\end{aligned}$$

Noticing that the right hand side (resp. the left hand side) of the above is a linear term (resp. a constant term) with respect to z, we obtain

$$(Tf)(a) = (i\psi(a,s))^{-1}(T(\mathrm{id})(a))f'(\psi(a,s)). \tag{3.13}$$

Applying (3.13) to $f(z) = z^2, z \in \mathbb{T}$, we see

$$(Tf)(a) = (i\psi(a,s))^{-1}(T(\mathrm{id})(a)) \cdot 2i(\psi(a,s))^2 = 2T(\mathrm{id})(a)\psi(a,s). \tag{3.14}$$

As is mentioned in Lemma 3.7, ψ is surjective. Taking two points $s_{\pm 1} \in \mathbb{T}$ so that $\psi(a, s_{\pm 1}) = \pm 1$ and using (3.14), we conclude that $T(\mathrm{id})(a) = 0$. By (3.13), we obtain $Tf(a) = 0$ for each $f \in C^1(\mathbb{T})$. This contradicts the surjectivity of T. Therefore we obtain $T\mathbf{1}(a) \neq 0$.

The equality (3.9) with Lemma 3.5 again implies $T\mathbf{1}(a) \cdot (T\mathbf{1})'(s) = 0$. Thus we have $(T\mathbf{1})' \equiv 0$ and $T\mathbf{1}$ is a constant function. Again by (3.9), we obtain $\alpha(r,s,z) \equiv T\mathbf{1}$ and hence α is a constant function.

This proves the conlusion in the second case and therefore completes the proof of the lemma. □

Let $\alpha(r, s, z) \equiv T\mathbf{1} \equiv \kappa$ from the above lemma. In view of Lemma 3.9, the formula (3.6) is reduced to

$$Tf(r) + z(Tf)'(s) = \kappa \left\{ f(\varphi(r, s)) + w(r, s, z) f'(\psi(r, s)) \right\} \tag{3.15}$$

for each $(r, s, z) \in X_D$ and for each $f \in C^1(\mathbb{T})$.

Lemma 3.10. (1) *For each $(r, s) \in D$, we have $T(\mathrm{id})(r) = \kappa\varphi(r, s)$. In particular $\varphi(r, s)$ does not depend on s and we may write $\varphi(r) := \varphi(r, s)$ and obtain a map $\varphi : I \to I$.*

(2) *We have $w(r, s, z) = zw(r, s, 1)$ for each $(r, s, z) \in X_D$ and we may write $w(r, s) := w(r, s, 1)$.*

Proof. (1) Fix $(r_0, s_0) \in D$ arbitrarily and let $g = \mathrm{id} - \varphi(r_0, s_0)$. We have $g(\varphi(r_0, s_0)) = 0$ and $g'(z) = iz$. Applying the above to g, we obtain

$$Tg(r_0) + z(Tg)'(s_0) = \kappa w(r_0, s_0, z) i\psi(r_0, s_0) \tag{3.16}$$

and in particular $|Tg(r_0) + z(Tg)'(s_0)| = 1$ for each $z \in \mathbb{T}$. By Lemma 3.5 (1), we have $Tg(r_0) \cdot (Tg)'(s_0) = 0$ and $|Tg(r_0)|^2 + |(Tg)'(s_0)|^2 = 1$.

Suppose that $(Tg)'(s_0) = 0$. Then $Tg(r_0) = i\kappa \cdot w(r_0, s_0, z)\psi(r_0, s_0)$ and $w(r_0, s_0, z)$ does not depend on z. Let $w_0 = w(r_0, s_0, z)$ for simplicity. By the surjectivity of T, we may take an $h \in C^1([0, 1])$ so that Th is a real-valued function and

$$Th(r_0) = 0 \text{ and } (Th)'(s_0) = 1.$$

Applying (3.15) to h, we have

$$z = z(Th)'(s_0) = \kappa \left\{ h(\varphi(r_0, s_0)) + w_0 h'(\psi(r_0, s_0)) \right\}.$$

The last term of the above does not depend on z, a contradiction. Therefore we have $(Tg)'(s_0) \neq 0$ and hence $Tg(r_0) = 0$.

By the $\mathbb{C}$-linearity of T we have

$$0 = Tg(r_0) = T(\mathrm{id})(r_0) - \varphi(r_0, s_0)T(\mathbf{1})(r_0) = T(\mathrm{id})(r_0) - \kappa\varphi(r_0, s_0).$$

Since (r_0, s_0) is an arbitrary point of D, we obtain $T(\mathrm{id})(r) = \kappa\varphi(r, s)$ and the map $\varphi(r, s)$ does not depend on s. Thus we may write $\varphi(r) := \varphi(r, s)$. Then we have

$$T(\mathrm{id})(r) = \kappa\varphi(r) \tag{3.17}$$

for each $r \in I$.

In order to prove (2), we start with

$$T(\mathrm{id})(r) + z(T(\mathrm{id}))'(s) = \kappa \left\{ \varphi(r) + w(r, s, z) i\psi(r, s) \right\}.$$

We use (3.17) to obtain

$$z(T(\mathrm{id}))'(s) = i\kappa \cdot w(r,s,z)\psi(r,s).$$

Thus

$$z = \frac{z(T(\mathrm{id}))'(s)}{(T(\mathrm{id}))'(s)} = \frac{w(r,s,z)}{w(r,s,1)}$$

which implies $w(r,s,z) = zw(r,s,1)$. This proves the lemma. □

Let us write $w(r,s) := w(r,s,1)$. Summing up these together, the equality (3.6) is reduced to the following:

$$Tf(r) + z(Tf)'(s) = \kappa \left\{ f(\varphi(r)) + zw(r,s)f'(\psi(r,s)) \right\}. \tag{3.18}$$

Comparing the z-linear and z-constant terms of the above, we obtain the following.

$$Tf(r) = \kappa f(\varphi(r)),\ r \in I, \tag{3.19}$$

$$(Tf)'(s) = \kappa w(r,s)f'(\psi(r,s)),\ s \in \mathbb{T}. \tag{3.20}$$

Now we finish the proof of Proposition 3.4.

Proof. (of Proposition 3.4) The equality (3.2) is exactly (3.19). It remains to show that the map w and ψ do not depend on r.

Suppose that $\psi(r_1,s) \neq \psi(r_2,s)$ for some $r_1, r_2 \in I$. We can choose $f \in C^1(\mathbb{T})$ such that $f'(\psi(r_1,s)) = 0 \neq f'(\psi(r_2,s))$. Then by (3.20)

$$0 = w(r_1,s)f'(\psi(r_1,s)) = (Tf)'(s) = w(r_2,s)f'(\psi(r_2,s)) \neq 0,$$

a contradiction. Thus $\psi(r,s)$ does not depend on r and we write $\psi(s) := \psi(r,s)$. Again by (3.20) we have

$$(T(\mathrm{id}))'(s) = i\kappa w(r,s)\psi(s).$$

Thus $w(r,s)$ does not depend on r either and we write $w(s) := w(r,s)$. Defining a map β by $\beta(s) = \kappa w(s)$ and combining it with (3.20), we obtain (3.3).

This completes the proof of Proposition 3.4. □

3.2. **Proof of Theorem 3.1.** Let $T : C^1(\mathbb{T}) \to C^1(\mathbb{T})$ be a $\mathbb{C}$-linear $\|\cdot\|_{<D>}$-isometry and apply Propostion 3.4 to find a constant $\kappa \in \mathbb{T}$, a continuous map $\beta : \mathbb{T} \to \mathbb{T}$, homeomorphisms $\varphi : I \to I$ and $\psi : \mathbb{T} \to \mathbb{T}$ such that

$$Tf(x) = \kappa f(\varphi(x)),\ x \in I, \tag{3.21}$$

$$(Tf)'(y) = \beta(y)f'(\psi(y)),\ y \in \mathbb{T}. \tag{3.22}$$

In order to prove Theorem 3.1, we examine the map and homeomorphisms above.

Here we recall some basics on the topology of $\mathbb{T}$. As before $\pi : \mathbb{R} \to \mathbb{T}$ denotes the covering map defined by $\pi(t) = \exp(2\pi it)$. For a map $\phi : \mathbb{T} \to \mathbb{T}$, let $\bar{\phi} : \mathbb{R} \to \mathbb{R}$ be a lift of ϕ, that is, a map satisfying $\pi\circ\bar{\phi} = \phi\circ\pi$. For two lifts ϕ_1, ϕ_2 of ϕ, there exists an integer $m \in \mathbb{Z}$ such that $\phi_1(t) - \phi_2(t) = m$ for each $t \in \mathbb{R}$. The degree $\deg(\phi) \in \mathbb{Z}$ of a map $\phi : \mathbb{T} \to \mathbb{T}$ is the unique integer γ_ϕ satisfying $\bar{\phi}(t+1) = \bar{\phi}(t) + \gamma_\phi$ for each $t \in \mathbb{R}$. If ϕ is a homeomorphism, then $\gamma_\phi = \pm 1$. If ϕ is further an isometry, then we have $\frac{d\bar{\phi}}{dt} \equiv \gamma_\phi$. If two maps $\phi_1, \phi_2 : \mathbb{T} \to \mathbb{T}$ are homotopic, in particular if they are sufficiently close, then we have $\deg(\phi_1) = \deg(\phi_2)$.

Lemma 3.11. *Assume that I is not a singleton. Then $\varphi : I \to I$ is an isometry and $\psi|I = \varphi$. Also $\beta|I$ is a constant function.*

Proof. Fix arbitrarily a lift $\bar{\varphi} : \mathbb{R} \to \mathbb{R}$ of φ. For $f \in C^1(\mathbb{T})$, let $\bar{f} = f \circ \pi$. Fix an interior point $e^{2\pi i\theta}$ of I. By differentiating (3.21) we obtain

$$\begin{aligned}
(Tf)'(e^{2\pi i\theta}) &= \frac{d}{dt}\Big|_{t=\theta} Tf(\pi(t)) = \frac{d}{dt}\Big|_{t=\theta} \kappa f(\varphi(\pi(t))) \\
&= \kappa \frac{d}{dt}\Big|_{t=\theta} \bar{f}(\bar{\varphi}(t)) = \kappa \frac{d\bar{\varphi}}{dt}\Big|_{t=\theta} \cdot \frac{d\bar{f}}{ds}\Big|_{s=\bar{\varphi}(\theta)} \\
&= \kappa \frac{d\bar{\varphi}}{dt}(\theta) f'(\varphi(e^{2\pi\theta})).
\end{aligned}$$

Comparing the above with (3.22) and using the continuity of the functions involved, we see the equality

$$\kappa \frac{d\bar{\varphi}}{dt}(\theta) f'(\varphi(x)) = \beta(x) f'(\psi(x)) \tag{3.23}$$

holds for each $x = e^{2\pi i\theta} \in I$ and for each $f \in C^1(\mathbb{T})$. If $\varphi(z) \neq \psi(z)$ for some $z \in I$, take a C^1-function f such that $f'(\varphi(z)) = 0 \neq f'(\psi(z))$. This contradicts (3.23) and hence we have $\psi|I = \varphi$. We then have, again by (3.23),

$$\beta(e^{2\pi i\theta}) = \kappa \frac{d\bar{\varphi}}{dt}(\theta)$$

for each θ. Since $|\beta(x)| = 1$ for each $x \in I$ and since $\frac{d\bar{\varphi}}{dt}$ is a real number, we have $\frac{d\bar{\varphi}}{dt} \equiv \gamma$, a real constant. Thus $\bar{\varphi}(t) = \gamma t + \text{constant}$. Since φ is a homeomorphism, we obtain $\gamma = \pm 1$ and thus φ is an isometry. Also $\beta(z) = \kappa\gamma$ for each $z \in I$ and hence $\beta|I \equiv \kappa\gamma$. □

The above lemma proves Theorem 3.1 when $p_1(D) = \mathbb{T}$. For the proof in the case $p_1(D) \neq \mathbb{T}$, we use the hypothesis $(*)$ to proceed as below.

Lemma 3.12. *Let $\psi : \mathbb{T} \to \mathbb{T}$ be a C^2-diffeomorphism and let $\beta : \mathbb{T} \to \mathbb{T}$ be a C^1-map such that, for each C^1-function $f \in C^1(\mathbb{T})$, there exists a C^1-function $F \in C^1(\mathbb{T})$ such that*

$$F'(z) = \beta(z) f'(\psi(z)), \ z \in \mathbb{T}.$$

Then β is a constant map and ψ is an isometry.

Proof. Fix a lift $\bar{\psi} : \mathbb{R} \to \mathbb{R}$ of ψ. Take an arbitrary $f \in C^1([0,1])$ and let $\bar{f} = f \circ \pi$, $\bar{\beta} = \beta \circ \pi$. For the function F as in the assumption and for each $z = e^{2\pi i\theta} \in \mathbb{T}$ we have

$$\begin{aligned} \frac{d}{dt}\Big|_{t=\theta} \bar{F}(t) &= 2\pi F'(z) = 2\pi\beta(z) f'(\psi(z)) \\ &= \bar{\beta}(\theta) \frac{d}{dt}\Big|_{t=\bar{\psi}(\theta)} \bar{f}(t), \end{aligned}$$

hence,

$$\frac{d}{dt}\bar{F}(t) = \bar{\beta}(t) \frac{d\bar{f}}{dt}(\bar{\psi}(t)). \tag{3.24}$$

Since ψ is a diffeomorphism, we notice that $\dot{\bar{\psi}}(t) := \frac{d}{dt}\bar{\psi}(t) \neq 0$ for each t. Integrating (3.24) and applying integration by parts, we see

$$\begin{aligned} 0 &= \bar{F}(1) - \bar{F}(0) = \int_0^1 \bar{\beta}(t) \frac{d\bar{f}}{dt}(\bar{\psi}(t)) dt \\ &= \left[\left(\frac{\bar{\beta}(t)}{\dot{\bar{\psi}}(t)} \right) \cdot \bar{f}(\bar{\psi}(t)) \right]_0^1 - \int_0^1 \frac{d}{dt} \left\{ \frac{\bar{\beta}(t)}{\dot{\bar{\psi}}(t)} \right\} (\bar{f} \circ \bar{\psi})(t) \\ &= -\int_0^1 \frac{d}{dt} \left\{ \frac{\bar{\beta}(t)}{\dot{\bar{\psi}}(t)} \right\} (\bar{f} \circ \bar{\psi})(t) dt \end{aligned}$$

for each $f \in C^1(\mathbb{T})$. For the last equality we use the equality $\bar{\psi}(1) = \bar{\psi}(0) \pm 1$ and $\dot{\bar{\psi}}(0) = \dot{\bar{\psi}}(1)$ and use also that $\bar{\beta}$ and $\bar{f}$ are both periodic functions of period 1. Every $g \in C^1(\mathbb{T})$ is written as $g = f \circ \psi$ and, for $\bar{g} = g \circ \pi$, we have $\bar{g} = \bar{f} \circ \bar{\psi}$. Hence for each C^1-periodic function $h : \mathbb{R} \to \mathbb{C}$ of period 1, we have:

$$\int_0^1 \frac{d}{dt} \left\{ \frac{\bar{\beta}(t)}{\dot{\bar{\psi}}(t)} \right\} h(t) dt = 0.$$

Therefore the function $\bar{\beta}(t)/\dot{\bar{\psi}}(t)$ is a constant function. Let $\bar{\beta}(t) = c\dot{\bar{\psi}}(t)$ for some constant c. Since $\bar{\psi}(t)$ is a real-valued function and $|\bar{\beta}(t)| \equiv 1$, we see that $\dot{\bar{\psi}}(t)$ is constant and let $\dot{\bar{\psi}}(t) \equiv \gamma$. Then we have $\bar{\psi}(t) = \gamma t + \text{const}$, and the equality $\bar{\psi}(t+1) = \bar{\psi}(t) \pm 1$ implies $\gamma = \pm 1$. Therefore ψ is an isometry and the map β is given by $\beta(z) \equiv c\gamma$, a constant function. This proves the lemma. $\square$

Lemma 3.13. *Let $\psi : \mathbb{T} \to \mathbb{T}$ be an isometry and let $\gamma = \deg\psi \in \{\pm 1\}$. Assume that C^1-functions F and $f \in C^1(\mathbb{T})$ satisfy*

$$F'(x) = \beta f'(\psi(x)),\ x \in \mathbb{T}.$$

Then we have, for an arbitrarily fixed $a \in \mathbb{T}$,

$$F(x) = \beta\gamma f(\psi(x)) + (F(a) - \beta\gamma\psi(a)),\ x \in \mathbb{T}.$$

Proof. For $\bar{F} = F \circ \pi$, we have, by $\frac{d\bar{\psi}}{dt} \equiv \gamma$

$$\frac{d}{dt}\bar{F}(t) = \beta\gamma\frac{d}{dt}(\bar{f} \circ \bar{\psi})(t).$$

Then the conlusion follows. $\square$

We are ready to prove Theorem 3.1.

Proof. (of Theorem 3.1) Let $T : C^1(\mathbb{T}) \to C^1(\mathbb{T})$ be a $\|\cdot\|_{<D>}$-isometry. If $p_1(D) = \mathbb{T}$, then as mentioned eariler, we already have a proof via Lemma 3.11. If it is not the case, T satisfies the condition $(*)$. By Proposition 3.4, we have a constant $\kappa \in \mathbb{T}$, a continuous map $\beta : \mathbb{T} \to \mathbb{T}$, and homeomorphisms $\varphi : I \to I$ and $\psi : \mathbb{T} \to \mathbb{T}$ such that

$$Tf(x) = \kappa f(\varphi(x)),\ x \in I, \tag{3.25}$$

$$(Tf)'(y) = \beta(x) f'(\psi(y)),\ y \in \mathbb{T} \tag{3.26}$$

for each $f \in C^1(\mathbb{T})$. Observe

$$(T(\mathrm{id}))'(z) = i\beta(z)\psi(z),\ (T(\overline{\mathrm{id}}))'(z) = -i\beta(z)\overline{\psi}(z). \tag{3.27}$$

Noticing $|\psi(z)| = 1$, we have

$$\beta(z)^2 = (T(\mathrm{id}))'(z)(T(\overline{\mathrm{id}}))'(z).$$

Thus β is a branch of the square-root function $\left\{(T(\mathrm{id}))'(z)(T(\overline{\mathrm{id}}))'(z)\right\}^{1/2}$. By the hypothesis $(*)$ we see that the function $(T(\mathrm{id}))' \cdot (T(\overline{\mathrm{id}}))'$ is of class C^2. Since $\beta(z) \neq 0$ for each $z \in \mathbb{T}$, we see that β is a C^2-map. From the first equality of (3.27) we see that ψ is a C^2-map as well.

Lemma 3.12 applies to conclude that β is a constant function and ψ is an isometry. Let $\gamma = \deg\psi \in \{\pm 1\}$. Fix a point $a \in I$. By Lemma 3.13, for each $f \in C^1(\mathbb{T})$, Tf is written as

$$Tf(x) = \beta\gamma f(\psi(x)) + \{Tf(a) - \beta\gamma(f(\psi(a)))\},\ x \in I. \tag{3.28}$$

Case 1. $I = \{a\}$. By (3.25) we have $Tf(a) = \kappa f(a)$. Substituting $Tf(a)$ by $\kappa f(a)$ in (3.28) and re-defining $\beta := \beta\kappa$, we obtain the required formula.

Case 2. $I \neq \{a\}$. By (3.25) we have $Tf(a) = \kappa\varphi(a)$. By Lemma 3.11 we see $\varphi = \psi|I$ and $\beta = \kappa\gamma$. Then $\kappa f(\varphi(a)) - \beta\gamma(f(\psi(a))) = 0$ for each $f \in C^1(\mathbb{T})$. Using this in (3.28) we obtain the conclusion. □

3.3. **Proof of Theorem 3.2.** Theorem 3.2 is proved just in the same way that Theorem 2.1 (2) is derived from the Main Theorem in [5] (cf. [4, Example 3.7]).

Proof. (of Theorem 3.2) First we identify $\mathbb{C}$ with the 2-dimensional real vector space $\mathbb{R}^2$ consisting of column vectors. The multiplication of a complex number of modulus 1 on $\mathbb{C}$ corresponds to the left action of a 2×2 matrix $\in SO(2)$, the special orthogonal group, on $\mathbb{R}^2$. Below $O(2)$ denotes the orthogonal group.

Using the above identification, we identify the space $C^1(\mathbb{T})$ with $C^1(\mathbb{T}, \mathbb{R}^2)$, the space of $\mathbb{R}^2$-valued C^1-functions on $\mathbb{T}$ with the C^1-topology. The same formula as that of (1.2) is used to define the derivative of f. Let $C(\mathbb{T}, \mathbb{R}^2)$ be the space of all continous $\mathbb{R}^2$-valued functions on $\mathbb{T}$ with the sup norm and let $D : C^1(\mathbb{T}, \mathbb{R}^2) \to C(\mathbb{T}, \mathbb{R}^2)$ be the operator defined $Df = f', f \in C^1(\mathbb{T}, \mathbb{R}^2)$.

Now let $T : C^1(\mathbb{T}) \to C^1(\mathbb{T})$ be a surjective $\mathbb{C}$-linear $\| \cdot \|_{[p]}$-isometry. By the above identification, T is regarded as a $SO(2)$-equivariant $\mathbb{R}$-linear $\| \cdot \|_{[p]}$-isometry $\hat{T} : C^1(\mathbb{T}, \mathbb{R}^2) \to C^1(\mathbb{T}, \mathbb{R}^2)$. As in the proof of Theorem 2.1 (2) we may apply the Main Theorem of [5] to find homeomorphisms $\varphi, \psi : \mathbb{T} \to \mathbb{T}$ and continuous maps $U, V : \mathbb{T} \to O(2)$ such that

$$\hat{T}f(z) = U(z)f(\varphi(z)), \tag{3.29}$$

$$(\hat{T}f)'(z) = V(z)f'(\psi(z)), \tag{3.30}$$

for each $f \in C^1(\mathbb{T}, \mathbb{R}^2)$ and for each $z \in \mathbb{T}$. By the hypothesis (const), T preserves the constant $\mathbb{R}^2$-valued functions and we see from this that U is a constant map taking a constant matrix $U_* \in O(2)$ as its value. By the $SO(2)$ equivariance of $\hat{T}$ with (3.29) we see U_* commutes with every element of $SO(2)$ and hence it is in $SO(2)$. Similarly we have from (3.30), $V(z) \in SO(2)$. Transforming (3.29) and (3.30) to the original isometry T, we obtain a constant $\kappa \in \mathbb{T}$ and a continuous map $\beta : \mathbb{T} \to \mathbb{T}$ such that

$$Tf(z) = \kappa f(\varphi(z)), \ z \in \mathbb{T}, \tag{3.31}$$

$$(Tf)'(z) = \beta(z)f'(\psi(z)), \ z \in \mathbb{T} \tag{3.32}$$

for each $f \in C^1(\mathbb{T})$. Now exacly the same proof as that of Lemma 3.11 works to completes the proof. □

Remark 3.14. We may apply the above argument to prove Theorem 3.1 when $D \cap \{(z, z) \mid z \in \mathbb{T}\}$ has no interior points. This restriction is caused by the applicability of the Main Theorem of [5].

4. Perturbations of isometry groups of $C^1(\mathbb{T})$

As a counterpart to Theorem 2.6, we make use of Theorem 3.1 to study perturbations of isometry groups associated with norms on $C^1(\mathbb{T})$. The space of all compact connected subsets D of $\mathbb{T}\times\mathbb{T}$ such that $p_2(D)=\mathbb{T}$ is denoted by $\mathcal{D}(\mathbb{T})$. As in Section 2, the space is endowed with the Hausdorff metric. The group $\mathrm{Isom}(\mathbb{T})$ of all isometries of $\mathbb{T}$ is assumed to have the compact-open topology. As in Section 2, an operator $T\in \mathrm{GL}(C^1(\mathbb{T}))$ of the form $Tf(z)=\beta f(\varphi(z))$ for a scalar $\beta\in\mathbb{T}$ and a map $\varphi:\mathbb{T}\to\mathbb{T}$ is denoted by

$$Tf=\beta\cdot(f\circ\varphi),\ \ f\in C^1(\mathbb{T}).$$

Also for a point $c\in\mathbb{T}$, let

$$\begin{aligned}\mathcal{WC}_{\mathrm{Isom}} &= \{T\in GL(C^1(\mathbb{T}))\mid Tf=\beta\cdot(f\circ\psi)\ \forall f\in C^1(\mathbb{T}),\\ &\quad\ \text{for some } \beta\in\mathbb{T},\psi\in\mathrm{Isom}(\mathbb{T})\}\\ \mathcal{WC}_{\mathrm{Isom},c} &= \{T\in \mathcal{WC}_{\mathrm{Isom}}\mid Tf=\beta\cdot(f\circ\psi)\ \forall f\in C^1(\mathbb{T}),\\ &\quad\ \text{for some } \beta\in\mathbb{T},\psi\in\mathrm{Isom}(\mathbb{T}) \text{ with } \psi(c)=c\}.\end{aligned}$$

We start with the following observation.

Lemma 4.1. *Let $(T_t)_{0\le t\le 1}$ be a collection of operators in $\mathcal{WC}_{\mathrm{Isom}}$ of the form*

$$T_tf=\beta_t\cdot(f\circ\psi_t),\ \ f\in C^1(\mathbb{T}).$$

Then $\lim_{t\to0}T_t=T_0$ if and only if $\lim_{t\to0}\beta_t=\beta_0$ and $\lim_{t\to0}\psi_t=\psi_0$.

Proof. Recall that, if$\varphi_1,\varphi_2:\mathbb{T}\to\mathbb{T}$ are sufficiently close, then the degrees of φ_1 and φ_2 coincide. It follows from this that, if $\lim_{\to0}\beta_t=\beta_0$ and $\lim_{t\to0}\psi_t=\psi_0$, then we have $\lim_{t\to0}T_tf=T_0f$ and $\lim_{t\to0}(T_tf)'=(T_0f)'$ for each $f\in C^1(\mathbb{T})$. That is, $\lim_{t\to0}T_t=T_0$.

Conversely if $\lim_{t\to0}T_t=T_0$, then applying the convergence $\lim_{t\to0}T_tf=T_0f$ and $\lim_{t\to0}(T_tf)'=(T_0f)'$ to $f_1=\mathbf{1}$ and $f_2=\mathrm{id}$ respectively, we have $\lim_{t\to0}\beta_t=\beta_0$ and $\lim_{t\to0}(i\beta_t\psi_t)=i\beta_0\psi_0$. From these two, we obtain $\lim_{t\to0}\psi_t=\psi_0$. □

The following is the main result of this section which is a counterpart to Theorem 2.6 for the space $C^1(\mathbb{T})$. For a compact connected subset $D\in\mathcal{D}(\mathbb{T})$, Theorem 3.1 states that each $\mathbb{C}$-linear $\|\cdot\|_{<D>}$-isometry $T: C^1(\mathbb{T})\to C^1(\mathbb{T})$ must be of the form

$$Tf(z)=\begin{cases}\beta f(\varphi(z))+\{\kappa f(c)-\beta f(\varphi(c))\}, & \text{if } p_1(D)=\{c\}\\ \beta f(\varphi(z)), & \text{if } p_1(D) \text{ is not a singleton}\end{cases}\tag{4.1}$$

where $\beta,\kappa\in\mathbb{T}$ and $\varphi\in\mathrm{Isom}(\mathbb{T})$.

Theorem 4.2. *Let $d:[-1,1]\to\mathcal{D}(\mathbb{T})$ be a continuous map and let $I(t)=p_1(d(t))$. Assume that*

(a) *for each* $t \in [-1, 0]$, $I(t) = \{a_t\}$, *a singleton.*
(b) *for each* $t \in (0, 1)$, $I(t)$ *is not a singleton,* $I(t) \neq I$ *and* $I(1) = \mathbb{T}$.

Under the notation of (4.1), we have the following.

(1) *For each* $\tau \in [-1, 0)$ *we have* $\lim_{t\to\tau}(\beta_t, \kappa_t, \varphi_t) = (\beta_\tau, \kappa_\tau, \varphi_\tau)$.
(2) *For each* $\tau \in (0, 1]$ *we have* $\lim_{t\to\tau}(\beta_t, \varphi_t) = (\beta_\tau, \varphi_\tau)$.
(3) $\beta_0 = \kappa_0$, $\varphi_0(a_0) = a_0$ *and* $\lim_{t\to 0}(\beta_t, \varphi_t, a_t) = (\kappa_0, \varphi_0, a_0)$. *In particular* $T_0 \in \mathcal{WC}_{\mathrm{Isom}}$.
(4) *Let* $J(t) = \overline{\mathbb{T} \setminus I(t)}$ *and let* $\{m\} = \lim_{t\to 1} J(t)$ *be the limit with respect to the Hausdorff metric. Then* $\psi_1(m) = m$ *and thus we have* $\psi_1 = \mathrm{id}$ *or* $\psi_1(z) = m^2\bar{z}$, $z \in \mathbb{T}$.

Proof. The statement (2) is a consequence of Lemma 4.1. We prove (1), (3) and (4) below.

(1) We have

$$\lim_{t\to\tau} \|T_t f - T_\tau f\|_\infty = 0, \tag{4.2}$$

$$\lim_{t\to\tau} \|(T_t f)' - (T_\tau f)'\|_\infty = 0, \tag{4.3}$$

for each $f \in C^1(\mathbb{T})$. Applying (4.2) to $f = \mathbf{1}$, we obtain $\lim_{t\to\tau} \kappa_t = \kappa_\tau$. Notice that $(T_t f)'(z) = \beta_t \gamma_t f'(\varphi_t(z))$, where $\gamma_t = \mathrm{deg}\varphi_t$. Hence (4.3) reduces to

$$\lim_{t\to\tau} \beta_t\gamma_t \cdot (f' \circ \varphi_t) = \beta_\tau\gamma_\tau \cdot (f' \circ \varphi_\tau), \ f \in C^1(\mathbb{T}). \tag{4.4}$$

Applying (4.4) to $g_1 = \mathrm{id}$ by noticing $g_1'(z) = iz$, we obtain the equality $\lim_{t\to\tau} \beta_t\gamma_t \cdot \varphi_t = \beta_\tau\gamma_\tau \cdot \varphi_\tau$. Note that $\deg(\beta_t\gamma_t \cdot \varphi_t) = \deg(\varphi_t) = \gamma_t$ and the same applies to $\deg(\beta_\tau \cdot \varphi_\tau)$. Hence we have

$$\gamma_t = \gamma_\tau \tag{4.5}$$

for t sufficiently close to τ. Thus we see

$$\lim_{t\to\tau} \beta_t \cdot \varphi_t = \beta_\tau \cdot \varphi_\tau. \tag{4.6}$$

Apply (4.4) to $g_2(z) = z^2$ by noticing $g_2'(z) = 2iz^2$. By (4.5) we have

$$\lim_{t\to\tau} \beta_t \cdot (\varphi_t)^2 = \beta_\tau \cdot (\varphi_\tau)^2. \tag{4.7}$$

Using (4.7) with (4.6) we obtain $\lim_{t\to\tau} \varphi_t = \varphi_\tau$. Then again by (4.6) we conclude $\lim_{t\to\tau} \beta_t = \beta_\tau$. This proves (1).

(3) We have

$$\lim_{t\downarrow 0} \|T_t f - T_0 f\|_\infty = 0, \tag{4.8}$$

$$\lim_{t\downarrow 0} \|(T_t f)' - (T_0 f)'\|_\infty = 0, \tag{4.9}$$

for each $f \in C^1(\mathbb{T})$. Applying (4.8) to $f = \mathbf{1}$ we have $\lim_{t\downarrow 0} \beta_t = \kappa_0$. Applying (4.8) to $f = \mathrm{id}$ and evaluating at a_0, we have $\lim_{t\downarrow 0} \beta_t \cdot \varphi_t(a_0) = \kappa_0 a_0$. Combining these two we have

$$\lim_{t\downarrow 0} \varphi_t(a_0) = a_0. \tag{4.10}$$

Let $\gamma_t = \mathrm{deg}\varphi_t$. As in (1) we have $\lim_{t\downarrow 0} \beta_t\gamma_t \cdot (f' \circ \varphi_t) = \beta_0\gamma_0 \cdot (f' \circ \varphi_0)$. Using this to g_1 and g_2 of (1) we have

$$\lim_{t\downarrow 0} \beta_t\gamma_t \cdot (\varphi_t) = \beta_0\gamma_0 \cdot (\varphi_0),\ \lim_{t\downarrow 0} \beta_t\gamma_t \cdot (\varphi_t)^2 = \beta_0\gamma_0 \cdot (\varphi_0)^2. \tag{4.11}$$

From the first equality, we conclude $\lim_{t\downarrow 0} \gamma_t = \gamma_0$. As in (1) we conclude from (4.11)

$$\lim_{t\downarrow 0} \varphi_t = \varphi_0,\ \lim_{t\downarrow 0} \beta_t = \beta_0. \tag{4.12}$$

Thus we have $\kappa_0 = \lim_{t\downarrow 0} \beta_t = \beta_0$. Also we have from (4.10) and (4.12) that $a_0 = \lim_{t\downarrow 0} \varphi_t(a_0) = \varphi_0(a_0)$.

(4). For each $t \in [0, 1)$ we have $\psi_t(J_t) = J_t$. Then by (2) we have $\psi_1(m) = \lim_{t\to 1} \psi_t(J(t)) = m$. The last statement is a direct consequence on a general fact on isometries on $\mathbb{T}$.

This completes the proof. □

Corollary 4.3. Let $d : [-1, 1] \to \mathcal{D}(\mathbb{T})$ be a continuous map satisfying the conditions (a),(b) of Theorem 4.2 and let $\| \cdot \|_t = \| \cdot \|_{<d(t)>}$. Let $\{m\} = \lim_{t\to 1} \overline{\mathbb{T} \setminus I(t)}$ be the limit with respect to the Hausdorff metric.

(1)(1-1) $\mathcal{WC}_{\mathrm{Isom}} \subset \mathcal{U}(\| \cdot \|_t)$ for each $t \in [-1, 1]$.

(1-2) For each $t \in [0, 1]$, we have the equality $\mathcal{WC}_{\mathrm{Isom}} = \mathcal{U}(\| \cdot \|_t)$.

(2) If $(T_t)_{0\le t\le 1}$ is a continuous collection of isometries, then $T_0 \in \mathcal{WC}_{\mathrm{Isom}}$ and $T_1 \in \mathcal{WC}_{\mathrm{Isom},m}$.

(3) For each $T \in \mathcal{U}(\| \cdot \|_{-1})$, there exists a continuous collection $(T_t)_{-1\le t\le 1}$ of isometries associated with d such that $T_{-1} = T$.

Proof. It remains to verify (3). Let $a = a_{-1}$ and take $T \in \mathcal{U}(\| \cdot \|_{-1})$ which is of the form

$$Tf(z) = \beta f(\varphi(z)) + (\kappa f(a) - \beta f(\varphi(a)))$$

for some $\beta, \kappa \in \mathbb{T}$ and $\varphi \in \mathrm{Iso}(\mathbb{T})$. We can find continuous maps $[0, 1] \to \mathbb{T}; t \mapsto \beta_t$ and $[0, 1] \to \mathrm{Isom}(\mathbb{T}); t \mapsto \varphi_t$ such that

(1) $\beta_{-1} = \beta, \beta_0 = \kappa$

(2) $\varphi_{-1} = \varphi, \varphi_0(a_0) = a_0$.

Then $T_t f(z) = \beta_t f(\varphi_t(z)) + (\kappa f(a) - \beta_t f(\varphi_t(a)))$ defines a continuous collection of isometries $(T_t)_{-1\le t\le 0}$ associated with $d|[-1, 0]$ such that $T_{-1} = T$ and $T_0 \in \mathcal{WC}_{\mathrm{Isom}}$. Then by defining $T_t \equiv T_0, 0 \le t \le 1$, we obtain a continuous collection $(T_t)_{-1\le t\le 1}$ of isometries associated with d. □

The situation on the interpolation between the norms $\|\cdot\|_\Sigma$ and $\|\cdot\|_M$ is exactly the same as that for the interval. The map

$$c : [1,\infty] \to \mathcal{N}(\mathbb{T}),\ c(p) = \|\cdot\|_{[p]},\ p \in [1,\infty],$$

defines a continuous interpolation between these norms. Theorem 3.1 and Theorem 3.2 yield the equality $\mathcal{U}(\|\cdot\|_{[p]})^{\mathrm{const}} = \mathcal{U}(\|\cdot\|_\Sigma) \cong \mathbb{T}\times\mathbb{T}\times\mathbb{Z}_2$. Thus each $T \in \mathcal{U}(\|\cdot\|_\Sigma)$ defines the trivial continuous collection of isometries associated with c. The full isometry group $\mathcal{U}(\|\cdot\|_M)$ and its behavior on the perturbation above is not known to the author.

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