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ON SPACES WITH RANK k-DIAGONALS OR ZEROSET DIAGONALS

WEI-FENG XUAN AND WEI-XUE SHI

Abstract. In this paper, we make some observations on spaces with rank k-diagonals or zeroset diagonals. In particular, we prove some cardinality inequalities on spaces with rank 2-diagonals or rank 3-diagonals. Moreover, we prove that if a space X has a zeroset diagonal and X^2 is star σ-compact then X is submetrizable.

1. Introduction

All spaces are assumed to be topological T_1-spaces. All notation and terminology not explained in the paper is given in [6]. If A is a subset of a space X and $\mathcal{U}$ is a family of subsets of X, then $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. We also put $\mathrm{St}^0(A,\mathcal{U}) = A$ and for a natural number n, $\mathrm{St}^{n+1}(A,\mathcal{U}) = \mathrm{St}(\mathrm{St}^{\mathrm{n}}(A,\mathcal{U}),\mathcal{U})$. If $A = \{x\}$ for some $x \in X$, then we write $\mathrm{St}^{\mathrm{n}}(x,\mathcal{U})$ instead of $\mathrm{St}^{\mathrm{n}}(\{x\},\mathcal{U})$.

A diagonal sequence of rank k on a space X, where $k \in \omega$, is a countable family $\{\mathcal{U}_n : n \in \omega\}$ of open covering of X such that $\{x\} = \bigcap\{\mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_n) : n \in \omega\}$ for each $x \in X$. A space X has a rank k-diagonal, where $k \in \omega$, if there is a diagonal sequence $\{\mathcal{U}_n : n \in \omega\}$ on X of rank k. A space X has a strong rank k-diagonal, where $k \in \omega$, if there is a diagonal sequence $\{\mathcal{U}_n : n \in \omega\}$ on X such that $\{x\} = \bigcap\{\overline{\mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_n)} : n \in \omega\}$ for each $x \in X$. The rank of the diagonal of X is defined as the greatest natural number k such that X has a rank k-diagonal, if such a number k exists. Note that every rank 3-diagonal implies regular G_δ-diagonal and every submetrizable space has a rank k-diagonal for each $k \in \omega$. For more details on rank k-diagonal, see [2].

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The following results on regular G_δ-diagonal or zeroset diagonal are due to Buzyakova.

Theorem 1.1. ([4]) *If a CCC space X has a regular G_δ-diagonal then the cardinality of X is at most $\mathfrak{c}$.*

Theorem 1.2. ([5]) *If X has a zero-set diagonal and X^2 has countable extent then X is submetrizable.*

In this paper, we make some observations on spaces with rank k-diagonals or zeroset diagonals. In particular, in Section 2, we prove some cardinality inequalities on spaces with rank 2-diagonals or rank 3-diagonals. In Section 3, we mainly prove that if a space X has a zeroset diagonal and X^2 is star σ-compact then X is submetrizable. The results in Section 2 can be compared to Theorem 1.1, and the main result in Section 3 is an improvement of Theorem 1.2, because countable extent implies star countability, which implies star σ-compactness.

We start with some easy consequences. It is known that, if a space X has a rank 1-diagonal then it is T_1 and if X has a rank 2-diagonal then it is Hausdorff. Moreover, we have the following observation.

Proposition 1.3. *If X has a strong rank 2-diagonal then it is Urysohn.*

Proof. Let x and y be two distinct points of X. Since X has a strong rank 2-diagonal, there exists a diagonal sequence $\{\mathcal{U}_n : n \in \omega\}$ on X such that $\{x\} = \bigcap\{\overline{\mathrm{St}^2(x,\mathcal{U}_n)} : n \in \omega\}$. It follows that there exists $n_0 \in \omega$ such that $y \notin \overline{\mathrm{St}^2(x,\mathcal{U}_{n_0})}$. So, there is an open neighborhood U of y such that $\overline{U} \cap \mathrm{St}^2(x,\mathcal{U}_{n_0}) = \emptyset$. Let $V = \mathrm{St}(x,\mathcal{U}_{n_0})$. It is evident that $\overline{V} = \overline{\mathrm{St}(x,\mathcal{U}_{n_0})} \subset \mathrm{St}^2(x,\mathcal{U}_{n_0})$ and hence $\overline{U} \cap \overline{V} = \emptyset$. This proves X is Urysohn. $\square$

A space X has a development if there is a sequence of open covers $\{\mathcal{U}_n : n \in \omega\}$ of X such that for every $x \in X$, the sequence $\{\mathrm{St}(x,\mathcal{U}_n) : n \in \omega\}$ is a base at x. We say that a space X is developable if it has a development. A Moore space is a developable regular space.

Proposition 1.4. *Let X be a developable space. Then for each $k \in \omega$, the following statements are equivalent: (i) X has a strong rank k-diagonal. (ii) X has a rank $(k+1)$-diagonal.*

Proof. Implication (ii) $\Rightarrow$ (i) is obvious, so that it suffices to prove that (i) $\Rightarrow$ (ii). Indeed, by (i) we can fix a development $\{\mathcal{U}_n : n \in \omega\}$ of X satisfying that $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$ and $\{x\} = \bigcap\{\overline{\mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_n)} : n \in \omega\}$ for each $x \in X$. Now let x, y be any two distinct points of X. We have to show that $y \notin \bigcap\{\mathrm{St}^{\mathrm{k}+1}(x,\mathcal{U}_n) : n \in \omega\}$.

Assume the contrary. Then $\mathrm{St}(y,\mathcal{U}_n) \cap \mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_n) \neq \emptyset$ for each $n \in \omega$. We can pick $z_n \in \mathrm{St}(y,\mathcal{U}_n) \cap \mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_n)$. Since the family $\{\mathrm{St}(y,\mathcal{U}_n) : n \in \omega\}$ forms a base at y, the sequence $\{z_n : n \in \omega\}$ converges to y. It is clear that for each $n \in \omega$, $y \in \overline{\{z_{m+n} : m \in \omega\}} \subset \overline{\mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_n)}$. This shows $y \in \bigcap\{\overline{\mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_n)} : n \in \omega\}$. A contradiction! □

2. Cardinality inequalities

In this section, we prove some cardinality inequalities on spaces with rank 2-diagonals or rank 3-diagonals. The cardinality of a set X is denoted by $|X|$, and $[X]^2$ will denote the set of two-element subsets of X. We write ω for the first infinite cardinal and $\mathfrak{c}$ for the cardinality of the continuum.

We say that a space X is weakly Lindelöf if every open cover of X has a countable subfamily whose union is dense in X. A space X has countable chain condition (abbreviated as CCC) if any disjoint family of open sets in X is countable, that is, the Souslin number (or cellularity) of X is at most ω. It is known that every CCC space is weakly Lindelöf. A space X is star countable if whenever $\mathcal{U}$ is an open cover of X, there is a countable subset A of X such that $\mathrm{St}(A,\mathcal{U}) = X$.

We will use the following countable version of a set-theoretic theorem due to Erdös and Radó.

Lemma 2.1. [7, Theorem 2.3] *Let X be a set with $|X| > \mathfrak{c}$ and suppose $[X]^2 = \bigcup\{P_n : n \in \omega\}$. Then there exists $n_0 < \omega$ and a subset S of X with $|S| > \omega$ such that $[S]^2 \subset P_{n_0}$.*

Lemma 2.2. *Let $\{\mathcal{U}_n : n \in \omega\}$ be a diagonal sequence on X of rank k, where $k \geq 1$. If $|X| > \mathfrak{c}$, then there exists an uncountable closed discrete subset S of X such that for any two distinct points $x, y \in S$ there exists $n_0 \in \omega$ such that $y \notin \mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_{n_0})$.*

Proof. By our assumptions, there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that $\{x\} = \bigcap\{\mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_n) : n \in \omega\}$ for every $x \in X$. We may suppose $\mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_{n+1}) \subset \mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_n)$ for any $n \in \omega$. For each $n \in \omega$ let

$$P_n = \left\{\{x,y\} \in [X]^2 : x \notin \mathrm{St}^{\mathrm{k}}(y,\mathcal{U}_n)\}\right\}.$$

Thus, $[X]^2 = \bigcup\{P_n : n \in \omega\}$. Then by Lemma 2.1 there exists a subset S of X with $|S| > \omega$ and $[S]^2 \subset P_{n_0}$ for some $n_0 \in \omega$. It is evident that for any two distinct points $x, y \in S$, $y \notin \mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_{n_0})$. Now we show that S is closed and discrete. If not, let $x \in X$ and suppose x were an accumulation point of S. Since X is T_1, each neighborhood $U \in \mathcal{U}_{n_0}$ of x meets infinitely many members of S. Therefore there exist distinct points y and z in $S \cap U$. Thus $y \in U \subset \mathrm{St}(z,\mathcal{U}_{n_0}) \subset \mathrm{St}^{\mathrm{k}}(z,\mathcal{U}_{n_0})$. It is a contradiction. Thus S has no accumulation points in X; equivalently, S is a closed and discrete subset of X. This completes the proof. □

In Lemma 2.2, if the diagonal rank of X is at least 2, i.e., $k \geq 2$, then S has a disjoint open expansion $\{\mathrm{St}(x,\mathcal{U}_{n_0}) : x \in S\}$. Indeed, if there exist $x, y \in S$ such that $\mathrm{St}(x,\mathcal{U}_{n_0}) \cap \mathrm{St}(y,\mathcal{U}_{n_0}) \neq \emptyset$, then $y \in \mathrm{St}^2(x,\mathcal{U}_{n_0}) \subset \mathrm{St}^{\mathrm{k}}(x,\mathcal{U}_{n_0})$. This is impossible. With the aid of this fact, we obtain

Corollary 2.3. *If X is a CCC space with a rank k-diagonal, where $k \geq 2$, then the cardinality of X is at most $\mathfrak{c}$.*

Moreover, by a well known fact [1, Proposition 2] that if X has uncountable closed discrete subset which has a disjoint open expansion then X is not star countable, we can deduce a further corollary.

Corollary 2.4. *If X is a star countable space with a rank k-diagonal, where $k \geq 2$, then the cardinality of X is at most $\mathfrak{c}$.*

Theorem 2.5. *If X is a weakly Lindelöf normal space and has a rank 2-diagonal, then the cardinality of X does not exceed $\mathfrak{c}$.*

Proof. Assume the contrary. It follows from Lemma 2.2 that there exist an uncountable closed and discrete subset S of X and an open cover $\mathcal{U}_{n_0}$ of X such that for any two distinct points x, y of S, $y \notin \mathrm{St}^2(x,\mathcal{U}_{n_0})$. Clearly, $S \subset \mathrm{St}(S,\mathcal{U}_{n_0})$. Since X is normal, there exists an open subset U of X such that $S \subset U \subset \overline{U} \subset \mathrm{St}(S,\mathcal{U}_{n_0})$. Let $\mathcal{U} = \{\mathrm{St}(x,\mathcal{U}_{n_0}) : x \in S\} \cup X \setminus \overline{U}$.

It is obvious that $\mathcal{U}$ is an open cover of X and hence there exists a countable subset A of S such that $\{\mathrm{St}(x,\mathcal{U}_{n_0}) : x \in A\} \cup X \setminus \overline{U}$ is dense in X, since X is weakly Lindelöf. Pick any point $x \in S \setminus A$. It is evident that $\mathrm{St}(x,\mathcal{U}_{n_0}) \cap U \cap (X \setminus \overline{U}) = \emptyset$. Moreover, $\mathrm{St}(x,\mathcal{U}_{n_0}) \cap U \cap \mathrm{St}(A,\mathcal{U}_{n_0}) = \emptyset$, otherwise there must exist $y \in A$ such that $\mathrm{St}(x,\mathcal{U}_{n_0}) \cap \mathrm{St}(y,\mathcal{U}_{n_0}) = \emptyset$, by symmetry, $y \in \mathrm{St}^2(x,\mathcal{U}_{n_0})$, a contradiction. Therefore, the neighborhood $\mathrm{St}(x,\mathcal{U}_{n_0}) \cap U$ of x witnesses that $x \notin \overline{\mathrm{St}(A,\mathcal{U}_{n_0}) \cup X \setminus \overline{U}} = X$. This is a contradiction and proves that $|X| \leq \mathfrak{c}$. □

Theorem 2.6. *If X is a weakly Lindelöf space and has a rank 3-diagonal, then the cardinality of X does not exceed $\mathfrak{c}$.*

Proof. Assume the contrary. It follows from Lemma 2.2 that there exist an uncountable closed and discrete subset S of X and an open cover $\mathcal{U}_{n_0}$ of X such that for any two distinct points x, y of S, $y \notin \mathrm{St}^3(x,\mathcal{U}_{n_0})$. Clearly, $\overline{\mathrm{St}(S,\mathcal{U}_{n_0})} \subset \mathrm{St}^2(S,\mathcal{U}_{n_0})$. In fact, pick any $x \in \overline{\mathrm{St}(S,\mathcal{U}_{n_0})}$. It is evident that $\mathrm{St}(x,\mathcal{U}_{n_0}) \cap \mathrm{St}(S,\mathcal{U}_{n_0}) \neq \emptyset$. By symmetry, $x \in \mathrm{St}^2(S,\mathcal{U}_{n_0})$. Let $\mathcal{U} = \{\mathrm{St}^2(x,\mathcal{U}_{n_0}) : x \in S\} \cup X \setminus \overline{\mathrm{St}(S,\mathcal{U}_{n_0})}$.

It is obvious that $\mathcal{U}$ is an open cover of X and hence there exists a countable subset A of S such that $\{\mathrm{St}^2(x,\mathcal{U}_{n_0}) : x \in A\} \cup X \setminus \overline{\mathrm{St}(S,\mathcal{U}_{n_0})}$ is dense in X, since X is weakly Lindelöf. Pick any point $x \in S \setminus A$. It is evident that $\mathrm{St}(x,\mathcal{U}_{n_0}) \cap (X \setminus \overline{\mathrm{St}(S,\mathcal{U}_{n_0})}) = \emptyset$. Moreover,

$\mathrm{St}(x,\mathcal{U}_{n_0}) \cap \mathrm{St}^2(A,\mathcal{U}_{n_0}) = \emptyset$, since otherwise there must exist $y \in A$ such that $\mathrm{St}(x,\mathcal{U}_{n_0}) \cap \mathrm{St}^2(y,\mathcal{U}_{n_0}) = \emptyset$, by symmetry, $y \in \mathrm{St}^3(x,\mathcal{U}_{n_0})$, a contradiction. Therefore, the neighborhood $\mathrm{St}(x,\mathcal{U}_{n_0})$ of x witnesses that $x \notin \overline{\mathrm{St}(A,\mathcal{U}_{n_0}) \cup X \setminus \overline{\mathrm{St}(S,\mathcal{U}_{n_0})}} = X$. This is a contradiction and proves that $|X| \leq \mathfrak{c}$. □

If we drop the condition "weakly Lindelöf" in Theorem 2.5 or Theorem 2.6, the conclusion is no longer true, as can be seen in the following example.

Example 2.7. Let D be a discrete space with $|D| = 2^{\mathfrak{c}}$. Clearly, for any $k \in \omega$, it has a rank k-diagonal and it is not weakly Lindelöf.

3. Zeroset diagonals

For any spaces X and Y, the symbol Y^X here denotes the set of all continuous mappings of X to Y. A space X is star σ-compact if whenever $\mathcal{U}$ is an open cover of X, there is a σ-compact subspace $A \subset X$ such that $\mathrm{St}(A,\mathcal{U}) = X$. Clearly, every star countable space is star σ-compact. But a star σ-compact space need not be star countable, see [3, Example 2.5].

We say that a space X is submetrizable if there exists a continuous injection of X into a metrizable space. A space X has a zeroset diagonal if there is a continuous mapping $f : X^2 \to [0,1]$ with $\Delta_X = f^{-1}(0)$, where $\Delta_X = \{(x,x) : x \in X\}$. In general, having a zeroset diagonal doesn't guarantee submetrizability [9]. In [8], Martin asks for what classes of spaces the presence of a zeroset diagonal implies submetrizability. In this section, we prove that if X has a zeroset diagonal and X^2 is star σ-compact then X is submetrizable.

Lemma 3.1. *Let X^2 be star σ-compact and C be a closed subset of X^2. If $\mathcal{U}$ is an open cover of C, then there exists a σ-compact subset A of X, such that $C \subset \mathrm{St}(X \times A,\mathcal{U})$.*

Proof. It is not difficult to see that $\mathcal{U} \cup \{X^2 \setminus C\}$ is an open cover of X^2. By virtue of star σ-compactness, there exists a σ-compact subset K of X^2, such that $\mathrm{St}(K,\mathcal{U} \cup \{X^2 \setminus C\}) = X^2$. Let $A = p(K)$, where p is the projection of X^2 onto the second factor. Since any continuous image of a compact set is compact, it follows that A is σ-compact. Therefore, we have $C \subset \mathrm{St}(K,\mathcal{U}) \subset \mathrm{St}(X \times A,\mathcal{U})$. □

Lemma 3.2. *Let $f : X^2 \to [0,1]$ be a continuous mapping and M be any subset of X. Then the mapping f_M from X to $[0,1]^M$ with the compact-open topology is continuous by defining $(f_M(x))(a) = f(x,a)$, where $x \in X$ and $a \in M$.*

Proof. We can apply Theorem 3.4.1 of [6] to conclude that the compact-open topology on $[0,1]^M$ is proper, i.e., for every space Z and any continuous mapping $g : Z \times M \to [0,1]$, the mapping $f_M : Z \to [0,1]^M$ defined by $(f_M(z))(a) = g(z,a)$ is continuous, where $z \in Z$ and $a \in M$. Let $Z = X$ and $g = f|_{X\times M}$. It is easy to check that f_M is continuous. □

Theorem 3.3. *If X has a zeroset diagonal and X^2 is star σ-compact then X is submetrizable.*

Proof. By our hypothesis, there exists a continuous mapping $f : X^2 \to [0,1]$ such that $\Delta_X = f^{-1}(0)$. Let $C_n = f^{-1}([\frac{1}{n},1])$ for each $n \in \mathbb{N}$. Obviously, C_n is closed in X^2 and $X^2 \setminus \Delta_X = \bigcup_{n\in\mathbb{N}} C_n$. For each $n \in \mathbb{N}$, we define $\mathcal{U}_n$ as following:

$$\mathcal{U}_n = \{U\times V : U\times V \subset f^{-1}((\frac{1}{2n},1]), V\times V \subset f^{-1}([0,\frac{1}{2n})); U,V \text{ are open in } X\}.$$

Note that $\mathcal{U}_n$ is an open cover of C_n. We apply Lemma 3.1 to conclude that there exists a σ-compact subset $K_n = \bigcup_{m\in\mathbb{N}} M_{n,m}$, where $M_{n,m}$ is compact in X, such that $C_n \subset \mathrm{St}(X \times K_n, \mathcal{U}_n)$.

Define the mapping $f_{M_{n,m}} : X \to [0,1]^{M_{n,m}}$ by $\big(f_{M_{n,m}}(x)\big)(a) = f(x,a)$ for each $m \in \mathbb{N}$, where $x \in X$ and $a \in M_{n,m}$. By Lemma 3.2, it is easy to see that $f_{M_{n,m}}$ is continuous. Next, we define $F = \Delta f_{M_{n,m}} : X \to \prod_{n,m\in\mathbb{N}}[0,1]^{M_{n,m}}$. Clearly, F is continuous and since every space $[0,1]^{M_{n,m}}$ with the compact-open topology is metrizable, it follows that the space $\prod_{n,m\in\mathbb{N}}[0,1]^{M_{n,m}}$ is metrizable.

To finish the proof, it is enough to check that F is an injection. Pick any two distinct points $x,y \in X$. Since $(x,y) \in X^2 \setminus \Delta_X = \bigcup_{n\in\mathbb{N}} C_n$, there exists some $n \in \mathbb{N}$ such that $(x,y) \in C_n$. Notice that $C_n \subset \mathrm{St}(X \times K_n, \mathcal{U}_n)$, so there exist $a \in K_n$ and $U \times V \in \mathcal{U}_n$ such that $(x,y) \in U \times V$ and $a \in V$, and hence $(x,a) \in U \times V$, $(y,a) \in V \times V$. By virtue of this fact and the definition of $\mathcal{U}_n$, we have $f(x,a) \neq f(y,a)$. Now let us consider the mapping $f_{M_{n,m}}$, where $m \in \mathbb{N}$ which makes $a \in M_{n,m} \subset K_n$. Since $(f_{M_{n,m}}(x))(a) = f(x,a) \neq f(y,a) = (f_{M_{n,m}}(y))(a)$, it follows that $f_{M_{n,m}}(x) \neq f_{M_{n,m}}(y)$, and thus $F(x) \neq F(y)$. □

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