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by

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QUANTALE-VALUED GENERALIZATIONS OF APPROACH SPACES: L-APPROACH SYSTEMS

GUNTHER JÄGER

Abstract. We define and study quantale-valued approach systems. We show that the resulting category is topological and study its relation to other categories of quantale-valued generalizations of approach spaces, such as the categories of quantale-valued approach spaces and of quantale-valued gauge spaces. We pay particular attention to the probabilistic case.

1. Introduction

The category of approach spaces, introduced in [13], is a common supercategory of the categories of metric and topological spaces. The theory of these spaces is far developed and has many applications as is demonstrated in e.g. [14, 15]. In simple terms one may say, that the theory of approach spaces is "metrical" in the sense that an approach space is often either defined by a *point-set distance function* or a suitable family of metrics (a so-called *gauge*) or by families of "local distances" (so-called *approach systems*). Therefore, the reservations that apply to metric spaces in terms of the precise knowledge of distances between elements apply also to approach spaces, and probabilistic generalizations seem natural. In [7], we introduced such a probabilistic generalization of approach spaces and suggested to even consider a further quantale-valued generalization. This was taken up in [10], who showed that such quantale-valued approach spaces fit nicely into the framework of monoidal topology [5]. In both the probabilistic case [7] and the quantale-valued case [10], the basic definition is in terms of a quantale-valued point-set distance function and also equivalent forms in terms of quantale-valued (ultra-)filter convergence are established.

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Classically, in many applications, the definition of an approach space in terms of gauges or approach systems is natural. It is therefore of interest to have similar definitions for the probabilistic case at our disposal. As is argued in [10], it is more transparent to develop the theory right from the start based on an arbitrary quantale and then recover special instances by suitable choices of the quantale. We then obtain e.g. Lowen's classical theory by choosing as quantale the extended half line $[0,\infty]$ ordered opposite to the natural order and addition as quantale operation. For the probabilistic case we choose as quantale the set Δ^+ of distance distribution functions with a sup-continuous triangle function as quantale operation.

In [8] we defined and studied quantale-valued gauge spaces. We showed that only under strong restrictions on the quantale, the extension to the quantale-valued case of the "classical" embedding functor from the category of approach spaces into the category of gauge spaces (cf. [14, 15]) is an isomorphism. In particular, in the probabilistic case these restrictions are in general not met. In this paper we address the problem of extending the definition of approach systems to the quantale-valued case. After collecting necessary results and notations in the preliminary section 2 — with a certain emphasis on the quantale of distance distribution functions, where we produce results that we did not find anywhere in the literature – we review in Section 3 the definitions of L-metric spaces, L-approach spaces and L-gauge spaces and their relations, where L is a quantale. Section 4 is then devoted to quantale-valued approach systems and we show that the resulting category is topological. In Section 5 we investigate the relations between the categories of quantale-valued gauge spaces and of quantale-valued approach system spaces and we show that the former can be reflectively embedded into the latter. We give two examples that show that, even for a linear order or in the probablistic case, the two categories may not be isomorphic under the natural embeddings. The final Section 6 then shows that the category of quantale-valued approach spaces can be coreflectively embedded into the category quantale-valued approach system spaces.

2. Preliminaries

Let $(L,\leq)$ be a complete lattice, where $\top \neq \bot$ for the top element $\top$ and the bottom element $\bot$. In any complete lattice L we can define the *well-below relation* $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. Then $\alpha \leq \beta$ whenever $\alpha \lhd \beta$, and $\alpha \lhd \bigvee_{j\in J}\beta_j$ iff $\alpha \lhd \beta_i$ for some $i \in J$. A complete lattice is *completely distributive* if and only if we have $\alpha = \bigvee\{\beta \,:\, \beta \lhd \alpha\}$ for any $\alpha \in L$, see e.g. Theorem 7.2.3 in [1]. Similarly, we can define the *well-above relation* $\beta \succ \alpha$ if for all subsets $D \subseteq L$ such that $\bigwedge D \leq \alpha$ there is $\delta \in D$ with $\delta \leq \beta$.

Then $\beta \succ \alpha$ implies $\beta \geq \alpha$, and $\alpha \succ \bigwedge_{j\in J} \beta_j$ iff $\alpha \succ \beta_j$ for some $j \in J$. L is completely distributive iff $\alpha = \bigwedge\{\beta \in L \ : \ \beta \succ \alpha\}$ for any $\alpha \in L$. Clearly, in a complete lattice L we have $\alpha \lhd \beta$ iff $\alpha \succ^{op} \beta$ in the opposite order. For more results on lattices we refer to [4].

The triple $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice, is called a *quantale* if $(L, *)$ is a semigroup, and the semigroup operation $*$ distributes over arbitrary joins, i.e. if for all $\alpha_j, \beta \in L$, $(j \in J)$ we have

$$\left(\bigvee_{j\in J} \alpha_j\right) * \beta = \bigvee_{j\in J} (\alpha_j * \beta) \quad \text{ and } \quad \beta * \left(\bigvee_{j\in J} \alpha_j\right) = \bigvee_{j\in J} (\beta * \alpha_j).$$

A quantale $\mathsf{L} = (L, \leq, *)$ is called *commutative* if $(L, *)$ is a commutative semigroup and it is called *integral* if the top element of L acts as the unit, i.e. if $\alpha * \top = \top * \alpha = \alpha$ for all $\alpha \in L$.

Standing assumption in this paper: *We consider in this paper only commutative and integral quantales* $\mathsf{L} = (L, \leq, *)$ *with completely distributive lattices* L.

In any such quantale we can define an implication $\rightarrow: L \times L \longrightarrow L$ by $\alpha \rightarrow \beta = \bigvee\{\gamma \in L \ : \ \alpha * \gamma \leq \beta\}$ for $\alpha, \beta \in L$. Then $\alpha * \beta \leq \gamma$ iff $\alpha \leq \beta \rightarrow \gamma$ for all $\alpha, \beta, \gamma \in L$. We say that the quantale $\mathsf{L} = (L, \leq, *)$ satisfies the *strong De Morgan law* if $(\bigwedge_{j\in J} \alpha_j) \rightarrow \beta = \bigvee_{j\in J}(\alpha_j \rightarrow \beta)$ for all $\alpha_j, \beta \in L$, $j \in J$. A *value quantale* [3] is a quantale L with an underlying completely distributive lattice $(L, \leq)$ such that $\alpha \vee \beta \lhd \top$ whenever $\alpha, \beta \lhd \top$. In a value quantale, if $\alpha \lhd \top$, then there is $\beta \lhd \top$ such that $\alpha \lhd \beta * \beta$, see [3].

Example 2.1. (1) **Left-continuous t-norms.** A *triangular norm* or *t-norm* is a binary operation $*$ on the unit interval $[0, 1]$ which is associative, commutative, non-decreasing in each argument and which has 1 as the unit. The triple $([0, 1], \leq, *)$ can be considered as a quantale if the t-norm is left-continuous, i.e. if $(\bigvee_{j\in J} \alpha_j) * \beta = \bigvee_{j\in J}(\alpha_j * \beta)$ for all $\alpha_j, \beta \in [0, 1], j \in J$. The three most commonly used (left-continuous) t-norms are:

- the minimum t-norm: $\alpha * \beta = \alpha \wedge \beta$,
- the product t-norm: $\alpha * \beta = \alpha \cdot \beta$,
- the Lukasiewicz t-norm: $\alpha * \beta = (\alpha + \beta - 1) \vee 0$.

Later, in some instances, we are also interested in *right-continuous* t-norms. These are t-norms, which distribute over arbitrary meets, i.e. that satisfy $(\bigwedge_{j\in J} \alpha_j) * \beta = \bigwedge_{j\in J}(\alpha_j * \beta)$ for all $\alpha_j, \beta \in [0, 1], j \in J$. An example of a left-continuous t-norm that is not right-continuous is the so-called *nilpotent minimum*, see e.g. [9].

(2) **Lawvere's quantale.** The interval $[0, \infty]$ with the opposite order and addition as the quantale operation $\alpha * \beta = \alpha + \beta$ (extended by $\alpha + \infty = \infty + a = \infty$ for all $\alpha, \beta \in [0, \infty]$) is a quantale, see e.g. [12, 3]. Lawvere's quantale is a value quantale [3], the quantale operation distributes over arbitrary meets and the strong De Morgan law is valid.
(3) A *frame* is a quantale with $* = \wedge$.
(4) A *complete MV-algebra* is a commutative and integral quantale $\mathsf{L} = (L, \leq, *)$ which satisfies $(\alpha \to \beta) \to \beta = \alpha \vee \beta$ for all $\alpha, \beta \in L$, [6]. In a complete MV-algebra the quantale operation distributes over arbitrary meets and the strong De Morgan law is valid.
(5) **The quantale of distance distribution functions.** A function $\varphi : [0, \infty] \longrightarrow [0, 1]$, which satisfies $\varphi(x) = \sup_{z<x} \varphi(z)$ for all $x \in [0, \infty]$ is called a *distance distribution function* [18]. We note that a distance distribution function is non-decreasing and satisfies $\varphi(0) = 0$. The set of all distance distribution functions is denoted by Δ^+. For example, for each $0 \leq a \leq \infty$ the functions

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a \\ 1 & \text{if } a < x \leq \infty \end{cases}$$

are in Δ^+. The set Δ^+ is ordered pointwise; the bottom element of Δ^+ is ε_∞ and the top element is ε_0. The set Δ^+ with this order then becomes a completely distributive lattice [3]. We note that joins and finite meets are computed pointwise but that $\bigwedge_{i \in I} \varphi_i$ is in general not the pointwise infimum. However, we have the following result.

Lemma 2.2. *Let $\psi_j \in \Delta^+$ for $j \in J$. Then*

$$(\bigwedge_{j \in J} \psi_j)(x) = \sup_{z<x} \inf_{j \in J} \psi_j(z),$$

with the pointwise infimum $\inf_{j \in J} \psi_j$.

Proof. We first note that for a function $\psi : [0, \infty] \longrightarrow [0, 1]$ if we define $\varphi : [0, \infty] \longrightarrow [0, 1]$ by $\varphi(x) = \sup_{z<x} \psi(z)$, then $\varphi \in \Delta^+$. Clearly $\bigwedge_{j \in J} \psi_j \leq \inf_{j \in J} \psi_j$ and therefore by left-continuity of $\bigwedge_{j \in J} \psi_j \in \Delta^+$, we have

$$\bigwedge_{j \in J} \psi_j(x) = \sup_{z \leq x} \bigwedge_{j \in J} \psi_j(z) \leq \sup_{z<x} \inf_{j \in J} \psi_j(z).$$

Furthermore, $\sup_{z<x} \inf_{j \in J} \psi_j(z) \leq \inf_{j \in J} \sup_{z<x} \psi_j(z) = \inf_{j \in J} \psi_j(x) \leq \psi_j(x)$ for all $j \in J$ and as the function $\varphi(x) = \sup_{z<x} \inf_{j \in J} \psi_j(z)$ is in Δ^+, the claim follows. $\square$

A binary operation, $* : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, which is commutative, associative, non-decreasing in each place and that satisfies the boundary condition $\varphi * \varepsilon_0 = \varphi$ for all $\varphi \in \Delta^+$, is called a *triangle function* [17, 18]. A triangle function is called *sup-continuous* [18], if $(\bigvee_{i\in I} \varphi_i) * \psi = \bigvee_{i\in I}(\varphi_i * \psi)$ for all $\varphi_i, \psi \in \Delta^+$, $(i \in I)$, i.e. if $(\Delta^+, \leq, *)$ is a quantale. It is shown in [3] that $(\Delta^+, \leq, *)$ is a value quantale.

The following two examples for sup-continuous triangle functions are induced by a left-continuous t-norm $*$ on $[0, 1]$ and are used later on. We define $\circledast : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ and $\boxast : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ by $\varphi \circledast \psi(x) = \bigvee_u \varphi(u) * \psi(x-u)$ and $\varphi \boxast \psi(x) = \varphi(x) * \psi(x)$.

In the sequel, we discuss the properties of the distributivity of these quantale operations over arbitrary meets and the strong De Morgan law.

Proposition 2.3. *Let the t-norm $*$ distribute over arbitrary meets and let $\varphi, \psi_j \in \Delta^+$ $(j \in J)$. Then $(\bigwedge\limits_{j\in J} \psi_j) \boxast \varphi = \bigwedge\limits_{j\in J}(\psi_j \boxast \varphi)$.*

Proof. We always have $(\bigwedge_{j\in J} \psi_j) \boxast \varphi \leq \bigwedge_{j\in J}(\psi_j \boxast \varphi)$. For the converse inequality, let $x \in (0, \infty)$. Then, using Lemma 2.2,

$$\begin{aligned}
\bigwedge_{j\in J}(\varphi_j \boxast \psi)(x) &= \sup_{z<x} \inf_{j\in J} (\varphi_j \boxast \psi(z)) \;=\; \sup_{z<x} \inf_{j\in J} (\varphi_j(z) * \psi(z)) \\
&= \sup_{z<x} (((\inf_{j\in J} \varphi_j(z)) * \psi(z)) \\
&\leq (\sup_{z<x} \inf_{j\in J} \varphi_j(z)) * (\sup_{z<x} \psi(z)) \\
&= (\bigwedge_{j\in J} \varphi_j(x)) * (\psi(x)) \;=\; (\bigwedge_{j\in J} \varphi) \boxast \psi(x). \qquad \square
\end{aligned}$$

For the triangle function $\circledast$ a similar result is not true, not even for finite meets. This can be seen by choosing the product t-norm and $\varphi_1(x) = \frac{1}{2}\varepsilon_0$, $\varphi_2(x) = \varepsilon_1$ and $\psi \in \Delta^+$ defined by $\psi(x) = x$ for $0 \leq x \leq 1$ and $\psi(x) = 1$ for $x > 1$. Then $(\varphi_1 \wedge \varphi_2) \odot \psi \ngeq (\varphi_1 \odot \psi) \wedge (\varphi_2 \odot \psi)$.

The following examples show, that in general $(\Delta^+, \circledast)$ and $(\Delta^+, \boxast)$ both do not satisfy the strong De Morgan law, even if the t-norm $*$ satisfies it. First we show the following result.

Lemma 2.4. *Let $*$ be a t-norm on $[0, 1]$ and let $\varphi, \psi \in \Delta^+$. If we denote the implication in $([0, 1], \leq, *)$ by $\alpha \overset{*}{\to} \beta$ and the implication in $(\Delta^+, \leq, \circledast)$ by $\varphi \overset{\circledast}{\to} \psi$, then for each $x \in [0, \infty]$ we* *have*

$$\varphi \overset{\circledast}{\to} \psi(x) = \sup_{z<x} \inf_u (\varphi(u) \overset{*}{\to} \psi(z+u)).$$

Proof. For $u \in [0,\infty]$ we define $\eta_u \in \Delta^+$ by $\eta_u(x) = \varphi(u) \overset{*}{\to} \psi(x+u)$ if $x > 0$ and $\eta_u(0) = 0$. Then

$$\begin{aligned} \varphi \overset{\circledast}{\to} \psi &= \bigvee\{\eta \in \Delta^+ \,:\, \varphi(u) * \eta(y) \leq \psi(y+u) \forall y, u\} \\ &= \bigvee\{\eta \in \Delta^+ \,:\, \eta \leq \eta_u \forall u\} \;=\; \bigwedge_u \eta_u. \end{aligned}$$

From Lemma 2.2 then the claim follows. □

We consider now the product t-norm and, for $n \in \mathbb{N}$, $\varphi_n = \frac{1}{n}\varepsilon_0 \vee \varepsilon_\infty$ and $\psi = \varepsilon_1$. Then for $x \leq 1$ we have

$$\varphi_n \overset{\odot}{\to} \psi(x) \leq \sup_{z<x} \inf_{u:z+u\leq 1} \left(\frac{\varepsilon(z+u)}{\varphi_n(u)} \wedge 1 \right) = 0.$$

However $\bigwedge_{n\in\mathbb{N}} \varphi_n = \varepsilon_\infty$ and hence $(\bigwedge_{n\in\mathbb{N}} \varphi_n) \overset{\odot}{\to} \psi = \varepsilon_0$. The same example can be used for $\boxdot$. Here we have $\varphi_n \overset{\boxdot}{\to} \psi = \varepsilon_1$ for all $n \in \mathbb{N}$ and again $(\bigwedge_{n\in\mathbb{N}} \varphi_n) \overset{\boxdot}{\to} \psi = \varepsilon_0$.

Later, the following condition will be crucial.

Definition 2.5. [7] A quantale $\mathsf{L} = (L, \leq, *)$ satisfies the condition (I) if

(I) we have $\beta \not\leq \gamma * \beta$ whenever $\beta \succ \bot, \gamma \lhd \top$.

We showed in [7] that if the quantale $\mathsf{L} = (L, \leq, *)$ satisfies the *strong cancellation law*

(SCL) for all $\gamma, \alpha \in L, \beta \succ \bot : \gamma * \beta \leq \alpha * \beta$ implies $\gamma \leq \alpha$

and if $\top \ntriangleleft \top$ then the condition (I) is satisfied.

Example 2.6. (1) The two-point chain $L = \{0,1\}$ does not satisfy the condition (I) as $1 \lhd 1$.

(2) In $\mathsf{L} = ([0,\infty], \geq, +)$ the strong cancellation law is valid and hence L satisfies the condition (I).
(3) For $\mathsf{L} = ([0,1], \leq, \cdot)$ with the product t-norm the strong cancellation law is satisfied and hence L satisfies the condition (I). In fact, for any *strict t-norm* $*$ on $[0,1]$ (see [9]), $\mathsf{L} = ([0,1], \leq, *)$ satisfies the strong cancellation law and hence the condition (I).
(4) For a *nilpotent t-norm* $*$ on $[0,1]$ (see [9]), $\mathsf{L} = ([0,1], \leq, *)$ satisfies the condition (I).
(5) A frame $(L, \leq, \wedge)$ does in general not satisfy (I).

(6) The 4-element Boolean algebra $\{\bot, \alpha, \beta, \top\}$ with $\alpha \wedge \beta = \bot$ and $\alpha \vee \beta = \top$ satisfies (I) but not the strong cancellation law.
(7) In an MV-algebra $(L, \leq, *)$ we have $\beta \leq \alpha * \beta$ iff $\beta \wedge (\alpha \to \bot) = \bot$. Hence an MV-algebra satisfies (I) if and only if $\beta \wedge (\alpha \to \bot) \neq \bot$ whenever $\alpha \ntriangleleft \top$ and $\beta \not\succ \bot$. In particular, if L has no zero-divisors for $\wedge$ and if $\top \ntriangleleft \top$ and $\bot \not\succ \bot$, then $(L, \leq, *)$ satisfies (I). This applies e.g. to a linearly ordered MV-algebra.
(8) As a final example we consider the lattice Δ^+. For $0 < \delta < \infty$ and $0 < \gamma \leq 1$ we define $f_{\delta\gamma} = \gamma \cdot \varepsilon_\delta \in \Delta^+$. Then if $\varphi \lhd \varepsilon_0$ there is $\epsilon < 1$ such that $\varphi \leq f_{\delta\epsilon}$. As a consequence, we can show the following result.

Lemma 2.7. *Let $*$ be a t-norm in $[0,1]$ that satisfies the property (I), i.e. for which $0 < \beta$ and $\epsilon < 1$ implies $\epsilon * \beta < \beta$. Then both $(\Delta^+, \circledast)$ and $(\Delta^+, \boxtimes)$ satisfy the condition (I).*

Proof. The case $(\Delta^+, \circledast)$ was shown in [8]. We prove the case for $(\Delta^+, \boxtimes)$. Let $\psi \succ \varepsilon_\infty$ and $\varphi \lhd \varepsilon_0$. Then there is $x \in [0, \infty)$ with $\psi(x) > 0$ and $\varphi \leq f_{\delta\epsilon}$. If we assume $\psi \leq \varphi \boxtimes \psi$, then $0 < \psi(x) \leq \varphi(x) * \psi(x)$ and hence $\varphi(x) = \epsilon$ and we obtain $\psi(x) \leq \epsilon * \psi(x)$, a contradiction. □

The condition (I) is needed for the following result.

Proposition 2.8. *Let L be a linearly ordered quantale that satisfies the condition (I) and for which $*$ distributes over arbitrary meets. If $\bigwedge_{j \in J} \delta_j \neq \bot$ and if $\alpha \lhd \top$, then there is $j_0 \in J$ such that $\alpha * \delta_{j_0} \leq \bigwedge_{j \in J} \delta_j$.*

Proof. We note that for a linearly ordered complete lattice, $\alpha > \beta$ implies $\alpha \succ \beta$. Hence if $\bigwedge_{j \in J} \delta_j \neq \bot$, then $\bigwedge_{j \in J} \delta_j \succ \bot$. We assume now that for all $i \in J$ we have $\bigwedge_{j \in J} \delta_j < \delta_i * \alpha$. Then $\bigwedge_{j \in J} \delta_j \leq \bigwedge_{j \in J} (\delta_j * \alpha) = (\bigwedge_{j \in J} \delta_j) * \alpha$, a contradiction to condition (I). □

Quantales that satisfy the assumptions of Proposition 2.8 are e.g. the unit interval $L = [0,1]$ with the product t-norm or Lawvere's quantale. That in general we cannot omit the assumptions of the proposition is shown by the following two examples.

Example 2.9. Let $L = [0,1] \cup \{\bot = -\infty, \top = \infty\}$ with the natural order. Then L is linearly ordered. We consider the product as quantale operation. As we have $\top \lhd \top$, the condition (I) is not satisfied. Let $\delta_n = \frac{1}{n}$ and $\alpha = 1 \lhd \top$. Then $\bigwedge_{n \in \mathbb{N}} \delta_n = 0 \succ \bot$ but for all $n \in \mathbb{N}$ we have $\frac{1}{n} * 1 > 0$. □

Example 2.10. We consider $L = \Delta^+$ and $\delta_n = \varepsilon_{1 - \frac{1}{n}}$ and $\alpha = f_{\frac{1}{2}, \frac{1}{2}} \lhd \varepsilon_0$. Then $\bigwedge_{n=2}^\infty \delta_n = \varepsilon_1 \neq \varepsilon_\infty$. For the triangle function $\boxdot$ induced by the product t-norm we have $f_{\delta,\epsilon} \boxdot f_{\gamma,\eta} = f_{\delta\vee\gamma,\epsilon\eta}$. Then the condition (I) is satisfied but $\alpha \boxdot \delta_n = f_{1-\frac{1}{n},\frac{1}{2}} \not\leq \varepsilon_1$ for all $n = 2, 3, 4, ...$ $\square$

We assume some familiarity with category theory and refer to the textbooks [2] and [16] for more details and notation. A *construct* is a category C with a faithful functor $U : \mathsf{C} \longrightarrow \mathsf{SET}$, from C to the category of sets. A functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ between the constructs C, D with faithful functors $U : \mathsf{C} \longrightarrow \mathsf{SET}$ and $V : \mathsf{D} \longrightarrow \mathsf{SET}$, respectively, is called a *concrete functor* if $U = V \circ F$. We always consider a construct as a category whose objects are structured sets (X, ξ) and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, i.e. if for every source $(f_i : X \longrightarrow (X_i, \xi_i))_{i\in I}$ there is a unique structure ξ on X, such that a mapping $g : (Y, \eta) \longrightarrow (X, \xi)$ is a morphism if and only if for each $i \in I$ the composition $f_i \circ g : (Y, \eta) \longrightarrow (X_i, \xi_i)$ is a morphism.

A pair of concrete functors $G : \mathsf{C} \longrightarrow \mathsf{D}$, $F : \mathsf{D} \longrightarrow \mathsf{C}$ between the constructs C, D, is a *Galois correspondence* if $id_{\mathsf{D}} \leq G \circ F$ and $F \circ G \leq id_{\mathsf{C}}$. The functor F is the called a *left-adjoint* and G is called a *right-adjoint.*

3. L-approach spaces, L-gauge spaces and L-metric spaces

In the sequel, let $\mathsf{L} = (L, \leq, *)$ be a commutative and integral quantale, where $(L, \leq)$ is completely distributive.

Definition 3.1. [8, 10, 11] A pair (X, δ) with a set X and an L-*distance* $\delta : X \times P(X) \longrightarrow L$ is called an L-*approach space* if, for all $x \in X$, $A, B \subseteq X$, the following axioms are satisfied.

(LD1) $\delta(x, \{x\}) = \top$;
(LD2) $\delta(x, \emptyset) = \bot$;
(LD3) $\delta(x, A) \vee \delta(x, B) = \delta(x, A \cup B)$ for all $A, B \subseteq X$;
(LD4) $\delta(x, A) \geq \delta(x, \overline{A}^{\alpha}) * \alpha$ for all $\alpha \in L$, where $\overline{A}^{\alpha} = \{x \in X : \delta(x, A) \geq \alpha\}$.

A mapping $f : (X, \delta) \longrightarrow (X', \delta')$ is called an L-*approach morphism* if $\delta(x, A) \leq \delta'(f(x), f(A))$ for all $x \in X, A \subseteq X$. The category with objects the L-approach spaces and morphisms the L-approach morphisms is denoted by L-AP.

For Lawvere's quantale $\mathsf{L} = ([0, \infty], \geq, +)$ we obtain Lowen's approach spaces [13, 14, 15]. For $\mathsf{L} = (\Delta^+, \leq, *)$ with a sup-continuous triangle function $*$, an L-approach space is a probabilistic approach space [7]. In [10, 11], L-approach spaces are called L-*valued topological spaces.* This name is justified as for the case $\mathsf{L} = (\{0, 1\}, \leq, \wedge)$, an L-approach space can be identified with a topological space (defined by its closure operator).

However, we prefer not to use this name in this paper because the terms L-topology and also L-valued topology are used for a different notion in the field of fuzzy topology.

Definition 3.2. [3] An L-*metric space* is a pair (X, d) of a set X and a mapping $d : X \times X \longrightarrow L$ which is

(LM1) *reflexive*, i.e. $d(x, x) = \top$ for all $x \in X$, and
(LM2) *transitive*, i.e. $d(x, y) * d(y, z) \leq d(x, z)$ for all $x, y, z \in X$.

A mapping between two L-metric spaces, $f : (X, d_X) \longrightarrow (Y, d_Y)$ is called an L-*metric morphism* if $d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. We denote the category of L-metric spaces with L-metric morphisms by L-MET. We further denote the fibre over X in L-MET by L-$\mathsf{MET}(X)$.

L-metric spaces are called *continuity spaces* in [3]. In case L is the two-point chain, an L-metric space is a preordered set. For Lawvere's quantale, $\mathsf{L} = ([0, \infty], \geq, +)$, an L-metric space is a quasimetric space. In the probabilistic case, $\mathsf{L} = (\Delta^+, \leq, *)$, an L-metric space is a probabilistic quasimetric space, see [3].

Definition 3.3. [8] Let $\mathcal{H} \subseteq \mathsf{L}$-$\mathsf{MET}(X)$ and $d \in \mathsf{L}$-$\mathsf{MET}(X)$.

(1) d is called *locally supported by* $\mathcal{H}$ if for all $x \in X$, $\alpha \lhd \top$, $\omega \succ \bot$ there is $e_x^{\alpha,\omega} \in \mathcal{H}$ such that $e_x^{\alpha,\omega}(x, \cdot) * \alpha \leq d(x, \cdot) \vee \omega$;
(2) $\mathcal{H}$ is called *locally saturated* if for $d \in \mathsf{L}$-$\mathsf{MET}(X)$ we have $d \in \mathcal{H}$ whenever d is locally supported by $\mathcal{H}$.

Definition 3.4. [8] A subset $\mathcal{G} \subseteq \mathsf{L}$-$\mathsf{MET}(X)$ is called an L-*gauge* if $\mathcal{G}$ is a filter in L-$\mathsf{MET}(X)$ and $\mathcal{G}$ is locally saturated. The pair $(X, \mathcal{G})$ is called an L-*gauge space*. A mapping $f : (X, \mathcal{G}) \longrightarrow (X', \mathcal{G}')$ is called an L-*gauge morphism* if $d' \circ (f \times f) \in \mathcal{G}$ whenever $d' \in \mathcal{G}'$. The category of L-gauge spaces with L-gauge morphisms is denoted by L-GS.

If $\mathsf{L} = ([0, \infty], \geq, +)$, then an L-gauge is a gauge in the original definition [13, 14, 15]. In case $\mathsf{L} = (\Delta^+, \leq, *)$ with a sup-continuous triangle function $*$, we speak of a *probabilistic gauge*.

In [8] we defined two functors $E : \mathsf{L}$-$\mathsf{AP} \longrightarrow \mathsf{L}$-$\mathsf{GS}$ and $R : \mathsf{L}$-$\mathsf{GS} \longrightarrow \mathsf{L}$-$\mathsf{AP}$ by $E((X, \delta)) = (X, \mathcal{G}^\delta)$, $E(f) = f$, with

$$\mathcal{G}^\delta = \{d \in \mathsf{L}\text{-}\mathsf{MET}(X) \ : \ \forall A \subseteq X, x \in X \ : \delta(x, A) \leq \bigvee_{a \in A} d(x, a)\},$$

and $R((X, \mathcal{G})) = (X, \delta^{\mathcal{G}})$, $R(f) = f$, with

$$\delta^{\mathcal{G}}(x, A) = \bigwedge_{d \in \mathcal{G}} \bigvee_{a \in A} d(x, a).$$

Proposition 3.5. [8]

(1) *The functor* $R : \mathsf{L\text{-}GS} \longrightarrow \mathsf{L\text{-}AP}$ *is right adjoint to the functor* $E : \mathsf{L\text{-}AP} \longrightarrow \mathsf{L\text{-}GS}$*, i.e. we have* $id_{\mathsf{L\text{-}AP}} \leq R \circ E$ *and* $E \circ R \leq id_{\mathsf{L\text{-}GS}}$.
(2) *If* L *satisfies the strong De Morgan law, then* E *is a full functor and* $id_{\mathsf{L\text{-}AP}} = R \circ E$*. Hence* $\mathsf{L\text{-}AP}$ *is bicoreflectively embedded in* $\mathsf{L\text{-}GS}$.
(3) *Let* L *be a linearly ordered value quantale that satisfies the condition (I) and the strong De Morgan law. Then* E *is a concrete isomorphism, i.e. the categories* $\mathsf{L\text{-}AP}$ *and* $\mathsf{L\text{-}GS}$ *are concretely isomorphic.*

We showed in [8] that in general we cannot omit the assumption of a linearly ordered value quantale that satisfies the condition (I) in Proposition 3.5 (3). In particular, the embedding functor E is not always an isomorphism between the categories of probabilistic approach spaces and probabilistic gauge spaces. This presents an interesting deviation from the classical case, $\mathsf{L} = ([0,\infty], \geq, +)$, where approach spaces can equivalently be described by either L-distances or by L-gauges, see [13, 14, 15].

4. The category of L-approach system spaces

Definition 4.1. Let $\mathcal{A} \subseteq L^X$ and let $\varphi \in L^X$.

(1) φ is *supported by* $\mathcal{A}$ if for all $\alpha \lhd \top$, $\omega \succ \bot$ there is $\varphi^\omega_\alpha \in \mathcal{A}$ such that $\varphi^\omega_\alpha * \alpha \leq \varphi \vee \omega$.
(2) $\mathcal{A}$ is *saturated* if $\varphi \in \mathcal{A}$ whenever φ is supported by $\mathcal{A}$.
(3) For $\mathcal{B} \subseteq L^X$ we call $\widehat{\mathcal{B}} = \{\varphi \in L^X \ : \ \varphi \text{ is supported by } \mathcal{B}\}$ the *saturation of* $\mathcal{B}$.

Definition 4.2. Let, for each $x \in X$, $\mathcal{A}(x) \subseteq L^X$. Then $\mathcal{A} = (\mathcal{A}(x))_{x\in X}$ is called an L-*approach system* if for all $x \in X$

(A0) $\mathcal{A}(x)$ is a filter in L^X;
(A1) $\varphi(x) = \top$ whenever $\varphi \in \mathcal{A}(x)$;
(A2) $\mathcal{A}(x)$ is saturated;
(A3) For all $\varphi \in \mathcal{A}(x)$, $\alpha \lhd \top$, $\omega \succ \bot$ there is a family $(\varphi_z)_{z\in X} \in \prod_{z\in X} \mathcal{A}(z)$ such that $\varphi_x(z) * \varphi_z(y) * \alpha \leq \varphi(y) \vee \omega$ for all $y, z \in X$.

The pair $(X, \mathcal{A})$ is called an L-*approach system space*. A mapping between two L-approach system spaces, $f : (X, \mathcal{A}) \longrightarrow (X', \mathcal{A}')$ is called an L-*approach system morphism* if $\varphi' \circ f \in \mathcal{A}(x)$ whenever $x \in X$ and $\varphi' \in \mathcal{A}'(f(x))$. The category of L-approach system spaces with L-approach system morphisms is denoted by $\mathsf{L\text{-}AS}$.

For Lawvere's quantale $\mathsf{L} = ([0,\infty], \geq, +)$ we obtain Lowen's approach systems [13, 14, 15]. For $\mathsf{L} = (\Delta^+, \leq, *)$ with a sup-continuous triangle function $*$, we speak of *probabilistic approach systems*.

Often it is sufficient – and more convenient – to work with simpler systems, where in particular (A2) is not demanded.

Definition 4.3. Let, for each $x \in X$, $\mathcal{B}(x) \subseteq L^X$. Then $(\mathcal{B}(x))_{x\in X}$ is called an L-*approach basis* if for all $x \in X$

(B0) $\mathcal{B}(x)$ is a filter basis in L^X;
(B1) $\varphi(x) = \top$ whenever $\varphi \in \mathcal{B}(x)$;
(B2) For all $\varphi \in \mathcal{B}(x)$, $\alpha \lhd \top$, $\omega \succ \bot$ there is a family $(\varphi_z)_{z\in X} \in \prod_{z\in X} \mathcal{B}(z)$ such that $\varphi_x(z) * \varphi_z(y) * \alpha \leq \varphi(y) \vee \omega$ for all $y, z \in X$.

Definition 4.4. Let $(\mathcal{A}(x))_{x\in X}$ be an L-approach system and $(\mathcal{B}(x))_{x\in X}$ be a collection of filter bases on L^X. $(\mathcal{B}(x))_{x\in X}$ is called a *basis for the* L-*approach system* $(\mathcal{A}(x))_{x\in X}$ if for all $x \in X$, $\widehat{\mathcal{B}(x)} = \mathcal{A}(x)$.

Lemma 4.5. *Let* L *be a value quantale.*

1. *If* $(\mathcal{B}(x))_{x\in X}$ *is an* L-*approach basis, then* $(\widehat{\mathcal{B}(x)})_{x\in X}$ *is an* L-*approach system with basis* $(\mathcal{B}(x))_{x\in X}$.
2. *If* $(\mathcal{B}(x))_{x\in X}$ *is a basis for an* L-*approach system, then* $(\mathcal{B}(x))_{x\in X}$ *is an* L-*approach basis.*

Proof. We first prove (1). In order to show the condition (A0) for $(\widehat{\mathcal{B}(x)})_{x\in X}$ it suffices to show that for each $x \in X$, $\widehat{\mathcal{B}(x)}$ is an upper system. Let $\varphi \in \widehat{\mathcal{B}(x)}$ and let $\psi \geq \varphi$. Then φ is supported by $\mathcal{B}(x)$ and hence, for $\alpha \lhd \top$ and $\omega \succ \bot$ there is $\varphi_\alpha^\omega \in \mathcal{B}(x)$ such that $\varphi_\alpha^\omega * \alpha \leq \varphi \vee \omega \leq \psi \vee \omega$ and hence ψ is supported by $\mathcal{B}(x)$, i.e. $\psi \in \widehat{\mathcal{B}(x)}$.

For (A1), let $\varphi \in \widehat{\mathcal{B}(x)}$. Then for all $\alpha \lhd \top$ and all $\omega \succ \bot$ there is $\varphi_\alpha^\omega \in \mathcal{B}(x)$ such that $\varphi_\alpha^\omega * \alpha \leq \varphi \vee \omega$. Then $\alpha = \varphi_\alpha^\omega(x) * \alpha \leq \varphi(x) \vee \omega$ and hence $\top = \bigvee\{\alpha \in L : \alpha \lhd \top\} \leq \varphi(x) \vee \omega$ and therefore also $\top \leq \bigwedge_{\omega\succ\bot}(\varphi(x) \vee \omega) = \varphi(x) \vee \bigwedge_{\omega\succ\bot} \omega = \varphi(x) \vee \bot = \varphi(x)$.

For (A2), let $\varphi \in L^X$ be supported by $\widehat{\mathcal{B}(x)}$. Then for $\alpha \lhd \top$ and $\omega \succ \bot$ there is $\varphi_\alpha^\omega \in \widehat{\mathcal{B}(x)}$ such that $\varphi_\alpha^\omega * \alpha \leq \varphi \vee \omega$. As L is a value quantale, there is $\beta \lhd \top$ such that $\alpha \lhd \beta * \beta$. Then φ_β^ω is dominated by $\mathcal{B}(x)$, i.e. there is $\psi_\beta^\omega \in \mathcal{B}(x)$ such that $\psi_\beta^\omega * \beta \leq \varphi_\beta^\omega \vee \omega$. We conclude

$$\psi_\beta^\omega * \alpha \leq (\psi_\beta^\omega * \beta) * \beta \leq (\varphi_\beta^\omega \vee \omega) * \beta \leq \varphi \vee \omega.$$

As $\alpha \lhd \top$ and $\omega \succ \bot$ are arbitrary this shows that φ is supported by $\mathcal{B}(x)$, i.e. $\varphi \in \widehat{\mathcal{B}(x)}$.

For (A3), let $\varphi \in \widehat{\mathcal{B}(x)}$, let $\alpha \lhd \top$ and let $\omega \succ \bot$. We choose $\beta \lhd \top$ such that $\alpha \lhd \beta * \beta$. Then there are $\varphi_\beta^\omega \in \mathcal{B}(x)$ such that $\varphi_\beta^\omega * \beta \leq \varphi \vee \omega$. Hence there are $(\varphi_z)_{z\in X} \in \prod_{z\in X} \mathcal{B}(z)$ such that $\varphi_x(z) * \varphi_z(y) * \beta \leq \varphi_\beta^\omega(y) \vee \omega$ and consequently

$$\varphi_x(z) * \varphi_z(y) * \alpha \leq \varphi_x(z) * \varphi_z(y) * \beta * \beta \leq (\varphi_\beta^\omega(y) \vee \omega) * \beta \leq \varphi(y) \vee \omega.$$

Finally, it is clear that $(\mathcal{B}(x))_{x\in X}$ is a basis for the approach system $(\widehat{\mathcal{B}(x)})_{x\in X}$.

We now prove (2). The properties (B0) and (B1) are easy and not shown. For (B2), let $\varphi \in \mathcal{B}(x)$, $\alpha \lhd \top$ and $\beta \succ \bot$. We can choose $\beta \lhd \top$ such that $\alpha \lhd (\beta * \beta) * (\beta * \beta)$. Then there are $(\varphi_z)_{z\in X} \in \prod_{z\in X} \mathcal{A}(z)$ such that $\varphi_x(z) * \varphi_z(y) * \beta \leq \varphi(y) \vee \omega$. For each $z \in X$, φ_z is dominated by $\mathcal{B}(z)$. Hence there is $\psi^{\omega}_{z\beta} \in \mathcal{B}(z)$ such that $\psi^{\omega}_{z\beta} * \beta \leq \varphi_z \vee \omega$. It follows that $\psi^{\omega}_{x\beta}(z) * \psi_{z\beta}(y) * \beta * \beta \leq (\varphi_x(z) \vee \omega) * (\varphi_z(y) \vee \omega)$ and we conclude

$$\psi^{\omega}_{x\beta}(z) * \psi^{\omega}_{z\beta}(y) * \alpha \leq (\psi^{\omega}_{x\beta}(z) * \psi^{\omega}_{z\beta}(y) * (\beta * \beta) * (\beta * \beta)$$

$$\leq (\varphi_x(z) \vee \omega) * \beta) * (\varphi_z(y) \vee \omega) * \beta)$$

$$\leq (\varphi_x(z) * \varphi_z(y) * \beta * \beta) \vee \omega \leq \varphi(y) \vee \omega.$$

□

Proposition 4.6. *Let* L *be a value quantale and let* $(X, \mathcal{A}), (X', \mathcal{A}') \in |\mathsf{L}\text{-}\mathsf{AS}|$ *and let* $\mathcal{B}'$ *be a basis for* $\mathcal{A}'$. *A mapping* $f : (X, \mathcal{A}) \longrightarrow (X', \mathcal{A}')$ *is an* L*-approach system morphism if and only if* $\varphi' \circ f \in \mathcal{A}(x)$ *whenever* $x \in X$ *and* $\varphi' \in \mathcal{B}'(f(x))$.

Proof. We only show necessity. Let $\varphi' \in \mathcal{A}'(f(x))$. Then φ' is supported by $\mathcal{B}'(f(x))$, i.e. for all $\alpha \lhd \top$ and all $\omega \succ \bot$ there is $\varphi^{\omega}_{\alpha} \in \mathcal{B}'(f(x))$ such that $\varphi^{\omega}_{\alpha} * \alpha \leq \varphi' \vee \omega$. Then $\varphi^{\omega}_{\alpha} \circ f \in \mathcal{A}(x)$. Moreover, it is not difficult to see that $(\varphi' \circ f) \vee \omega \geq (\varphi^{\omega}_{\alpha} \circ f) * \alpha$. This shows that $\varphi' \circ f$ is supported by $\mathcal{A}(x)$ and, $\mathcal{A}(x)$ being saturated, we conclude $\varphi' \circ f \in \mathcal{A}(x)$. □

Theorem 4.7. *For a value quantale* L*, the construct* $\mathsf{L}\text{-}\mathsf{AS}$ *is topological.*

Proof. Let $(f_i : X \longrightarrow (X_i, \mathcal{A}_i))_{i\in J}$ be a source. We define, for $x \in X$,

$$\mathcal{B}(x) = \{ \bigwedge_{j\in K} \varphi_j \circ f_j \ : \ K \subseteq J \text{ finite }, \varphi_j \in \mathcal{A}_j(f_j(x))\}.$$

Then $\mathcal{B}(x)$ is an L-approach basis. The properties (B0) and (B1) are easy and not shown. For (B2), let $\bigwedge_{j\in K} \varphi_j \circ f_j \in \mathcal{B}(x)$, $\alpha \lhd \top$ and $\omega \succ \bot$. For each $j \in K$ there is a family $(\psi^j_z)_{z\in X_j} \in \prod_{z\in X_j} \mathcal{A}_j(z)$ with

$$\psi^j_{f_j(x)}(z) * \psi^j_z(y) * \alpha \leq \varphi_j(y) \vee \omega.$$

We denote, for $t \in X$, $\eta_t = \bigwedge_{j\in K} \psi^j_{f_j(t)} \circ f_j$. Then $\eta_t \in \mathcal{B}(t)$ and for any $t, s \in X$ we have

$$\begin{aligned}
\eta_x(t) * \eta_t(s) * \alpha &\leq \bigwedge_{j\in K} \left(\left[\psi^j_{f_j(x)} \circ f_j(t)\right] * \left[\psi^j_{f_j(t)} \circ f_j(s)\right] * \alpha \right) \\
&= \bigwedge_{j\in K} \left(\psi^j_{f_j(x)}(f_j(t)) * \psi^j_{f_j(t)}(f_j(s)) * \alpha \right) \\
&\leq \bigwedge_{j\in K} (\varphi_j \circ f_j(s) \vee \omega) \;=\; \left(\bigwedge_{j\in K} \varphi_j \circ f_j(s) \right) \vee \omega.
\end{aligned}$$

Hence (B2) is satisfied and $\mathcal{A}(x) = \widehat{\mathcal{B}(x)}$ is an L-approach system on X.

Clearly, each $f_j : (X, \mathcal{A}) \longrightarrow (X_j, \mathcal{A}_j)$ is an L-approach system morphism. Consider now a further L-approach system space $(Z, \mathcal{C})$ and a mapping $g : Z \longrightarrow X$ such that for each $j \in J$ the composition $f_j \circ g : (Z, \mathcal{C}) \longrightarrow (X_j, \mathcal{A}_j)$ is an L-approach system morphism. Let $z \in Z$ and let $\bigwedge_{j\in K} \varphi \circ f_j \in \mathcal{A}(g(z))$. Then

$$\left(\bigwedge_{j\in K} \varphi \circ f_j \right) \circ g = \bigwedge_{j\in K} \varphi \circ (f_j \circ g) \in \mathcal{C}(z)$$

and hence $g : (Z, \mathcal{C}) \longrightarrow (X, \mathcal{A})$ is an L-approach system morphism. □

5. L-gauge spaces as L-approach system spaces

Proposition 5.1. *Let $\mathcal{G}$ be an L-gauge on X and define, for $x \in X$, $\mathcal{B}^{\mathcal{G}}(x) = \{d(x, \cdot) \,:\, d \in \mathcal{G}\}$. Then $(\mathcal{B}^{\mathcal{G}}(x))_{x\in X}$ is an L-approach basis.*

Proof. (B0) and (B1) are easy and not shown. For (B2), let $\varphi = d(x, \cdot) \in \mathcal{B}^{\mathcal{G}}(x)$ and let $\alpha \lhd \top$ and $\omega \succ \bot$. As d is an L-metric, we know $d(x, z) * d(z, y) \leq d(x, y)$. Hence with $\varphi_z = d(z, \cdot) \in \mathcal{B}^{\mathcal{G}}(x)$ for $z \in X$ we have

$$\varphi_x(z) * \varphi_z(y) * \alpha = d(x, z) * d(z, y) * \alpha \leq d(x, y) = \varphi(y) \leq \varphi(y) \vee \omega.$$

□

We denote the L-approach system with basis $\mathcal{B}^{\mathcal{G}}$ by $\mathcal{A}^{\mathcal{G}}$.

Proposition 5.2. *Let L be a value quantale. Then $F : \mathsf{L}\text{-}\mathsf{GS} \longrightarrow \mathsf{L}\text{-}\mathsf{AS}$, defined by $F((X, \mathcal{G})) = (X, \mathcal{A}^{\mathcal{G}})$ and $F(f) = f$, is an embedding functor.*

Proof. Let $f : (X, \mathcal{G}) \longrightarrow (X', \mathcal{G}')$ be an L-gauge morphism and let $x \in X$ and let $\varphi' \in \mathcal{B}^{\mathcal{G}'}(f(x))$. Then $\varphi' = d'(f(x), \cdot)$ with $d' \in \mathcal{G}'$. Noting that for $y \in X$ we have $\varphi' \circ f(y) = \varphi'(f(y)) = d'(f(x), f(y)) = d' \circ (f \times f)(x, y) = d' \circ (f \times f)(x, \cdot)(y)$ and as $d' \circ (f \times f) \in \mathcal{G}$ by assumption, we see that $\varphi' \circ f \in \mathcal{B}^{\mathcal{G}}(x)$. Hence $f : (X, \mathcal{A}^{\mathcal{G}}) \longrightarrow (X', \mathcal{A}^{\mathcal{G}'})$ is an L-approach system morphism. This shows that F is a functor.

We next show that F is injective on objects. Let $\mathcal{G} \neq \mathcal{G}'$. Without loss of generality there is $d \in \mathcal{G}$ such that $d \notin \mathcal{G}'$. Then for all $x \in X$ we have that $d(x,\cdot) \in \mathcal{A}^{\mathcal{G}}(x)$. Assume that for all $x \in X$, $d(x,\cdot) \in \mathcal{A}^{\mathcal{G}'}(x)$. Then for each $x \in X$, $\alpha \lhd \top$ and $\omega \succ \bot$ there is $e_x^{\alpha,\omega} \in \mathcal{G}'$ such that $e_x^{\alpha,\omega}(x,\cdot) * \alpha \leq d(x,\cdot) \vee \omega$. Hence d is locally supported by $\mathcal{G}'$, i.e. $d \in \mathcal{G}'$, a contradiction. Hence $\mathcal{A}^{\mathcal{G}} \neq \mathcal{A}^{\mathcal{G}'}$. As a concrete functor, F is faithful, and hence it is an embedding. □

The functor F has a left adjoint. We define, for $(X,\mathcal{A}) \in |\mathsf{L\text{-}AS}|$,

$$\mathcal{G}^{\mathcal{A}} = \{d \in L\text{-}MET(X) \,:\, \forall x \in X, d(x,\cdot) \in \mathcal{A}(x)\}.$$

Proposition 5.3. (1) *$S : \mathsf{L\text{-}AS} \longrightarrow \mathsf{L\text{-}GS}$ defined by $S((X,\mathcal{A})) = (X,\mathcal{G}^{\mathcal{A}})$ and $S(f) = f$ is a functor.*

(2) *If L is a value quantale, then $S \circ F = id_{\mathsf{L\text{-}GS}}$ and $id_{\mathsf{L\text{-}AS}} \leq F \circ S$, i.e. S is left adjoint to F. Consequently, $\mathsf{L\text{-}GS}$ is a full bireflective subcategory of $\mathsf{L\text{-}AS}$.*

Proof. (1) We first show that if $(\mathcal{A}(x))_{x\in X}$ is an L-approach system on X, then $\mathcal{G}^{\mathcal{A}} = \{d \in L\text{-}MET(X) \,:\, \forall x \in X, d(x,\cdot) \in \mathcal{A}(x)\}$ is an L-gauge. It follows easily from the fact that all $\mathcal{A}(x)$ are filters that $\mathcal{G}^{\mathcal{A}}$ is a filter in $\mathsf{L\text{-}MET}(X)$. We show that $\mathcal{G}^{\mathcal{A}}$ is locally saturated. Let $d \in \mathsf{L\text{-}MET}(X)$ be locally supported by $\mathcal{G}^{\mathcal{A}}$. Then for all $x \in X$, $\alpha \lhd \top$ and $\omega \succ \bot$ there is $e_{x\alpha}^{\omega} \in \mathcal{G}^{\mathcal{A}}$ such that $e_{x\alpha}^{\omega}(x,\cdot) * \alpha \leq d(x,\cdot) \vee \omega$. By definition of $\mathcal{G}^{\mathcal{A}}$, $e_{x\alpha}^{\omega}(x,\cdot) \in \mathcal{A}(x)$ for all $x \in X$ and hence by saturation of $\mathcal{A}(x)$ we have $d(x,\cdot) \in \mathcal{A}(x)$ for all $x \in X$. But this means $d \in \mathcal{G}^{\mathcal{A}}$.

Next we show that if $(X,\mathcal{A}),(X',\mathcal{A}') \in |\mathsf{L\text{-}AS}|$ and if $f : (X,\mathcal{A}) \longrightarrow (X',\mathcal{A}')$ is an L-approach system morphism, then $f : (X,\mathcal{G}^{\mathcal{A}}) \longrightarrow (X',\mathcal{G}^{\mathcal{A}'})$ is an L-gauge morphism. Let $d' \in \mathcal{G}^{\mathcal{A}'}$. Then for all $x' \in X'$ we have $d'(x',\cdot) \in \mathcal{A}'(x')$. In particular $d'(f(x),\cdot) \in \mathcal{A}'(f(x))$, for all $x \in X$. As f is an L-approach system morphism, $d'(f(x),\cdot) \circ f \in \mathcal{A}(x)$ for all $x \in X$. Noting that for $z \in X$ we have $d'(f(x),\cdot) \circ f(z) = d'(f(x),f(z)) = d' \circ (f \times f)(x,z)$ we have that, for all $x \in X$, $d' \circ (f \times f)(x,\cdot) \in \mathcal{A}(x)$, i.e. $d' \circ (f \times f) \in \mathcal{G}^{\mathcal{A}}$.

(2) Let L be a value quantale and let $\mathcal{G}$ be an L-gauge on X. We show that $\mathcal{G}^{(\mathcal{A}^{\mathcal{G}})} = \mathcal{G}$. Let first $d \in \mathcal{G}$. Then $d(x,\cdot) \in \mathcal{B}^{\mathcal{G}}(x) \subseteq \mathcal{A}^{\mathcal{G}}(x)$ for all $x \in X$ and hence $d \in \mathcal{G}^{(\mathcal{A}^{\mathcal{G}})}$. Conversely, let $d \in \mathsf{L\text{-}MET}(X)$ with $d(x,\cdot) \in \widehat{\mathcal{B}^{\mathcal{G}}(x)}$ for all $x \in X$. Then, for all $x \in X$, $d(x,\cdot)$ is supported by $\mathcal{B}^{\mathcal{G}}(x)$. Hence, for $\alpha \lhd \top$ and $\omega \succ \bot$ there is $e_{\alpha}^{\omega}(x,\cdot) \in \mathcal{B}^{\mathcal{G}}(x)$ such that $e_{\alpha}^{\omega}(x,\cdot) * \alpha \leq d(x,\cdot) \vee \omega$. By definition all $e_{\alpha}^{\omega} \in \mathcal{G}$, i.e. d is locally supported by $\mathcal{G}$ and hence $d \in \mathcal{G}$. It follows from this, that F is a full functor.

Let now $\mathcal{A}$ be an L-approach system on X. We show that $\mathcal{A}^{(\mathcal{G}^{\mathcal{A}})}(x) \subseteq \mathcal{A}(x)$ for all $x \in X$. Let $\varphi \in \widehat{\mathcal{B}^{(\mathcal{G}^{\mathcal{A}})}(x)}$ and let $\alpha \lhd \top$ and $\omega \succ \bot$. Then there is $d^{\omega}_{\alpha} \in \mathcal{G}^{\mathcal{A}}$ such that $d^{\omega}_{\alpha}(x,\cdot) * \alpha \leq \varphi \vee \omega$. As $d^{\omega}_{\alpha}(x,\cdot) \in \mathcal{A}(x)$ we see that φ is supported by $\mathcal{A}(x)$ and hence $\varphi \in \mathcal{A}(x)$. □

Corollary 5.4. *Let L be a value quantale. The category L-GS is topological with initial structures constructed as in L-AS.*

The fact that L-GS is topological was already shown in [8].

Under certain assumptions, the categories L-GS and L-AS are even isomorphic. We need the following results.

Lemma 5.5. *Let $\mathcal{B} = (\mathcal{B}(x))_{x\in X}$ be an L-approach basis on X and denote, for each $x \in X$, $\mathcal{A}(x) = \widehat{\mathcal{B}(x)}$. Let further $Z \subseteq X$ and $\omega \succ \bot$. Then $d^{\omega}_{Z} : X \times X \longrightarrow L$ defined by*

$$d^{\omega}_{Z}(x,y) = \left(\bigwedge_{\psi\in\mathcal{B}(y)} \bigvee_{z\in Z} (\psi(z) \vee \omega) \right) \to \left(\bigwedge_{\varphi\in\mathcal{B}(x)} \bigvee_{z\in Z} (\varphi(z) \vee \omega) \right)$$

is an L-metric.

Furthermore, if L is a linearly ordered value quantale that satisfies the condition (I) and for which $*$ distributes over arbitrary meets, then $d^{\omega}_{Z} \in \mathcal{G}^{\mathcal{A}}$.

Proof. Clearly $d^{\omega}_{Z}(x,x) = \top$ for all $x \in X$. The transitivity follows as, in any quantale, for all $\alpha, \beta, \gamma \in L$ we have $(\alpha \to \beta) * (\gamma \to \alpha) \leq \gamma \to \beta$. Let now L be a linearly ordered value quantale and satisfy the condition (I) and let the quantale operation distribute over arbitrary meets. We fix $\alpha \lhd \top$ and $x \in X$ and note that $\bigwedge_{\varphi\in\mathcal{B}(x)} \bigvee_{z\in Z} (\varphi(z) \vee \omega) \geq \omega \succ \bot$. There is $\beta \lhd \top$ such that $\alpha \lhd \beta * \beta$. We choose $\varphi_0 \in \mathcal{B}(x)$ such that

$$\bigvee_{z\in Z} (\varphi_0(z) \vee \omega) * \beta \leq \bigwedge_{\varphi\in\mathcal{B}(x)} \bigvee_{z\in Z} (\varphi(z) \vee \omega),$$

and we choose $(\psi_u)_{u\in X} \in \prod_{u\in X} \mathcal{B}(u)$ such that $\psi_x(y) * \psi_y(z) * \beta \leq \varphi_0(z) \vee \omega$. Then

$$\begin{aligned} d^{\omega}_{Z}(x,y) \;&\geq\; \bigvee_{z\in Z} (\psi_y(z) \vee \omega) \to \bigvee_{z\in Z} (\varphi_0(z) \vee \omega) \vee \omega \\ &\geq\; \left(\left(\bigvee_{z\in Z} \psi_y(z) \right) \vee \omega\right) \to \left(\left(\bigvee_{z\in Z} \psi_x(y) * \psi_y(z) * \beta * \beta \right) \vee \omega\right) \\ &\geq\; \left(\bigvee_{z\in Z} \psi_y(z) \right) \to \left(\psi_x(y) * \beta * \beta * \bigvee_{z\in Z} \psi_y(z) \right) \\ &\geq\; \psi_x(y) * \beta * \beta \;\geq\; \psi_x(y) * \alpha. \end{aligned}$$

Hence $d_Z^\omega(x,\cdot)$ is supported by $\mathcal{B}(x)$, i.e. $d_Z^\omega(x,\cdot) \in \mathcal{B}(x)$ and we have $d_Z^\omega \in \mathcal{G}^{\mathcal{A}}$. $\square$

For the second part of the lemma, we cannot omit the assumptions. We give the following examples for this.

Example 5.6. Let $L = [0,1] \cup \{-\infty, \infty\}$ with the natural order and the minimum as quantale operation. Then $\mathsf{L} = (L, \leq, \wedge)$ is a linearly ordered value quantale that does not satisfy the condition (I). Let $X = (0,1)$. For $x \in X$ and $0 < \alpha < 1$ we define $\varphi_{\alpha,x}(y) = \begin{cases} \top & \text{if } y = x \\ \alpha & \text{if } y \neq x \end{cases}$ and $\mathcal{B}(x) = \{\varphi_{\alpha,x} \,:\, 0 < \alpha < 1\}$. We show that $(\mathcal{B}(x))_{x\in X}$ is an L-approach basis. We first note that $\varphi_{\alpha,x} \wedge \varphi_{\beta,x} = \varphi_{\alpha\wedge\beta,x}$ and hence $\mathcal{B}(x)$ is a filter basis, i.e. we have (B0). Clearly $\varphi_{\alpha,x}(x) = \top$ and (B1) is true. Finally, we have $\varphi_{\alpha,x}(z) \wedge \varphi_{\alpha,z}(y) = \top$ iff $x = y = z$ and then also $\varphi_{\alpha,x}(y) = \top$. Hence also condition (B2) is satisfied. We denote $\mathcal{A}(x) = \widehat{\mathcal{B}(x)}$.

We consider now, for a fixed point $x_0 \in X$ the subset $Z = X \setminus \{x_0\}$. We then have $\bigwedge_{\varphi\in\mathcal{B}(x_0)} \bigvee_{z\neq x_0} \varphi(z) = \bigwedge_{\alpha\in(0,1)} \alpha = 0$ and for $y \neq x_0$ we have $\bigwedge_{\psi\in\mathcal{B}(y)} \bigvee_{z\neq x_0} \psi(z) \geq \bigwedge_{\psi\in\mathcal{B}(y)} \psi(y) = \top$. Hence $d_Z^\perp(x_0,y) = \top \to 0 = 0$ if $y \neq x_0$ and $d_Z^\perp(x_0,y) = \top$ if $y = x_0$. Therefore, for all $0 < \alpha < 1$ we have $\varphi_{\alpha,x_0} \wedge \top \not\leq d_Z^\perp(x_0,\cdot) \vee \perp$ and $d_Z^\perp(x_0,\cdot)$ is not supported by $\mathcal{B}(x_0)$. Consequently also $d_Z^\perp(x_0,\cdot) \notin \mathcal{A}(x_0)$ and $d_Z^\perp \notin \mathcal{G}^{\mathcal{A}}$. $\square$

Example 5.7. Let $\mathsf{L} = (\Delta^+, \leq, \boxdot)$ with the triangle function $\boxdot$ induced by the product t-norm. Let S be a set and, for $0 < \alpha < 1$ and $p \in S$, we define the mapping $\Phi_{p\alpha} : S \longrightarrow \Delta^+$ by $\Phi_{p\alpha}(q) = \varepsilon_{1-\alpha}$ if $p \neq q$ and $\Phi_{p\alpha}(p) = \varepsilon_0$. We note that $\Phi_{p\alpha} \wedge \Phi_{p\beta} = \Phi_{p\alpha\wedge\beta}$. Hence $\mathcal{B}(p) = \{\Phi_{p\alpha} \,:\, 0 < \alpha < 1\}$ is a filter basis. Also we have $\Phi_{p\alpha}(p) = \varepsilon_0$ and hence (B1) is satisfied. Noting that $\varepsilon_\alpha \boxdot \varepsilon_\beta = \varepsilon_{\alpha\vee\beta}$, we see that (B2) is satisfied as follows. Let $\Phi_{p\alpha} \in \mathcal{B}(p)$. We choose $\Phi_{q\alpha} \in \mathcal{B}(q)$ for all $q \in S$ and then have $\Phi_{p\alpha}(q) \boxdot \Phi_{q\alpha}(r) \leq \Phi_{p\alpha}(r)$. The only interesting case is here that $p \neq q$ and $q \neq r$ and $p \neq r$. But then we have $\varepsilon_{1-\alpha} \boxdot \varepsilon_{1-\alpha} = \varepsilon_{1-\alpha}$. Hence $(\mathcal{B}(p))_{p\in S}$ is an L-approach basis.

We fix now $p_0 \in S$ and $Z = S \setminus \{p_0\}$ and $\omega = g_{1,1/4} \succ \epsilon_\infty$ with $g_{1,1/4} = \frac{1}{4}\varepsilon_0 \vee \varepsilon_1$. Then $\displaystyle\bigwedge_{\Phi\in\mathcal{B}(p_0)} \bigvee_{q\neq p_0} \Phi(q) \vee \omega = \bigwedge_{\alpha\in(0,1)} \varepsilon_{1-\alpha} \vee \omega = \varepsilon_1 \vee \omega$ and for $p \neq p_0$ we have $\displaystyle\bigwedge_{\Psi\in\mathcal{B}(p)} \bigvee_{q\neq p_0} \Psi(q) \vee \omega = \varepsilon_0 \vee \omega = \varepsilon_0$. Hence for $q = p_0$ we have $d_Z^\omega(p_0,q) = \varepsilon_0$ and for $q \neq p_0$ we have $d_Z^\omega(p_0,q) = \varepsilon_0 \to (\varepsilon_1 \vee \omega) = \varepsilon_1 \vee \omega$. If we choose $\varphi = f_{1/2,1/2} \lhd \varepsilon_0$, then for all $0 < \alpha < 1$ we have, for $q \neq p_0$, $\Phi_{p_0\alpha}(q) \boxdot \varphi(x) = \varepsilon_{1-\alpha}(x) \cdot f_{1/2,1/2}(x)$ $= \frac{1}{2}\varepsilon_{(1-\alpha)\vee 1/2}(x) \not\leq g_{1,1/4}(x) = \varepsilon_1 \vee g_{1,1/4}(x)$ for some $0 < x < 1$. Hence $d_Z^\omega(p_0,\cdot)$ is not supported by $\mathcal{B}(p_0)$, i.e. $d_Z^\omega(p_0,\cdot) \notin \mathcal{A}(p_0)$, and therefore $d_Z^\omega \notin \mathcal{G}^{\mathcal{A}}$. $\square$

Proposition 5.8. *Let* L *be a linearly ordered value quantale that satisfies the condition (I) and let the quantale operation distribute over arbitrary meets and let the strong De Morgan law hold. Let* $(X, \mathcal{A}) \in |\mathsf{L}\text{-}\mathsf{AS}|$. *Then*

$$\bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{a \in A} \varphi(a) = \bigwedge_{d \in \mathcal{G}^{\mathcal{A}}} \bigvee_{a \in A} d(x, a).$$

Proof. For $d \in \mathcal{G}^{\mathcal{A}}$ we have $d(x, \cdot) \in \mathcal{A}(x)$ and hence $\bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{a \in A} \varphi(a) \leq \bigwedge_{d \in \mathcal{G}^{\mathcal{A}}} \bigvee_{a \in A} d(x, a)$. For the converse, we use Lemma 5.5. We have

$$\begin{aligned}
& \bigwedge_{d \in \mathcal{G}^{\mathcal{A}}} \bigvee_{a \in A} d(x, a) \\
\leq \; & \bigwedge_{\omega \succ \bot} \bigvee_{a \in A} \left(\left(\bigwedge_{\psi \in \mathcal{A}(a)} \bigwedge_{z \in A} (\psi(z) \vee \omega) \right) \to \left(\bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{z \in A} (\varphi(z) \vee \omega) \right) \right) \\
\leq \; & \bigwedge_{\omega \succ \bot} \left(\left(\bigwedge_{a \in A} \bigwedge_{\psi \in \mathcal{A}(a)} \bigwedge_{z \in A} (\psi(z) \vee \omega) \right) \to \left(\bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{z \in A} (\varphi(z) \vee \omega) \right) \right) \\
\leq \; & \bigwedge_{\omega \succ \bot} \left(\underbrace{\left(\bigwedge_{a \in A} \bigwedge_{\psi \in \mathcal{A}(a)} (\psi(a) \vee \omega) \right)}_{=\top} \to \left(\bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{z \in A} (\varphi(z) \vee \omega) \right) \right) \\
= \; & \bigwedge_{\omega \succ \bot} \bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{z \in A} (\varphi(z) \vee \omega) \; = \; \bigwedge_{\omega \succ \bot} \bigwedge_{\varphi \in \mathcal{A}(x)} ((\bigvee_{z \in A} \varphi(z)) \vee \omega) \\
= \; & \bigwedge_{\omega \succ \bot} ((\bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{z \in A} \varphi(z)) \vee \omega) \\
= \; & (\bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{z \in A} \varphi(z)) \vee \bigwedge_{\omega \succ \bot} \omega \; = \; \bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{z \in A} \varphi(z).
\end{aligned}$$

□

Theorem 5.9. *Let* L *be a linearly ordered value quantale that satisfies the condition (I) and let the quantale operation distribute over arbitrary meets and let the strong De Morgan law hold. Then* $id_{\mathsf{L}\text{-}\mathsf{AS}} = F \circ S$, *i.e. the categories* $\mathsf{L}\text{-}\mathsf{AS}$ *and* $\mathsf{L}\text{-}\mathsf{GS}$ *are isomorphic.*

Proof. We show that $\mathcal{A} = \mathcal{A}^{(\mathcal{G}^{\mathcal{A}})}$. We have seen above that for all $x \in X$ we have $\mathcal{A}^{(\mathcal{G}^{\mathcal{A}})}(x) \subseteq \mathcal{A}(x)$. Assume now that $\psi \in \mathcal{A}(x)$ but $\psi \notin \mathcal{A}^{(\mathcal{G}^{\mathcal{A}})}(x)$.

Then ψ is not dominated by $\mathcal{B}^{(\mathcal{G}^{\mathcal{A}})}(x)$ and hence there is $\alpha \lhd \top$ and $\omega \succ \perp$ such that for $d \in \mathcal{G}^{\mathcal{A}}$ we have $d(x,\cdot) * \alpha \not\leq \psi \vee \omega$. For $d \in \mathcal{G}^{\mathcal{A}}$ we define $\mathcal{D}(d) = \{y \in X \ : \ \psi(y) \vee \omega < d(x,y) * \alpha\}$. Then, for all $d, e \in \mathcal{G}^{\mathcal{A}}$, $\mathcal{D}(d) \neq \emptyset$ and $\mathcal{D}(d \wedge e) = \mathcal{D}(d) \cap \mathcal{D}(e)$. Hence

$$
\begin{aligned}
& \left(\left(\bigwedge_{d \in \mathcal{G}^{\mathcal{A}}} \bigwedge_{e \in \mathcal{G}^{\mathcal{A}}} \bigvee_{y \in \mathcal{D}(d)} e(x,y) \right) \vee \omega \right) * \alpha \\
\geq \ & \left(\bigwedge_{d \in \mathcal{G}^{\mathcal{A}}} \bigwedge_{e \in \mathcal{G}^{\mathcal{A}}} \bigvee_{y \in \mathcal{D}(d \wedge e)} (d \wedge e)(x,y) \right) * \alpha \\
= \ & \left(\bigwedge_{d \in \mathcal{G}^{\mathcal{A}}} \bigvee_{y \in \mathcal{D}(d)} d(x,y) \right) * \alpha \ \geq \ \bigwedge_{d \in \mathcal{G}^{\mathcal{A}}} \bigvee_{y \in \mathcal{D}(d)} (d(x,y) * \alpha) \\
\geq \ & \bigwedge_{d \in \mathcal{G}^{\mathcal{A}}} \bigvee_{y \in \mathcal{D}(d)} (\psi(y) \vee \omega) \ = \ \left(\bigwedge_{d \in \mathcal{G}^{\mathcal{A}}} \bigvee_{y \in \mathcal{D}(d)} \psi(y) \right) \vee \omega \\
\geq \ & \left(\bigwedge_{d \in \mathcal{G}^{\mathcal{A}}} \bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{y \in \mathcal{D}(d)} \varphi(y) \right) \vee \omega \\
\overset{5.8}{\geq} \ & \left(\bigwedge_{d \in \mathcal{G}^{\mathcal{A}}} \bigwedge_{e \in \mathcal{G}^{\mathcal{A}}} \bigvee_{y \in \mathcal{D}(d)} e(x,y) \right) \vee \omega.
\end{aligned}
$$

This is a contradiction to condition (I). □

The assumptions of the theorem are satisfied by a linearly ordered MV-algebra L with $\perp \not\succ \perp$ and $\top \not\lhd \top$ and also by $\mathsf{L} = ([0,\infty], \geq, +)$. This latter case is treated in Lowen's theory of approach spaces [13, 14, 15]. In general we cannot omit the assumptions in Theorem 5.9. This is shown by the following two examples.

Example 5.10. Let $\mathsf{L} = (\{0,1\}, \leq, \wedge)$ and $X = [0,1]$. Then the condition (I) is not satisfied. We define for each $x \in X$ and $n = 1,2,3,...$ the function $\varphi_{n,x} : X \longrightarrow \{0,1\}$ by $\varphi_{n,x}(y) = 1$ if $|x-y| \leq \frac{1}{n}$ and $\varphi_{n,x}(y) = 0$ if $|x-y| > \frac{1}{n}$. Define, for each $x \in X$, $\mathcal{B}(x) = \{\varphi_{n,x} \ : \ n = 1,2,3,...\}$. We show that $(\mathcal{B}(x))_{x \in X}$ is an L-approach basis. As $\varphi_{n,x} \wedge \varphi_{m,x} = \varphi_{n \vee m,x}$ we see that condition (B0) is satisfied. Also (B1) is true. For (B2), let $\varphi_{n,x} \in \mathcal{B}(x)$. We choose for all $y \in X$, $\varphi_{2n,y} \in \mathcal{B}(y)$. If $\varphi_{2n,x}(y) \wedge \varphi_{2n,y}(z) = 1$, then $|x-z| \leq |x-y| + |y-z| \leq \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$ and hence $\varphi_{n,x}(z) = 1$. This shows $\varphi_{2n,x}(y) \wedge \varphi_{2n,y}(z) \leq \varphi_{n,x}(z)$ for all $y, z \in X$ and (B2) is satisfied. We denote $\mathcal{A}(x) = \widehat{\mathcal{B}(x)}$. We now show that $d \in \mathcal{G}^{\mathcal{A}}$ implies that $d \equiv 1$. If $d \in \mathcal{G}^{\mathcal{A}}$, then for each $x \in X$ we have $d(x,\cdot) \in \mathcal{A}(x)$. Hence for all $x \in X$ there is $n_x \in \{1,2,3,...\}$ such that $\varphi_{n_x,x} \le d(x,\cdot)$, i.e. for all $x \in X$ we have $d(x,y) = 1$ whenever $|x-y| \le \frac{1}{n_x}$. The set of open intervals $\mathbb{U} = \{I_x = (x - \frac{1}{n_x}, x + \frac{1}{n_x}) \ : \ 0 \le x \le 1\}$ is an open cover of X. Let now $x, z \in X$ and $x \ne z$. As the interval $[x,z]$ is compact and connected, we can choose a finite subset $\mathbb{U}_0 = \{I_{x_1}, I_{x_2}, ..., I_{x_n}\} \subseteq \mathbb{U}$ with the properties $x \in I_{x_1}$, $z \in I_{x_n}$ and $I_{x_k} \cap I_{x_{k+1}} \ne \emptyset$ for $k = 1,2,...,n-1$. Let $y_k \in I_{x_k} \cap I_{x_{k+1}}$ for $k = 1,2,...,n-1$. Then by the triangle inequality, we obtain

$$1 = d(x,x_1) \wedge d(x_1,y_1) \wedge d(y_1,x_2) \wedge \cdots \wedge d(x_n,z) \le d(x,z).$$

Hence for all $x, z \in X$ we have $d(x,z) = 1$. If now $\varphi \in \mathcal{A}^{(\mathcal{G}^{\mathcal{A}})}(x)$, then there is $d \in \mathcal{G}^{\mathcal{A}}$ such that $1 \equiv d(x,\cdot) \le \varphi$ and hence $\varphi_{n,x} \notin \mathcal{A}^{(\mathcal{G}^{\mathcal{A}})}(x)$ for $n > 2$. This shows that $\mathcal{A}(x) \nsubseteq \mathcal{A}^{(\mathcal{G}^{\mathcal{A}})}(x)$. $\square$

Example 5.11 (Probabilistic case). Let $\mathsf{L} = (\Delta^+, \le, \wedge)$ with the pointwise minimum as triangle function. Let $S = [0,1]$ and define, for $p \in S$ and $n = 1,2,3,...$ the mappings $\Phi_{pn} : S \longrightarrow \Delta^+$ by $\Phi_{pn}(q) = \varepsilon_0$ if $|p-q| < \frac{1}{n}$ and $\Phi_{pn}(q) = \varepsilon_\infty$ if $|p-q| \ge \frac{1}{n}$. It is then not difficult to show that $\Phi_{pn} \wedge \Phi_{pm} = \Phi_{p(m \vee n)}$ and hence $\mathcal{B}(p) = \{\Phi_{np} \ : \ n = 1,2,3,...\}$ is a filter basis. Moreover $\Phi_{p,2n}(q) \wedge \Phi_{q,2n}(r) \le \Phi_{pn}(r)$ and hence $(\mathcal{B}(p))_{p \in S}$ is an L-approach basis. We denote $\mathcal{A}(p) = \widehat{\mathcal{B}(p)}$ for all $p \in S$. Let now $D \in \mathcal{G}^{\mathcal{A}}$. We note that $f_{\frac{1}{k},1-\frac{1}{k}} = (1-\frac{1}{k})\varepsilon_{\frac{1}{k}} \vee \varepsilon_\infty \lhd \varepsilon_0$ and $g_{k,\frac{1}{k}} = \frac{1}{k}\varepsilon_0 \vee \varepsilon_k \succ \varepsilon_\infty$ for all $k \in \mathbb{N}$. Then for all $p \in S$ and all $k \in \mathbb{N}$ there is $n = n(p,k) \in \mathbb{N}$ such that

$$\Phi_{pn} \wedge f_{\frac{1}{k},1-1\frac{1}{k}} \le D(p,\cdot) \vee g_{k,\frac{1}{k}}.$$

Hence, for $|p-q| \le \frac{1}{n}$ and $x > \frac{1}{k}$ we have $1-\frac{1}{k} \le D(p,q)(x)$, i.e. $f_{\frac{1}{k},1-\frac{1}{k}} \le D(p,q)$. Like in the previous example, we will show that $D(p,q) = \varepsilon_0$. We fix $k \in \mathbb{N}$ and consider the set of intervals $\mathbb{U} = \{I_p = (p-\frac{1}{n(p,k)}, p+\frac{1}{n(p,k)}) :$ $p \in S\}$. For $p, q \in S$ then there are finitely many $I_{p_1}, ..., I_{p_m}$ with the properties $p \in I_{p_1}, q \in I_{p_m}$ and $I_{p_k} \cap I_{p_{k+1}} \ne \emptyset$ for $k = 1,2,...,m-1$. Again, by the triangle inequality, then

$$f_{\frac{1}{k},1-\frac{1}{k}} \le D(p,p_1) \wedge D(p_1,q_1) \wedge ... \wedge D(p_m,q) \le D(p,q),$$

where $q_k \in I_{p_k} \cap I_{p_{k+1}}$ for $k = 1,2,...,m-1$. This is true for all $k \in \mathbb{N}$ and hence $D(p,q)(x) = 1$ for $x > 0$. Assume now that $\varphi_{pn} \in \mathcal{A}^{(\mathcal{G}^{\mathcal{A}})}(p)$ for $n \ge 2$. Then for all $k \in \mathbb{N}$ there is $D = D^k \in \mathcal{G}^{\mathcal{A}}$ such that

$$f_{\frac{1}{k},1-\frac{1}{k}} = D(p,\cdot) \wedge f_{\frac{1}{k},1-\frac{1}{k}} \le \Phi_{pn} \vee g_{k,\frac{1}{k}}.$$

For $|p-q| \ge \frac{1}{n}$ then $f_{\frac{1}{k},1-\frac{1}{k}} \le g_{k,\frac{1}{k}}$, a contradiction. Hence also here $\mathcal{A}(p) \nsubseteq \mathcal{A}^{(\mathcal{G}^{\mathcal{A}})}(p)$.

Remark 5.12. We are at present unable to provide an example for $\mathcal{A}^{(\mathcal{G}^{\mathcal{A}})} \neq \mathcal{A}$ in the probabilistic case for a triangle function other than the largest triangle function. In particular, we do not have an example using a triangle function of the form $\circledast$ for a t-norm $*$ that satisfies the condition (I). We have to leave this as an open problem.

6. L-approach spaces versus L-approach system spaces

In order to embed $\mathsf{L\text{-}AP}$ into $\mathsf{L\text{-}AS}$, we consider the composition $G = F \circ E$ of the functors E and F from Sections 3 and 5, i.e. we consider the following commutative diagram.

$$\begin{array}{ccc} \mathsf{L\text{-}AP} & \xrightarrow{E} & \mathsf{L\text{-}GS} \\ & {\scriptstyle G=F\circ E}\searrow & \downarrow F \\ & & \mathsf{L\text{-}AS} \end{array}$$

Proposition 6.1. *Let* L *be a value quantale. Then for* $(X, \delta) \in |\mathsf{L\text{-}AP}|$ *we have* $G((X, \delta)) = (X, \mathcal{A}^{\delta})$*, where for* $x \in X$ *we define*

$$\mathcal{A}^{\delta}(x) = \{\varphi \in L^X \ : \ \delta(x, A) \leq \bigvee_{a \in A} \varphi(a) \ \forall A \subseteq X\}.$$

Proof. We show that $\mathcal{A}^{\delta}(x) = \mathcal{A}^{(\mathcal{G}^{\delta})}(x)$ with the L-gauge $\mathcal{G}^{\delta} = \{d \in \mathsf{L\text{-}MET}(X) \ : \ \delta(x, A) \leq \bigvee_{a \in A} d(x, a) \forall x \in X, A \subseteq X\}$, see [8]. Let first $\varphi \in \mathcal{A}^{(\mathcal{G}^{\delta})}(x)$. Then φ is supported by $\mathcal{B}^{(\mathcal{G}^{\delta})}(x)$, i.e. for all $\alpha \lhd \top$ and all $\omega \succ \perp$ there is $d^{\alpha\omega}(x, \cdot) \in \mathcal{B}^{(\mathcal{G}^{\delta})}(x)$ such that $d^{\alpha\omega}(x, \cdot) * \alpha \leq \varphi \vee \omega$. Then $\delta(x, A) * \alpha \leq \bigvee_{a \in A} d^{\alpha\omega}(x, a) * \alpha \leq \bigvee_{a \in A} \varphi(a) \vee \omega$ for all $\alpha \lhd \top$ and all $\omega \succ \perp$. Taking the join over all $\alpha \lhd \top$ and the meet over all $\omega \succ \perp$ we obtain $\delta(x, A) \leq \bigvee_{a \in A} \varphi(a)$ and hence $\varphi \in \mathcal{A}^{\delta}(x)$. Conversely, let $\varphi \in \mathcal{A}^{\delta}(x)$ and assume $\varphi \notin \mathcal{A}^{(\mathcal{G}^{\delta})}(x)$. Then φ is not supported by $\mathcal{B}^{(\mathcal{G}^{\delta})}$, i.e. for all $d(x, \cdot) \in \mathcal{B}^{(\mathcal{G}^{\delta})}(x)$ we have $d(x, \cdot) \not\leq \varphi$. Hence there is $a \in X$ such that $d(x, a) \not\leq \varphi(a)$ and with $A = \{a\}$ we obtain $\delta(x, \{a\}) \leq d(x, a) \not\leq \varphi(a) = \bigvee_{b \in \{a\}} \varphi(b)$, a contradiction to $\varphi \in \mathcal{A}^{\delta}(x)$. $\square$

The functor G has a right adjoint, T.

Proposition 6.2. (1) $T : \mathsf{L\text{-}AS} \longrightarrow \mathsf{L\text{-}AP}$ *defined by* $T((X, \mathcal{A})) = (X, \delta^{\mathcal{A}})$, $T(f) = f$ *with*

$$\delta^{\mathcal{A}}(x, A) = \bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{a \in A} \varphi(a),$$

is a functor.

(2) *If* L *is a value quantale, then* $id_{\mathsf{L\text{-}AP}} \leq T \circ G$ *and* $G \circ T \leq id_{\mathsf{L\text{-}AS}}$*, i.e.* T *is right adjoint for* G.

(3) *If* L *is a value quantale that satisfies the strong De Morgan law, then* $id_{\mathsf{L}\text{-}\mathsf{AP}} = T \circ G$. *Hence* L-AP *is a full bicoreflective subcategory of* L-AS.

Proof. (1) Let $(X, \mathcal{A}) \in |\mathsf{L}\text{-}\mathsf{AS}|$. We show that $(X, \delta^{\mathcal{A}}) \in |\mathsf{L}\text{-}\mathsf{AP}|$. (LD1) and (LD2) are easy and not shown. For (LD3) the inequality $\delta^{\mathcal{A}}(x, A \cup B) \geq \delta^{\mathcal{A}}(x, A) \vee \delta^{\mathcal{A}}(x, B)$ is clear. For the converse inequality, let $\delta^{\mathcal{A}}(x, A) \vee \delta^{\mathcal{A}}(x, B) \prec \alpha$. Then there are $\varphi_A, \varphi_B \in \mathcal{A}(a)$ such that $\bigvee_{a \in A} \varphi_A(a) \leq \alpha$ and $\bigvee_{b \in B} \varphi_B(b) \leq \alpha$. Then $\varphi_A \wedge \varphi_B \in \mathcal{A}(x)$ and as $\mathcal{A}(x)$ is saturated there is, for $\beta \lhd \top$ and $\omega \succ \bot$, $\varphi_\beta^\omega \in \mathcal{A}(x)$ such that $\varphi_\beta^\omega * \beta \leq (\varphi_A \wedge \varphi_B) \vee \omega$. We conclude

$$
\begin{aligned}
\delta^{\mathcal{A}}(x, A \cup B) * \beta \;&=\; \Big(\bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{a \in A \cup B} \varphi(a)\Big) * \beta \\
\leq\; & \bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{a \in A \cup B} \varphi(a) * \beta \;\leq\; \bigvee_{a \in A \cup B} \varphi_\beta^\omega(a) * \beta \\
=\; & \Big(\bigvee_{a \in A} \varphi_\beta^\omega(a) \vee \bigvee_{b \in B} \varphi_\beta^\omega(b)\Big) * \beta \;=\; \Big(\bigvee_{a \in A} \varphi_\beta^\omega(a) * \beta\Big) \vee \Big(\bigvee_{b \in B} \varphi_\beta^\omega(b) * \beta\Big) \\
\leq\; & \Big(\bigvee_{a \in A} \varphi_A(a) \vee \omega\Big) \vee \Big(\bigvee_{b \in B} \varphi_B(b) \vee \omega\Big) \;\leq\; \alpha \vee \omega.
\end{aligned}
$$

Taking the join over all $\alpha \lhd \top$ and the meet over all $\omega \succ \bot$ we obtain $\delta^{\mathcal{A}}(x, A \cup B) \leq \alpha$, from which, using the complete distributivity of L, $\delta^{\mathcal{A}}(x, A \cup B) \leq \delta^{\mathcal{A}}(x, A) \vee \delta^{\mathcal{A}}(x, B)$ follows.

We now prove (LD4). First we note that if $\delta \lhd \alpha$, then $\delta \lhd \alpha = \alpha * \top = (\bigvee_{\beta \lhd \alpha} \beta) * (\bigvee_{\gamma \lhd \top} \gamma) = \bigvee_{\beta \lhd \alpha, \gamma \lhd \top} (\beta * \gamma)$ and hence there is $\beta \lhd \alpha$ and $\gamma \lhd \top$ such that $\delta \lhd \beta * \gamma$. Let now $\varphi \in \mathcal{A}(x)$, $\alpha \in L$, $\delta \lhd \alpha$ and $\omega \succ \bot$. As we have just noted, then there are $\beta \lhd \alpha, \gamma \lhd \top$ such that $\delta \lhd \beta * \gamma$. There is a family $(\varphi_z)_{z \in X} \prod_{z \in X} \mathcal{A}(z)$ such that

$$\varphi_x(z) * \varphi_z(y) * \gamma \leq \varphi(y) \vee \omega \quad \forall z, y \in X.$$

For $b \in \overline{A}^\alpha$ we have $\bigwedge_{\psi \in \mathcal{A}(b)} \bigvee_{a \in A} \psi(a) \geq \alpha \rhd \beta$ and for $\varphi_b \in \mathcal{A}(b)$ there is $a_\beta^\varphi \in A$ such that $\varphi_b(a_\beta^\varphi) \rhd \beta$. Hence $\varphi(a_\beta^\varphi) \vee \omega \geq \varphi_x(b) * \varphi_b(a_\beta^\varphi) * \gamma \geq \varphi_x(b) * \beta * \gamma$ and we conclude

$$\bigvee_{a \in A} \varphi(a) \vee \omega \geq \bigvee_{b \in \overline{A}^\alpha} \varphi_x(b) * \beta * \gamma \geq \delta^{\mathcal{A}}(x, \overline{A}^\alpha) * \delta.$$

As $\omega \succ \bot$ was arbitrary, we conclude $\bigvee_{a \in A} \varphi(a) \geq \delta^{\mathcal{A}}(x, \overline{A}^\alpha) * \delta$. This is true for all $\delta \lhd \alpha$ and therefore also $\bigvee_{a \in A} \varphi(a) \geq \bigvee_{\delta \lhd \alpha} \delta^{\mathcal{A}}(x, \overline{A}^\alpha) * \delta = \delta^{\mathcal{A}}(x, \overline{A}^\alpha) * \alpha$. As $\varphi \in \mathcal{A}(x)$ was arbitrary, we conclude $\delta^{\mathcal{A}}(x, A) = \bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{a \in A} \varphi(a) \geq \delta^{\mathcal{A}}(x, \overline{A}^\alpha) * \alpha$.

Let now $f : (X, \mathcal{A}) \longrightarrow (X', \mathcal{A}')$ be an L-approach system morphism. We show that $f : (X, \delta^{\mathcal{A}}) \longrightarrow (X', \delta^{\mathcal{A}'})$ is an L-approach morphism. For $\psi \in \mathcal{A}'(f(x))$ we have $\psi \circ f \in \mathcal{A}(x)$. Hence

$$\delta^{\mathcal{A}}(x, A) = \bigwedge_{\varphi \in \mathcal{A}(x)} \bigvee_{a \in A} \varphi(a) \leq \bigwedge_{\psi \circ f \in \mathcal{A}(x)} \bigvee_{a \in A} \psi(f(a))$$

$$\leq \bigwedge_{\psi \in \mathcal{A}'(f(x))} \bigvee_{b \in f(A)} \psi(b) = \delta^{\mathcal{A}'}(f(x), f(A)).$$

(2) Let first $(X, \delta) \in |\mathsf{L\text{-}AP}|$. Then

$$\delta^{(\mathcal{A}^{\delta})}(x, A) = \bigwedge_{\varphi : \delta(x,B) \leq \bigvee_{b \in B} \varphi(b) \forall B \subseteq X} \bigvee_{a \in A} \varphi(a) \geq \delta(x, A).$$

Hence $id_{\mathsf{L\text{-}AP}} \leq T \circ G$. Let now $(X, \mathcal{A}) \in |\mathsf{L\text{-}AS}|$ and let $\varphi \in \mathcal{A}(x)$. Then $\delta^{\mathcal{A}}(x, A) = \bigwedge_{\psi \in \mathcal{A}(x)} \bigvee_{a \in A} \psi(a) \leq \bigvee_{a \in A} \varphi(a)$ and hence $\varphi \in \mathcal{A}^{(\delta^{\mathcal{A}})}(x)$. This shows the other inequality.

(3) Under the assumptions on L, both E, F are full, and so is the composition $G = F \circ E$ and hence we even have $id_{\mathsf{L\text{-}AP}} = T \circ G$. □

Corollary 6.3. *Let* L *be a value quantale that satisfies the strong De Morgan law. The category* $\mathsf{L\text{-}AP}$ *is topological and we can construct initial structures by applying the coreflector* T *to the initial structures obtained in* $\mathsf{L\text{-}AS}$.

The fact that $\mathsf{L\text{-}AP}$ is topological was, under weaker requirements on the quantale, already shown in [10, 11].

We note that $\delta^{\mathcal{A}}(x, A) \leq \bigwedge_{d \in \mathcal{G}^{\mathcal{A}}} \bigvee_{a \in A} d(x, a) = \delta^{(\mathcal{G}^{\mathcal{A}})}(x, A)$ for all $x \in X, A \subseteq X$. Theorem 5.9 tells us further, that if L is a linearly ordered value quantale for which the quantale operation distributes over arbitrary meets and the strong De Morgan law holds, then $\delta^{\mathcal{A}} = \delta^{(\mathcal{G}^{\mathcal{A}})}$. In this case, as Proposition 3.5 and Theorem 5.9 show, the categories $\mathsf{L\text{-}AP}$, $\mathsf{L\text{-}GS}$ and $\mathsf{L\text{-}AS}$ are isomorphic.

7. Conclusions

In this paper, we generalized one of the definitions of an approach space in terms of approach systems to the quantale-valued case. We obtained a topological category of quantale-valued approach system spaces. The functors, that show in the case $L = [0, \infty]$ that approach systems, gauges and approach distances are equivalent concepts, provide only in a very restricted case, that L is a linearly ordered value quantale that satisfies a weak cancellation condition (I) and for which the quantale operation distributes over arbitrary meets and satisfies the strong De Morgan law, an isomorphism between the categories of quantale-valued approach spaces, quantale-valued gauge spaces and quantale-valued approach system spaces. In particular in the probabilistic case, the embedding functors are in general not isomorphisms, and one cannot simply "translate" the theory of approach spaces to probabilistic approach spaces.

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