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STUDY OF TRANSITIVITY THROUGH STEEPNESS

V. KANNAN, R. BASU, K. SANNYASI, AND N. UNNIKRISHNAN

Abstract. The aim of this work is to provide some simple sufficient conditions for topological transitivity of piecewise monotone maps on $[0,1]$. Here we introduce a steepness condition that will imply that the map is expanding (in the sense that for every interval, the length of its image is greater than the length of that interval, unless the image is the whole space), and then we prove that these expanding maps are transitive. The theorems stated in this paper improve some known recent results. Moreover, they are simpler to state.

1. Introduction

Discrete dynamical systems arise as mathematical models of any motion obeying a rule that does not change with time. The rough idea behind transitivity is that we like to require any point in the phase space to visit every portion of the space in the course of time. Because points are seldom handled accurately, due to round-off errors and computational errors, we modify our requirement: Every neighborhood of every point visits every region at some time or other. Consequently, such a dynamical system cannot be decomposed into two disjoint sets with nonempty interiors which do not interact under the transformation. Thus, transitivity in some sense is irreducibility.

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Moreover, for interval maps transitivity is the same as the well-known property of chaos (according to Robert L. Devaney's definition) (see [3], [7], [8]).

Sufficient conditions for topological transitivity of a dynamical system have been studied by many persons. In [5], Peter Raith and Angela Stachelberger found some sufficient conditions for the topological transitivity of interval maps. Also in [6], the authors found that "A subshift of finite type is topologically transitive if and only if the associated matrix M is irreducible in the sense that for every (i, j), the $(i, j)^{th}$ entry of the matrix power M^n is strictly positive for some positive integer n."

It has been observed that transitivity of a dynamical system can be inferred through the steepness of the graph of the function. Raith found some sufficient conditions for the transitivity of a function [4, Theorem 1]. Later, in [2], Anima Nagar, V. Kannan, and Karanam Srinivas improved the results obtained by Raith (see section 4 of this paper).

In this paper, we try to improve the results further and examples have been provided to show that these new theorems are more powerful than the previous ones. In addition, our results simplify the condition for transitivity. Since it is not always possible to exhibit dense orbits or to construct a topological conjugacy with known topologically transitive maps, these results have acquired practical value.

The main theorem of this paper paves the way for writing an algorithm to check in some cases whether a given piecewise linear map is transitive. This is because there are only finitely many intervals of surjection where the steepness condition is to be verified.

2. Definitions

A self-map f on a topological space X is said to be *topologically transitive* if for every pair of non-empty open sets U and V in X, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \phi$.

A function f on X is said to be *locally eventually onto* (*leo*) if for every non-empty open set U there exists $n \in \mathbb{N}$ such that $f^n(U) = X$.

A set $A \subset X$ is said to be *forward invariant* under f if $f(A) \subseteq A$.

A *lap* is defined as a maximal subinterval in the domain (here it is $[0, 1]$) of f, on which f is monotonic.

A function is said to be *piecewise monotone* (*p.m.*) if the domain is the union of finitely many subintervals on each of which f is monotonic.

A function is said to be *piecewise linear* (*p.l.*) if the domain is the union of finitely many subintervals on each of which f is linear.

A function is said to be *admissible piecewise linear* (*a.p.l.*) if it is piecewise linear and slopes are alternately positive and negative. This is equivalent to saying that f is linear on each lap. It is obvious that an a.p.l. map $\Rightarrow$ a p.l. map $\Rightarrow$ a p.m. map.

A point x is said to be a *critical point* of f if in every neighborhood of x, f fails to be one-one.

An a.p.l. map is completely specified by its values at the end points of the laps. By an a.p.l. map, defined by

$$f(a_i) = b_i \quad \text{for} \quad i \in \{0, 1, \dots, r+1\}, \tag{2.1}$$

we mean the map f satisfying (2.1), where

$$0 = a_0 < a_1 < \cdots < a_r < a_{r+1} = 1$$

such that f is linear in $[a_i, a_{i+1}]$ for $i \in \{0, 1, \dots, r\}$. For example, the tent map is defined by $f(0) = 0$, $f(\frac{1}{2}) = 1$, and $f(1) = 0$.

3. Transitivity Through Expansion

The next theorem makes precise the following rough idea: If f admits several intervals of surjection on which the steepness condition holds, then f has to be transitive.

Theorem 3.1. *Let $f : I(= [0,1]) \to I$ be a p.m. interval map differentiable at all points outside a finite set F. Let $J_1, J_2, \dots, J_k$ be subintervals of I in the increasing order of their left end-points such that*

(1) $f(J_i) = [0,1]$ *for all* $i \in \{1, 2, \dots, k\}$
(2) *If c is an end point of J_i, then either c is a critical point or c belongs to* $\{0,1\}$.
(3) $\inf\{|f'(x)| : x \in (J_i \cup J_{i+1}) - F\} >$ *no of laps in* $(J_i \cup J_{i+1})$ *for* $i \in \{1, 2, \dots, k-1\}$.

If $\{J_i | i = 1, 2, \dots, k\}$ covers the whole I, then f is topologically transitive.

Proof. Consider an open interval K in I. Our present purpose is to find a lower bound for $\frac{|f(K)|}{|K|}$.

Case 1: Let K lie completely inside some lap in I. If $K = (a, b)$, then $|K| = b - a$. Because f is monotonic on K, the length of $f(K)$ is equal to $|f(b) - f(a)|$. Applying MVT to every one of the open intervals $K_1, K_2, \dots, K_p$ which are components of $K - F$, we can conclude that there exist $c_j \in K_j$ for all $j \in \{1, 2, \dots, p\}$ such that $|f(K_j)| = |f'(c_j)||K_j|$. Hence,

$$\begin{aligned} |f(K)| &= |f(b) - f(a)| \\ &= |f'(c_1)||K_1| + |f'(c_2)||K_2| + \cdots + |f'(c_p)||K_p| \\ &\geq \inf\{|f'(x)| : x \in K - F\}|K|. \end{aligned}$$

Case 2: Let $K \subset (J_i \cup J_{i+1})$ for some $i \in \{1, \dots, k-1\}$. Let $L_1, L_2, \dots L_n$ be the laps that meet K. Let $K_l = K \cap L_l$ for $1 \leq l \leq n$. Choose l such that $|K_l| \geq |K_j|$ for all j. Hence,

$$\begin{aligned}
|f(K)| \geq |f(K_l)| &\geq \inf\{|f'(x)| : x \in K_l - F\}|K_l| \\
&\geq \inf\{|f'(x)| : x \in (J_i \cup J_{i+1}) - F\}|K_l| \\
&\geq \frac{\inf\{|f'(x)| : x \in (J_i \cup J_{i+1}) - F\}}{n}|K|.
\end{aligned}$$

Case 3: Let $K \nsubseteq (J_i \cup J_{i+1})$ and a_i and b_i be the end points of J_i for all $i = 1, 2, \dots, k-1$. Take the largest i such that a_i is less than or equal to the left end point of K. (Note that a_{i+1} is not less than or equal to the left end point of K.) Two cases are possible:

(i): If $b_{i+1} \in K$, then $K \supset J_{i+1}$. Hence, $f(K) = [0, 1]$.

(ii): If b_{i+1} is greater than or equal to every element in K, then $K \subset J_i \cup J_{i+1}$, a contradiction to our assumption.

Thus, we have found lower bounds for $\frac{|f(K)|}{|K|}$. In Case 1, it was

$$\inf\{|f'(x)| : x \in K - F\}$$

and in Case 2, it was

$$\frac{\inf\{|f'(x)| : x \in (J_i \cup J_{i+1}) - F\}}{n}.$$

Now we find a common lower bound that works in all cases. Let

$$\delta = \min_{1 \leq i < k} \left(\frac{\inf\{|f'(x)| : x \in (J_i \cup J_{i+1}) - F\}}{\{\text{no of laps in } (J_i \cup J_{i+1})\}} \right) - 1.$$

Note that assumption (3) implies $\delta > 0$.

We shall prove

$$\frac{|f(K)|}{|K|} \geq (1 + \delta)$$

in all cases, unless $f(K) = [0, 1]$.

We may note that in Case 1,

$$\begin{aligned}
\inf\{|f'(x)| : x \in K - F\} &\geq \inf\{|f'(x)| : x \in (J_i \cup J_{i+1}) - F\} \\
&\quad \text{for that } i \text{ such that } K \subset J_i \\
&\geq \frac{\inf\{|f'(x)| : x \in (J_i \cup J_{i+1}) - F\}}{\{\text{no of laps in } (J_i \cup J_{i+1})\ \}} \\
&\geq (1 + \delta),
\end{aligned}$$

and in Case 2,

$$\begin{aligned}\frac{\inf\{|f'(x)| : x \in (J_i \cup J_{i+1}) - F\}}{n} &\geq \frac{\inf\{|f'(x)| : x \in (J_i \cup J_{i+1}) - F\}}{\{\text{no of laps in } (J_i \cup J_{i+1})\}} \\ &\geq (1+\delta).\end{aligned}$$

Thus, in all the cases,

$$|f(K)| \geq (1+\delta)|K| \quad \text{or} \quad f(K) = [0,1].$$

Proceeding in the same way, we can show that for all positive intergers n

$$|f^n(K)| \geq (1+\delta)^n|K| \quad \text{or} \quad f^n(K) = [0,1].$$

Since $(1+\delta)^n \to \infty$ as $n \to \infty$, there exists $n \in N$ such that $f^n(K) = [0,1]$. So the function is transitive and, in fact, leo. Hence, the theorem is proved. □

Remark 3.2. If $F = \phi$ (empty set), then by using Rolle's theorem, we can show that f is not expanding.

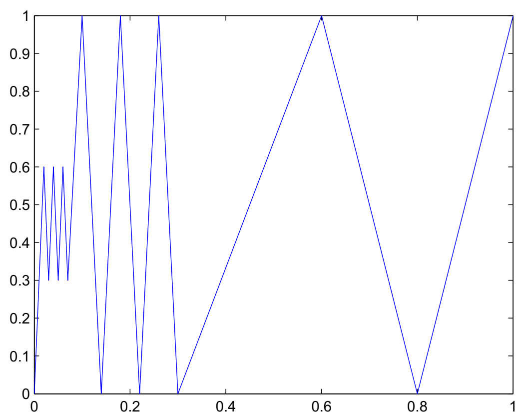
4. Theorem 1 of [2] as a Corollary of Theorem 3.1

In our statement of Theorem 1 of [2], let $n \geq 2$ be in $\mathbb{N}$. Let f be a p.m. map differentiable at all points outside a finite set F. Assume (1) if an interval J contains n critical points, then $f(J) = [0,1]$, and (2) $\inf\{|f'(x)| : x \notin F\} > n$. Then f is topologically transitive.

Proof. Let $c_1, c_2, \dots, c_k$ be the critical points of f such that $0 = c_0 < c_1 < \cdots < c_k < c_{k+1} = 1$. We claim that $k+1 > n$, that is, $k \geq n$. If not, let us assume that $k+1 \leq n$. Consider the lap L of maximum length, say $[c_i, c_{i+1}]$ for some $i \in \{0, 1, \dots, k\}$, then $|L| > \frac{1}{k+1}|[0,1]| = \frac{1}{k+1}$ which implies that $|f(L)| > \frac{n}{k+1} \geq 1$, a contradiction to the fact that f is a self-map. So we can conclude that $0 < c_1 < c_2 < \cdots < c_n \leq c_k < 1$. Hence, there are intervals containing n critical points. Let $J_1 = [0, c_n]$, $J_2 = [c_1, c_n]$, $J_3 = [c_2, c_{n+1}]$, …, and $J_l = [c_{k-n+1}, 1]$. We find that condition (1) in our statement of [2, Theorem 1] implies that $f(J_1) = f(J_2) = f(J_3) = \cdots = f(J_l) = [0,1]$. Hence, condition (1) of Theorem 3.1 is verified. Condition (2) of Theorem 3.1 is obvious. Next, we find that condition (2) in our statement of [2, Theorem 1] implies condition (3) of Theorem 3.1, as $inf\{|f'(x)| : x \in (J_1 \cup J_2) - F\} > n =$ number of laps in $J_1 \cup J_2$. Similar results can be proved for $J_i \cup J_{i+1}$ for all i. Therefore, we have shown that assumptions (1) and (2) of our statement of [2, Theorem 1] imply assumptions (1), (2) and (3) of Theorem 3.1. Hence, by Theorem 3.1, f is topologically transitive. □

5. Example

Consider the function depicted in the graph shown below. Every interval of surjection containing zero should contain seven critical points; hence, n mentioned in [2, Theorem 1] will be greater than or equal to 7. So the steepness of the graph has to be more than or equal to 8, but in another interval of surjection (between $0.3-0.6$), the steepness is 3.33 (not more than 8), so [2, Theorem 1] is not applicable. But Theorem 3.1 of this paper is applicable as the minimum steepness is allowed to vary here.



6. Estimation for the Ratio $\frac{|f(K)|}{|K|}$

This section provides results supplementing the previous theorem in a manner that is easier for practical applications.

Theorem 6.1. *If $s_1, s_2, \ldots, s_n$ and $l_1, l_2, \ldots, l_n$ are positive real numbers, then*

$$\frac{s_1 l_1 \vee s_2 l_2 \vee \cdots \vee s_n l_n}{l_1 + l_2 + \cdots + l_n} \geq \frac{1}{\frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_n}} \tag{6.1}$$

and equality occurs when $s_1 l_1 = s_2 l_2 = \cdots = s_n l_n$, where $\vee$ represents the maximum. In fact, if $s_1, s_2, \ldots, s_n (> 0)$ are fixed and $l_1, l_2, \ldots, l_n (> 0)$ are allowed to vary, then

$$\inf_{l_1, l_2, \ldots, l_n} \left(\frac{s_1 l_1 \vee s_2 l_2 \vee \cdots \vee s_n l_n}{l_1 + l_2 + \cdots + l_n} \right) = \frac{1}{\frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_n}}. \tag{6.2}$$

It follows that if f is an a.p.l. continuous function from $\mathbb{R}$ to $\mathbb{R}$ and $s_1, s_2, \ldots, s_n$ are moduli of slopes in the graph of f, then for any interval

K meeting n laps, we have

$$\frac{|f(K)|}{|K|} \geq \frac{1}{\frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_n}}. \tag{6.3}$$

Proof. We prove (6.2) by induction. The result is trivially true for $n = 1$. Let us prove it for $n = 2$; that is,

$$\inf_{l_1, l_2} \left(\frac{s_1 l_1 \vee s_2 l_2}{l_1 + l_2} \right) = \frac{1}{\frac{1}{s_1} + \frac{1}{s_2}}.$$

Let $t = \frac{l_1}{l_2}$, then the last equality becomes

$$\inf_{t} \left(\frac{s_1 t \vee s_2}{t + 1} \right) = \frac{1}{\frac{1}{s_1} + \frac{1}{s_2}}.$$

Since $\frac{s_1 t}{1+t}$ is an increasing function and $\frac{s_2}{1+t}$ is a decreasing function, the point of intersection of these two curves gives the infimum value for $\frac{s_1 t \vee s_2}{t+1}$. We can easily show that the point of intersection is at $t = \frac{s_2}{s_1}$. Hence,

$$\begin{aligned} \inf_{t} \left(\frac{s_1 t \vee s_2}{t + 1} \right) &= \frac{s_1 s_2}{s_1 + s_2} \\ &= \frac{1}{\frac{1}{s_1} + \frac{1}{s_2}}. \end{aligned} \tag{6.4}$$

Let us assume that (6.2) holds for $n = k$. Next, we will prove (6.2) for $n = k + 1$. Let $s = \frac{1}{\frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_k}}$ and $l = l_1 + l_2 + \cdots + l_k$. As (6.2) is true for $n = k$, so

$$\inf_{l_1, l_2, \ldots, l_k} \left(\frac{s_1 l_1 \vee s_2 l_2 \vee \cdots \vee s_k l_k}{l} \right) = s,$$

which implies

$$s_1 l_1 \vee s_2 l_2 \vee \cdots \vee s_k l_k \geq s l.$$

Hence, we can write

$$\frac{s_1 l_1 \vee s_2 l_2 \vee \cdots \vee s_{k+1} l_{k+1}}{l_1 + l_2 + \cdots + l_{k+1}} \geq \frac{\{s l \vee s_{k+1} l_{k+1}\}}{l + l_{k+1}}.$$

By using the results for $n = 2$, we get

$$\begin{aligned} \frac{\{s l \vee s_{k+1} l_{k+1}\}}{l + l_{k+1}} &\geq \frac{1}{\frac{1}{s} + \frac{1}{s_{k+1}}} \\ &= \frac{1}{\frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_{k+1}}}. \end{aligned}$$

Equality occurs when $l_i = \frac{1}{s_i}$ for all i. So,

$$\inf_{l_1,l_2,\ldots,l_{k+1}} \left(\frac{s_1 l_1 \vee s_2 l_2 \vee \cdots \vee s_{k+1} l_{k+1}}{l_1 + l_2 + \cdots + l_{k+1}} \right) \quad = \quad \frac{1}{\frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_{k+1}}}.$$

Hence, by induction, the result follows. Inequality (6.1) follows from (6.2). We will prove (6.3).

Let K be an interval meeting n laps namely $L_1, L_2, \ldots, L_n$ and suppose $l_i = |K \cap L_i|$ for $i \in \{1, \ldots, n\}$. Therefore,

$$K = (K \cap L_1) \cup (K \cap L_2) \cup \cdots \cup (K \cap L_n).$$

Hence, $f(K) \supset f(K \cap L_i)$ for all i, which implies $|f(K)| \geq |f(K \cap L_i)| = s_i l_i$ for all i. Therefore, $|f(K)| \geq s_1 l_1 \vee s_2 l_2 \vee \cdots \vee s_n l_n$. Since $|K| = l_1 + l_2 + \cdots + l_n$, we can conclude that

$$\frac{|f(K)|}{|K|} \geq \frac{s_1 l_1 \vee s_2 l_2 \vee \cdots \vee s_n l_n}{l_1 + l_2 + \cdots + l_n}.$$

From (6.1), it follows that

$$\frac{|f(K)|}{|K|} \geq \frac{1}{\frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_n}}.$$

Hence, the result follows. □

Remark 6.2. The previous theorem is also true for piecewise monotone maps. For a piecewise monotone map, the steepness on a lap L is defined as $\inf\{|f'(x)| : x \in L - F\}$, where f is differentiable at all points ouside a finite set F.

Theorem 6.3. *Let f be a p.m. interval map on I differentiable at all points outside a finite set F. Let $L_1, L_2, \ldots, L_m$ be the laps and let $s_1, s_2, \ldots, s_m$ be the steepnesses there (assume steepness s_i is always > 1 and on a lap L_i it is defined as $s_i = \inf\{|f'(x)| : x \in L_i - F\}$). If for all $i < j$, $\frac{1}{s_i} + \cdots + \frac{1}{s_j} < 1$ or $f(L_i \cup \cdots \cup L_j) = [0, 1]$ (this will be hereafter called the steepness condition), then f is topologically transitive.*

Proof. Consider an open interval K in I.

Case 1: Let K lie completely inside some lap in I with steepness s_i. Because f is monotonic on K, the length of $f(K)$ is greater than or equal to $s_i|K|$. Therefore,

$$\frac{|f(K)|}{|K|} \geq s_i > 1.$$

Hence,

$$\begin{aligned} |f(K)| &\geq \frac{1}{\frac{1}{s_i}}|K| \\ &\geq \frac{1}{\frac{1}{s_1}+\frac{1}{s_2}+\cdots+\frac{1}{s_m}}|K| \\ &= (1+\delta)|K|, \end{aligned}$$

where

$$\delta = \frac{1}{\frac{1}{s_1}+\frac{1}{s_2}+\cdots+\frac{1}{s_m}} - 1.$$

We can easily verify that $\delta > 0$.

Case 2: Let K meet n laps, namely $L_1, L_2, \ldots, L_n$ with steepnesses $s_1, s_2, \ldots, s_n$, and suppose $l_i = |K \cap L_i|$ for $i = 1, \ldots, n$. Then we obtain $|f(K)| \geq |f(K \cap L_i)| \geq s_i l_i$ for all i. Therefore,

$$\frac{|f(K)|}{|K|} \geq \frac{1}{\frac{1}{s_1}+\frac{1}{s_2}+\cdots+\frac{1}{s_n}} \geq 1+\delta,$$

where $|K| = l_1 + l_2 + \cdots + l_n$. Thus, our assumption gives $|f(K)| \geq (1+\delta)|K|$ unless $f(K) = [0,1]$. Proceeding as in Theorem 3.1, we can conclude that f is topologically transitive. □

7. Theorem 1 of [2] and Theorem 3.1 as a Corollary of Theorem 6.3

We now show that the assumptions of [2, Theorem 1] and Theorem 3.1 imply the assumptions of Theorem 6.3.

Proof. Let $i < j$. Let $K = L_i \cup ... \cup L_j$ and $s_i, \ldots, s_j$ be the steepnesses in $L_i, \ldots, L_j$, respectively. We will deduce from these assumptions that either

$$\frac{1}{s_i} + \cdots + \frac{1}{s_j} < 1$$

or $f(L_i \cup \cdots \cup L_j) = [0,1]$. First, we will prove [2, Theorem 1] as a corollary of Theorem 6.3. Let k be the number of critical points in K. Let n be as in [2, Theorem 1]. If $k < n$, then

$$\frac{1}{s_i} + \frac{1}{s_{i+1}} + \cdots + \frac{1}{s_j} < \frac{1}{n} + \cdots + \frac{1}{n}(k \quad times) = \frac{k}{n} < 1,$$

(as each $s_i > n$). If $k \geq n$, then by the assumptions of [2, Theorem 1], $f(K) = [0,1]$. Hence, by Theorem 6.3, f is topologically transitive.

Next, we will prove Theorem 3.1 of this paper as a corollary of Theorem 6.3. If $K \supset J_l$, then $f(K) = [0,1]$ as $f(J_l) = [0,1]$. Next, if $K \subset$

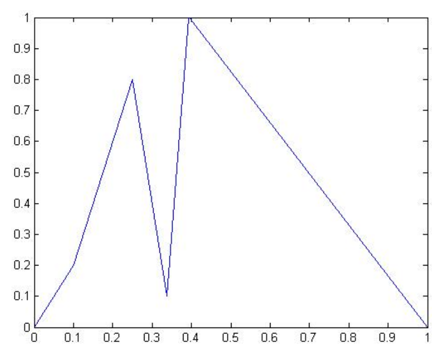
$J_l \cup J_{l+1}$ for some l, then it follows from assumption (3) of Theorem 3.1 that steepness in each $L_r (i \leq r \leq j)$ (in fact in $J_l \cup J_{l+1}$) is greater than the number of laps in $J_l \cup J_{l+1}$, say m (m depends on l). So,

$$\begin{aligned} \frac{1}{s_i} + \frac{1}{s_{i+1}} + \cdots + \frac{1}{s_j} &< \frac{1}{m} + \cdots + \frac{1}{m} \\ &= \frac{\textit{number of laps in } K}{m} \leq \frac{m}{m} = 1. \end{aligned}$$

Hence, by Theorem 6.3, f is topologically transitive. □

8. More on the Steepness Condition

Consider the function depicted in the following graph. In this example, the critical points are 0.1, 0.25, 0.3375, and 0.39375, and the steepnesses of the graph in the first, second, third, and fourth laps are 2, 4, 8, and 16, respectively.



Suppose J_1 is an interval of surjection containing the laps with slopes 2, 4, 8, and 16, and $J_2 (= I - J_1)$ is another interval of surjection containing one lap.

Note that Theorem 3.1 is not applicable here as the steepness in the first lap is 2 which is not greater than the number of laps in $J_1 \cup J_2$, but Theorem 6.3 is applicable since the sum of the reciprocals of the steepnesses on J_1 is less than 1.

Remark 8.1. In both [4] and [2], the steepness was required to be greater than a fixed lower bound everywhere. In this paper, Theorem 3.1 is also similar, but the lower bound of the steepness is allowed to vary at different places.

Remark 8.2. Consider the a.p.l. map $f : [0, 1] \to [0, 1]$ defined by $f(0) = 0$, $f(\frac{1}{4}) = \frac{3}{4}$, $f(\frac{1}{3}) = \frac{1}{2}$, $f(\frac{7}{12}) = 1$, $f(\frac{3}{4}) = 0$, and $f(1) = 1$. We can easily verify that for the function f, the sum of the reciprocals of the steepnesses in the first three laps is $\frac{1}{3}+\frac{1}{3}+\frac{1}{2} = \frac{7}{6} > 1$, so Theorem 6.3 is not applicable to f. But from the graph of fof, we can easily verify that the sum of the reciprocals of the steepnesses in any interval outside the laps of surjection, i.e., outside those laps L for which $f(L) = [0, 1]$, is less than 1; hence, by applying Theorem 6.3 to fof, we can conclude that fof is topologically transitive. Consequently, f is topologically transitive (by [1, Theorem 2.3]).

We would like to know whether for every a.p.l. transitive map f, should some power of f satisfy the steepness condition? While this is open, the next theorem gives some other kind of converse.

Theorem 8.3. *Suppose $s_1, s_2, ...s_k$ are positive numbers such that every a.p.l. map on $\mathbb{R}$ with these as moduli of slopes is expanding. Then the sum of their reciprocals is less than 1.*

Proof. Consider this example. Suppose the critical points for a function are $\frac{1}{s_1}, \frac{1}{s_1} + \frac{1}{s_2}, \dots, \frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_k}$. The function takes the value 0 at 0, then 1 at $\frac{1}{s_1}$, and alternately 0 and 1 at these critical points. The a.p.l. map defined by these is also expanding by our assumption. Let s be the sum of the reciprocals of $s_1, s_2, \dots, s_k$. Then, because the closed interval $[0, s]$ under this map goes to the closed interval $[0, 1]$, it follows that $s < 1$. □

Remark 8.4. If the steepness condition fails for some map, then there exists a non-transitive map with the same sequence of steepnesses. Therefore, among all steepness conditions implying transitivity, the one studied in this paper is the best.

Theorem 8.5. *Any continuous a.p.l. function $f : [0, \frac{1}{2}] \to [0, 1]$ satisfying*

$$\frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_n} < 1,$$

where $s_1, s_2, \dots, s_n$ are moduli of slopes in the graph of f, can be extended to a function $g : [0, 1] \to [0, 1]$ such that g is transitive.

Proof. Choose $s > 0$ such that

$$\frac{1}{s} + \frac{1}{s_1} + \cdots + \frac{1}{s_n} < 1.$$

Next choose α in $[\frac{1}{2}, 1]$ such that $\frac{1-f(\frac{1}{2})}{\alpha-\frac{1}{2}} = s$. (If $f(\frac{1}{2}) = 1$, we take $\alpha = \frac{1}{2}$.) Now by extending f as an a.p.l. map such that $f(\alpha) = 1$ and $f(1) = 0$, the result follows from Theorem 6.3. □

Remark 8.6. An open map from $[0, 1]$ to $[0, 1]$ is transitive if and only if it is not one-one.

As for every open map, the critical values are either 0 or 1. So every lap is mapped onto $[0, 1]$; hence, by applying Theorem 6.3, we get the results.

Remark 8.7. Note that an a.p.l. continuous function f from $\mathbb{R}$ to $\mathbb{R}$ with steepnesses $s_1, s_2, \dots, s_n$ is expanding if

$$\frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_n} < 1. \tag{8.1}$$

Remark 8.8. An a.p.l. continuous function f from $\mathbb{R}$ to $\mathbb{R}$ with steepnesses $s_1, s_2, \dots, s_n$ is expanding if the harmonic mean of $s_1, s_2, \dots, s_n$ is greater than n.

References

[1] Anima Nagar and V. Kannan, *Topological transitivity for discrete dynamical systems* in Applicable Mathematics in the Golden Age. Ed. J. C. Misra. New Delhi, India: Narosa Publishing House, 2003. 410–454.

[2] Anima Nagar, V. Kannan, and Karanam Srinivas, *Some simple conditions implying topological transitivity for interval maps*, Aequationes Math. **67** (2004), no. 3, 201–204.

[3] Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems.* 2nd ed. Addison-Wesley Studies in Nonlinearity. Redwood City, CA: Addison-Wesley Publishing Company, Advanced Book Program, 1989.

[4] Peter Raith, *Topological transitivity for expanding piecewise monotonic maps on the interval*, Aequationes Math. **57** (1999), no. 2-3, 303–311.

[5] Peter Raith and Angela Stachelberger, *Topological transitivity for a class of monotonic mod one transformations*, Aequationes Math. **82** (2011), no. 1-2, 91–109.

[6] Mark Pollicott and Michiko Yuri, *Dynamical Systems and Ergodic Theory.* London Mathematical Society Student Texts, 40. Cambridge: Cambridge University Press, 1998.

[7] Stephen Silverman, *On maps with dense orbits and the definition of chaos*, Rocky Mountain J. Math. **22** (1992), no. 1, 353–375.

[8] Michel Vellekoop and Raoul Berglund, *On intervals, transitivity = chaos*, Amer. Math. Monthly **101** (1994), no. 4, 353–355.

School of Mathematics and Statistics; University of Hyderabad; Hyderabad - 500 046 Telangana INDIA

E-mail address, Kannan: `vksm@uohyd.ernet.in`

E-mail address, Basu: `rakhee.research@gmail.com`

E-mail address, Sannyasi: `kunal.hcumath@gmail.com`

E-mail address, Unnikrishnan: `nidhishunnikrishnan@yahoo.com`