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DARIA MICHALIK

Abstract. We prove that if X and Y are locally connected curves not being ANRs, then X and Y are homeomorphic if and only if $\mathrm{Cone}(X)$ and $\mathrm{Cone}(Y)$ are homeomorphic.

1. Introduction

Let X be a topological space. The *cone* of X is the quotient space defined by

$$\mathrm{Cone}(X) = X \times \mathbb{I} \diagup (X \times \{1\}).$$

The *cylinder* of X is the Cartesian product $X \times \mathbb{I}$ and the *suspension* of X is the quotient space

$$\mathrm{Sus}(X) = X \times \mathbb{I} \diagup (X \times \{0\},\ X \times \{1\}).$$

It is well known that cones of non-homeomorphic spaces can be homeomorphic, e.g., $\mathrm{Cone}(S^1)$ and $\mathrm{Cone}(\mathbb{I})$.

Example 1.1. Let A_i be a cone over i-point space and B_i be a suspension over i-point space. Then, for every $i \in \mathbb{N}$, $\mathrm{Cone}(A_i)$ and $\mathrm{Cone}(B_i)$ are homeomorphic, but A_i and B_i are not homeomorphic.

The main theorem of this note follows.

Theorem 1.2. *Let us assume that X and Y are locally connected curves not being ANRs. Then* $\mathrm{Cone}(X)$ *and* $\mathrm{Cone}(Y)$ *are homeomorphic if and only if X is homeomorphic to Y.*

Remark 1.3. By [3], if X and Y are locally connected curves, then the cylinder of X is homeomorphic to the cylinder of Y if and only if X is homeomorphic to Y.

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Using the same methods as in the proof of Theorem 1.2 (see §5), one can prove the following theorem.

Theorem 1.4. *Let us assume that X and Y are locally connected curves not being ANRs. Then* $\mathrm{Sus}(X)$ *and* $\mathrm{Sus}(Y)$ *are homeomorphic if and only if X is homeomorphic to Y.*

An important step toward proving our main result is Theorem 4.1 about the uniqueness decomposition for the products of a locally connected curve and the interval $[0,1)$. Let us recall that, in general, the cancellation law in Cartesian products does not hold, especially for non-compact factors, e.g.,

$$[0,1] \times [0,1) \approx [0,1) \times [0,1) \approx (0,1) \times [0,1).$$

Thus, Theorem 4.1 seems to be interesting. It is also one of the crucial steps in the proof of the main result of [4]. The proof of Theorem 4.1 involves techniques and ideas developed in [1] and employs the notation of isotopic components. This part of our work is contained in §3 and the proof of Theorem 4.1 may be found in §4. Finally, §5 contains the proof of Theorem 1.2.

2. Notation and Tools

Our terminology follows [2]. All spaces are assumed to be metric. All maps in this paper are continuous. A *curve* is a one-dimensional continuum.

By $\alpha(X)$ we denote the set of Euclidean points of X, i.e., the points having a neighborhood homeomorphic to Euclidean space E^n. By $\beta(X)$ we denote the set of semi-Euclidean points of $X \setminus \alpha(X)$, namely the points $(x_1, \ldots, x_n) \in E^n$ with $x_n \geq 0$ and $\gamma(X) = X \setminus (\alpha(X) \cup \beta(X))$. The components of $\alpha(X)$ are called *Euclidean components* of X.

A space M is a *manifold* if M is compact and connected space such that $\gamma(M) = \emptyset$.

Remark 2.1. If C is a locally connected curve, then

$$\alpha(C \times [0,1)) = \alpha(C) \times (0,1).$$

A point $p \in X$ is *approximately Euclidean* if, for every $\epsilon > 0$, there exists a map $f : X \times \mathbb{I} \to X$ such that

(1) $f(x,0) = x$,
(2) $\mathrm{dist}(f(x,t),x) < \epsilon$ for every $(x,t) \in X \times \mathbb{I}$,
(3) $p \in \alpha(f(X \times \{1\}))$, and
(4) the dimension of $f(X \times \{1\})$ in point p is equal to the dimension of X in p.

Let κ be a cardinal number. A point $x \in X$ is *of order less than or equal to* κ provided that x has a basis of open neighborhoods whose boundaries have at most κ elements. The smallest cardinal number κ with the above property is called the *order of point* x in X.

Since the property of a point being approximately Euclidean is a local one (see [1, p. 145]), by [1, theorems 7 and 9], we obtain following proposition.

Proposition 2.2. *Let C be a locally connected curve. A point $(x,t) \in C \times [0,1)$ is approximately Euclidean in $C \times [0,1)$ if and only if x is of order 2, C is locally contractible at x, and $t \in (0,1)$.*

For every pair of points x and y in a curve C, let us denote by $\nu_C(x,y)$ the number (finite or not) of Euclidean components A in C such that the boundary of A contains only the points x and y.

In the proof of Theorem 4.1, we will use the following result.

Lemma 2.3 ([1, p. 155])**.** *Let C and C' be two locally connected curves and $h : \beta(C) \cup \gamma(C) \to \beta(C') \cup \gamma(C')$ be a homeomorphism. Then h can be extended to a homeomorphism between C and C' if and only if $\nu_C(x,y) = \nu_{C'}(h(x),h(y))$ for every pair of points $x, y \in \beta(C) \cup \gamma(C)$.*

3. Isotopic Components

A continuous mapping $h\colon X \times \mathbb{I} \to X$ is an *isotopic deformation in* X if $h(x,0) = x$ for every $x \in X$, and a map $h_t(x) = h(x,t)$ is a homeomorphic embedding in X for every $t \in \mathbb{I}$. If, for every $t \in [0,1]$, a map $h_t(x) = h(x,t)$ is a homeomorphism on X, then $h\colon X \times \mathbb{I} \to X$ is an *isotopic deformation on* X. Two points x_1 and x_2 are *isotopic* in X (in symbols $x_1 \sim x_2$) if there exists an isotopic deformation $h\colon X \times \mathbb{I} \to X$ on X such that $h(x_1, 1) = x_2$.

A point $p \in X$ is *isotopically labile* if, for every $\epsilon > 0$, there exists an isotopic deformation $h(x,t)$ in X satisfying the following conditions:

(1) $\operatorname{dist}(h(x,t), x) < \epsilon$, for every $(x,t) \in X \times \mathbb{I}$ and
(2) $h(x,1) \neq p$, for every $x \in X$.

The points which are not isotopically labile are said to be *isotopically stable.*

Remark 3.1 (see [1, p. 149])**.** If C is a locally connected curve, then the isotopically labile points are the same as semi-Euclidean points.

The following result is analogous to [1, Lemma 11].

Lemma 3.2. *Let C be a locally connected curve. The set of isotopically labile points in $C \times [0,1)$ is the same as the set $C \times \{0\} \cup \beta(C) \times [0,1)$.*

Proof. Obviously, every point in $C \times \{0\} \cup \beta(C) \times [0,1)$ is isotopically labile.

To prove the inverse implication, let $p \in C \times [0,1)$ be an isotopically labile point in $C \times [0,1)$. First, we will prove that p is also an isotopically labile point in $C \times [0,1]$. Fix $\epsilon > 0$. There exists an isotopic deformation $h : C \times [0,1) \times \mathbb{I} \to C \times [0,1)$ such that $\mathrm{dist}(h(x,t),x) < \epsilon/2$ for every $(x,t) \in C \times [0,1) \times \mathbb{I}$, and $h(x,1) \neq p$ for every $x \in C \times [0,1)$. Let $g : C \times [0,1] \times \mathbb{I} \to C \times [0,1]$ be an isotopic deformation into $C \times [0,1]$ such that $g(C \times [0,1] \times \{1\}) \subseteq C \times [0,1)$ and $\mathrm{dist}(g(x,t),x) < \epsilon/2$ for $x \in C \times [0,1]$ and $t \in \mathbb{I}$. One can see that $h_g(x,t) : C \times [0,1] \times \mathbb{I} \to C \times [0,1]$, defined by the formula

$$h_g(x,t) = \begin{cases} g(x,2t) & \text{for } t \in [0,1/2] \\ h(g(x,1),2t-1) & \text{for } t \in (1/2,1] \end{cases}$$

in an isotopic deformation, $\mathrm{dist}(h_g(x,t),x) < \epsilon$ for $(x,t) \in C \times [0,1] \times \mathbb{I}$, and $h_g(x,1) \neq p$ for $x \in C \times [0,1]$. Hence, p is an isotopically labile point in $C \times [0,1]$. By [1, Lemma 11], $p \in C \times \{0\} \cup \beta(C) \times [0,1)$. □

The following result will be used in the proof of Lemma 3.9. Its statement and proof are analogous to [1, Lemma 13].

Lemma 3.3. *Let C be a locally connected curve. Two points $(x_0,y_0) \in \gamma(C) \times (0,1)$ and $(x_1,y_1) \in C \times [0,1)$ are isotopic in $C \times [0,1)$ if and only if $x_0 = x_1$ and $y_1 \in (0,1)$.*

Proof. Obviously, if $x_0 = x_1$ and $y_1 \in (0,1)$, then (x_0,y_0) and (x_1,y_1) are isotopic.

Assume now that $(x_0,y_0) \in \gamma(C) \times (0,1)$ and $(x_1,y_1) \in C \times [0,1)$ are isotopic in $C \times [0,1)$. Hence, there exists an isotopic deformation $\phi : C \times [0,1) \times \mathbb{I} \to C \times [0,1)$ such that

$$\phi(x,y,t) = (\phi_C(x,y,t), \phi_{[0,1)}(x,y,t)),$$
$$\phi(x_0,y_0,1) = (x_1,y_1).$$

Since (x_0,y_0) is not Euclidean and, by Lemma 3.2, is isotopically stable, a point $\phi(x_0,y_0,t) = (\phi_C(x,y,t), \phi_{[0,1)}(x,y,t))$ is also isotopically stable and is not Euclidean for every $t \in [0,1]$. By Lemma 3.2 and Remark 2.1, $\phi_C(x_0,y_0,t) \in \gamma(C)$ and $\phi_{[0,1)}(x,y,t) \in (0,1)$ for every $t \in [0,1]$.

Assume that $x_0 \neq x_1$. Hence, $x_0 \neq \phi_C(x_0,y_0,1) = x_1$.

Let us observe that $\phi_C(x,y_0,t)$ is a homotopic deformation of C and $\phi_C(x_0,y_0,1) \neq x_0$. Since, in a locally connected curve, the homotopically fixed points are the same as the points in which the curve is not locally dendrite, x_0 has a neighborhood U in C being a dendrite. The set of points of order ≥ 3 of a dendrite is always finite or countable. Since $\phi_C(x_0,y_0,t) \in \gamma(C)$ for every $t \in [0,1]$, there exist t_1 and $t_2 \in [0,1]$ such that $\phi_C(x_0, y_0, t_1) \in U$ is of order 2 and $\phi_C(x_0, y_0, t_2) \in U$ is of order ≥ 3. By Proposition 2.2, the point $\phi_C(x_0, y_0, t_1)$ is approximately Euclidean and the point $\phi_C(x_0, y_0, t_2)$ is not approximately Euclidean in $C \times [0, 1)$. Since $\phi(x_0, y_0, t_1)$ and $\phi(x_0, y_0, t_2)$ are isotopic, we obtain a contradiction. Hence, $x_0 = x_1$. $\square$

We recall that two points x and y are *isotopic* in X if there exists an isotopic deformation $h\colon X \times \mathbb{I} \to X$ on X such that $h(x, 1) = y$. It is easy to observe that the relation $\sim$ is reflexive and transitive. We prove that $\sim$ is also symmetrical. Let $x \sim y$. Hence, there exists an isotopic deformation $h : X \times \mathbb{I} \to X$ such that $h(x, 1) = y$. Let $h^{-1}(x, t_0)$ denote the inverse of the mapping $h(x, t_0)$, for every $t_0 \in \mathbb{I}$. Then $h^{-1}(x, t)$ is an isotopic deformation satisfying $h^{-1}(y, 1) = x$. Thus, the relation $\sim$ is an equivalence relation on X. The equivalence class of the relation $\sim$ is called an *isotopic component* of X and the equivalence class of a point x is called an *isotopic component of* x. We denote it by $K(x)$.

The idea of isotopic components comes from [1], where isotopic components are used in the proof of the decomposition uniqueness for the products of a locally connected curve and a manifold. Our proofs of Lemma 3.9, Lemma 3.11, and Theorem 4.1 are based on K. Borsuk's proofs from [1], where reader can find more properties of isotopic components.

Remark 3.4. Let X be a topological space.

(a) Every isotopic component of X is arcwise connected.
(b) If $x \in X$ and $h\colon X \to Y$ is a homeomorphism then the image of the isotopic component of a point x is an isotopic component of the point $h(x)$.
(c) If x and y belong to the same isotopic component of X then X is locally homeomorphic at x and y.

Remark 3.5. If $(x_1, x_2) \in X_1 \times X_2$, then the isotopic component of the point (x_1, x_2) contains the product of the isotopic component of the point x_1 and the isotopic component of the point x_2. The inverse inclusion does not hold. If $\mathbb{I} = [0, 1]$, then $\{0\}$ is the isotopic component of the point $0 \in \mathbb{I}$, but $\partial \mathbb{I}^2$ is the isotopic component of the point $(0, 0) \in \mathbb{I}^2$.

Remark 3.6. Every Euclidean component of a space X is an isotopic component of X.

Proposition 3.7 ([1, p. 152])**.** *Let C be a locally connected curve. The isotopy components of C containing at least two points are identical with the Euclidean components of C.*

Corollary 3.8. *A locally connected curve C has, as isotopic components,*

- *Euclidean components,*

- *the individual points of* $\beta(C)$*,*
- *the individual points of* $\gamma(C)$*.*

Let C be a locally connected curve, $S^1 \neq C \neq \mathbb{I}$, and $\mathcal{A}$ be a set of all Euclidean components of C. Observe that every Euclidean component $A \subset C$ is homeomorphic to the open interval $(0,1)$ and $\bar{A} \cap \gamma(C)$ contains one or two points. Hence, there are three types of Euclidean components of C. Strictly speaking, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, where $\mathcal{A}_i$ are pairwise disjoint and

(1) $A \in \mathcal{A}_1$ if $\bar{A} \cap \beta(C) = \emptyset$ and $|\bar{A} \cap \gamma(C)| = 1$,
(2) $A \in \mathcal{A}_2$ if $\bar{A} \cap \beta(C) = \emptyset$ and $|\bar{A} \cap \gamma(C)| = 2$,
(3) $A \in \mathcal{A}_3$ if $\bar{A} \cap \beta(C) \neq \emptyset$.

Now, we will use the introduced above sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ to classify the isotopic components of the product $C \times [0,1)$.

Lemma 3.9. *Assume that C is a locally connected curve, $S^1 \neq C \neq \mathbb{I}$, and $\mathcal{B}$ is a set of isotopic components of $C \times [0,1)$. Then $\mathcal{B} = \bigcup_{i=1}^{7} \mathcal{B}_i$, where $\mathcal{B}_i$ are pairwise disjoint and*

1.–3. $B \in \mathcal{B}_i$ *if* $B = A \times (0,1)$*, where* $A \in \mathcal{A}_i$*, for* $i = 1,2,3$*;*
4. $B \in \mathcal{B}_4$*, if* $B = A \times \{0\}$*, where* $A \in \mathcal{A}_1 \cup \mathcal{A}_2$*;*
5. $B \in \mathcal{B}_5$*, if* $B = (\bar{A} \cap \beta(C)) \times [0,1) \cup A \times \{0\}$*, where* $A \in \mathcal{A}_3$*;*
6. $B \in \mathcal{B}_6$*, if* $B = \{x\} \times (0,1)$*, where* $x \in \gamma(C)$*;*
7. $B \in \mathcal{B}_7$*, if* $B = \{x\} \times \{0\}$*, where* $x \in \gamma(C)$*.*

Proof. Obviously, for every set $B \in \bigcup_{i=1}^{7} \mathcal{B}_i$, there exists an isotopic component of $C \times [0,1)$ containing B.

By remarks 2.1 and 3.6, every $B \in \bigcup_{i=1}^{3} \mathcal{B}_i$ is an isotopic component of $C \times [0,1)$.

Observe that $\beta(C \times [0,1)) = \bigcup_{B \in \mathcal{B}_4 \cup \mathcal{B}_5} B$. By Remark 3.4(a)(c), every $B \in \mathcal{B}_4 \cup \mathcal{B}_5$ is an isotopic component of $C \times [0,1)$.

By Lemma 3.3, B is an isotopic component of $C \times [0,1)$ for every $B \in \mathcal{B}_6$. Since every set $B \in \mathcal{B}_7$ is a boundary of some set $B' \in \mathcal{B}_6$, every B from $\mathcal{B}_7$ is also an isotopic component of $C \times [0,1)$. $\square$

Remark 3.10. If $B \in \bigcup_{i=1}^{3} \mathcal{B}_i$, then B is 2-dimensional; if $B \in \bigcup_{i=4}^{6} \mathcal{B}_i$, then B is 1-dimensional; and if $B \in \mathcal{B}_7$, then B is a singleton.

If $B \in \mathcal{B}_i$, we say that B is an isotopic component of type i for $i = 1,2,\ldots,7$. Now, we shall prove that every homeomorphism preserves types of isotopic components.

Lemma 3.11. *Let us assume that C and C' are locally connected curves, $S^1 \neq C$, $C' \neq \mathbb{I}$, $\mathcal{B} = \bigcup_{i=1}^{7} \mathcal{B}_i$, and $\mathcal{B}' = \bigcup_{i=1}^{7} \mathcal{B}'_i$ are sets of isotopic components of $C \times [0,1)$ and $C' \times [0,1)$ defined in Lemma 3.9, respectively. If $h : C \times [0,1) \to C' \times [0,1)$ is a homeomorphism, then h maps each $B \in \mathcal{B}_i$ onto $h(B) \in \mathcal{B}'_i$ for every $i = 1, 2, \dots, 7$.*

Proof. By Remark 3.4(b), if $B \in \mathcal{B}$, then $h(B) \in \mathcal{B}'$. It is enough to prove that if $B \in \mathcal{B}_i$, then $h(B) \in \mathcal{B}'_i$ for $i = 1, \dots, 7$.

Let $B \in \mathcal{B}_1$. Then $\overline{B}$ is homeomorphic to $S^1 \times [0,1)$ and $\overline{B} \cap \overline{(C \times [0,1)) \setminus \overline{B}}$ is homeomorphic to $[0,1)$.

If $B \in \mathcal{B}_2$, then $\overline{B}$ is homeomorphic to $[0,1] \times [0,1)$ and $\overline{B} \cap \overline{(C \times [0,1)) \setminus \overline{B}}$ is homeomorphic to $\{0,1\} \times [0,1)$.

If $B \in \mathcal{B}_3$, then $\overline{B}$ is homeomorphic to $[0,1] \times [0,1)$ and $\overline{B} \cap \overline{(C \times [0,1)) \setminus \overline{B}}$ is homeomorphic to $[0,1)$.

Since only isotopic components from $\bigcup_{i=1}^{3} \mathcal{B}_i$ are 2-dimensional and the above topological properties distinguish each of them, we can conclude that h preserves types i of isotopic components for $i = 1, 2, 3$.

If $B \in \mathcal{B}_4 \cup \mathcal{B}_5$, then B lies on the boundary of exactly one isotopic component C_B of $C \times [0,1)$. Moreover, if $B \in \mathcal{B}_4$, then $C_B \in \mathcal{B}_1 \cup \mathcal{B}_2$ and if $B \in \mathcal{B}_5$, then $C_B \in \mathcal{B}_3$. Since h preserves types i of isotopic components for $i = 1, 2, 3$, it also preserves types 4 and 5.

Since B is 0-dimensional only for $B \in \mathcal{B}_7$, we can conclude that h preserves type 7 of isotopic components.

Let us observe that, for $i = 1, 2, 3, 4, 5, 7$, the type i of isotopic components is preserved by the homeomorphism h. Thus, type 6 is preserved by h as well. □

4. Decomposition Uniqueness for the Products of a Locally Connected Curve and the Interval $[0,1)$

Theorem 4.1. *If C and C' are locally connected curves and $C \times [0,1)$ is homeomorphic to $C' \times [0,1)$, then C is homeomorphic to C'.*

Proof. If $\gamma(C \times [0,1)) = \emptyset$, then C is an arc or a simply closed curve, and the assertion of the theorem holds.

We assume now that $\gamma(C \times [0,1)) \neq \emptyset$. Then $\mathbb{I} \neq C \neq S^1$. Since there are only two 1-dimensional compact manifolds (S^1 and $\mathbb{I}$), C is not a manifold. Hence, $\gamma(C) \neq \emptyset$. Let $h : C \times [0,1) \to C' \times [0,1)$ be a homeomorphism. We will construct a homeomorphism $h_0 : C \times \{0\} \to C' \times \{0\}$. First, we will construct a homeomorphism between $(\gamma(C) \cup \beta(C)) \times \{0\}$ and $(\gamma(C') \cup \beta(C')) \times \{0\}$. By Lemma 3.9, for every

$x \in \gamma(C) \times \{0\}$, the set $\{x\}$ is an isotopic component of $C \times [0,1)$. Hence, by Lemma 3.11, the homeomorphism h maps $\gamma(C) \times \{0\}$ onto $\gamma(C') \times \{0\}$. Define

$$h_0(x) = h(x) \quad \text{for } x \in \gamma(C) \times \{0\}.$$

We observe now that for every $x \in \beta(C) \times \{0\}$ there exists an isotopic component $K(x) \in \mathcal{B}_5$ such that $x \in K(x)$, where $\mathcal{B}_5$ is the set of isotopic components in $C \times [0,1)$ of type 5 defined in Lemma 3.9. By Lemma 3.11, $h(K(x)) \in \mathcal{B}'_5$ and $h(K(x)) = (\bar{A} \cap \beta(C')) \times [0,1) \cup A \times \{0\}$ for some $A \in \mathcal{A}'_3$. We observe that $\beta(C') \times \{0\} \cap h(K(x)) = (\bar{A} \cap \beta(C')) \times \{0\}$. Since $C' \neq \mathbb{I}$, $|\bar{A} \cap \beta(C')| = 1$. Hence, there exists exactly one point $x' \in \beta(C') \times \{0\} \cap h(K(x))$. Define

$$h_0(x) = x' \text{ for } x \in \beta(C) \times \{0\}.$$

Let us prove that the map h_0 defined above is continuous. Obviously, $h_{0|\gamma(C)\times\{0\}}$ is continuous. Take a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \beta(C) \times \{0\}$ and assume that $x_n \to x_0 = (x_0^C, x_0^{[0,1)})$. Obviously, $x_0^{[0,1)} = 0$. Since in every neighborhood of x_0^C there are points from $\beta(C)$, the point x_0^C belongs to $\gamma(C)$. For every $n \in \mathbb{N}$, there exists an isotopic component $K(x_n) \in \mathcal{B}_5$ such that

$$x_n \in K(x_n) = (\bar{A}_n \cap \beta(C)) \times [0,1) \cup A_n \times \{0\},$$

where A_n is a Euclidean component of C and $A_n \in \mathcal{A}_3$. Since C is a locally connected curve, the diameters of Euclidean components of C tend to zero. Hence, $\operatorname{diam}(A_n) \to 0$ and $K(x_n)$ tends to $\{x_0^C\} \times [0,1)$. The sequence of sets $h(K(x_n))$ tends to $h(\{x_0^C\} \times [0,1))$, and $h_0(x_n) \to h(x_0) = h_0(x_0)$.

Since $\gamma(C) \cup \beta(C)$ is compact, h_0 defined in this way is a homeomorphism between $(\gamma(C) \cup \beta(C)) \times \{0\}$ and $(\gamma(C') \cup \beta(C')) \times \{0\}$.

Recall that $\nu_C(x,y)$ denotes the number of Euclidean components A in C such that the boundary of A contains only the points x and y and $K(x)$ denotes the isotopic component of the point x. Now, observe that, for $x, y \in (\gamma(C) \cup \beta(C)) \times \{0\}$, the number $\nu_{C\times\{0\}}(x,y)$ is equal to the number of Euclidean components of $C \times [0,1)$ for which the boundary contains both sets $K(x)$ and $K(y)$ and does not contain any other sets $K(z)$ for $z \in (\gamma(C) \cup \beta(C)) \times \{0\}$. The same equality holds for $\nu_{C'\times\{0\}}(x',y')$, where $x', y' \in (\gamma(C') \cup \beta(C')) \times \{0\}$. Again by Lemma 3.11, we obtain $\nu_{C\times\{0\}}(x,y) = \nu_{C'\times\{0\}}(h_0(x), h_0(y))$. Now, using Lemma 2.3, we conclude the proof. □

5. Proof of the Main Theorem

Obviously, if X and Y are homeomorphic, then the cones over X and Y are also homeomorphic. In the proof of the inverse implication we use the following lemma.

Lemma 5.1. *If C and C' are locally connected curves not being an ANR and $h\colon \mathrm{Cone}(C) \to \mathrm{Cone}(C')$ is a homeomorphism, then h maps the vertex of $\mathrm{Cone}(C)$ onto the vertex of $\mathrm{Cone}(C')$.*

Proof. Assume that C is a locally connected curve. There are three kinds of points in $\mathrm{Cone}(C)$:

(1) the points $(c,t) \in C \times [0,1)$ such that c has a neighborhood in C being a dendrite,
(2) points $(c,t) \in C \times [0,1)$ such that every neighborhood of c in C contains a simply closed curve,
(3) the vertex of $\mathrm{Cone}(C)$.

Let $x \in \mathrm{Cone}(C)$. Obviously, if x satisfies (1), then $\mathrm{Cone}(C)$ is locally contractible at x and at all points sufficiently closed to x.

If x satisfies (2), then $\mathrm{Cone}(C)$ is not locally contractible at x.

If x is a vertex of $\mathrm{Cone}(C)$, then $\mathrm{Cone}(C)$ is locally contractible at x, but, since C is not an ANR, in every neighborhood of x, there is a point y such that $\mathrm{Cone}(C)$ is not locally contractible at y.

Since the vertex of $\mathrm{Cone}(C)$ is the unique point in $\mathrm{Cone}(C)$ having the last property, every homeomorphism of $\mathrm{Cone}(C)$ onto $\mathrm{Cone}(C')$ maps the vertex of $\mathrm{Cone}(C)$ onto the vertex of $\mathrm{Cone}(C')$. □

Proof of Theorem 1.2. Assume that $h : \mathrm{Cone}(X) \to \mathrm{Cone}(Y)$ is a homeomorphism. By Lemma 5.1, $h(x_0) = y_0$, where x_0 and y_0 are vertices of $\mathrm{Cone}(X)$ and $\mathrm{Cone}(Y)$, respectively. Thus, $h(X \times [0,1)) = Y \times [0,1)$. Using Theorem 4.1, we conclude that X is homeomorphic to Y. □

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