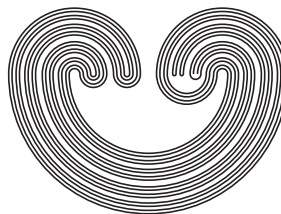


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DYNAMICS OF NON-AUTONOMOUS DISCRETE DYNAMICAL SYSTEMS

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DYNAMICS OF NON-AUTONOMOUS DISCRETE DYNAMICAL SYSTEMS

PUNEET SHARMA AND MANISH RAGHAV

ABSTRACT. We study the dynamics of a general non-autonomous dynamical system generated by a family of continuous self-maps on a compact space X . We derive necessary and sufficient conditions for the system to exhibit complex dynamical behavior. In the process we discuss properties like transitivity, weakly mixing, topologically mixing, minimality, sensitivity, topological entropy, and Li-Yorke chaoticity for the non-autonomous system. We also give examples to prove that the dynamical behavior of the non-autonomous system in general cannot be characterized in terms of the dynamical behavior of its generating functions.

1. INTRODUCTION

Let (X, d) be a compact metric space and let $\mathbb{F} = \{f_n : n \in \mathbb{N}\}$ be a family of continuous self-maps on X . Any such family \mathbb{F} generates a non-autonomous dynamical system via the relation $x_n = f_n(x_{n-1})$; such a dynamical system will be denoted by (X, \mathbb{F}) . For any $x \in X$, $\{f_n \circ f_{n-1} \circ \dots \circ f_1(x) : n \in \mathbb{N}\}$ defines the orbit of x . The objective of study of a non-autonomous dynamical system is to investigate the orbit of an arbitrary point x in X . For notational convenience, let $\omega_n(x) = f_n \circ f_{n-1} \circ \dots \circ f_1(x)$ be the state of the system after n iterations. If $y = \omega_n(x) = f_n \circ f_{n-1} \circ \dots \circ f_1(x)$, then $x \in f_1^{-1} \circ f_2^{-1} \circ \dots \circ f_n^{-1}(y) = \omega_n^{-1}(y)$ and, hence, ω_n^{-1} traces the point n units back in time.

A point x is called *periodic* for \mathbb{F} if there exists $n \in \mathbb{N}$ such that $\omega_{nk}(x) = x$ for all $k \in \mathbb{N}$. The least such n is known as the period

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of the point x . The system (X, \mathbb{F}) is *transitive* (or \mathbb{F} is transitive) if for each pair of open sets U and V in X , there exists $n \in \mathbb{N}$ such that $\omega_n(U) \cap V \neq \phi$. The system (X, \mathbb{F}) is said to be *minimal* if it does not contain any proper non-trivial subsystems. The system (X, \mathbb{F}) is said to be *weakly mixing* if for any two pairs U_1, U_2 and V_1, V_2 of non-empty open subsets of X , there exists a natural number n such that $\omega_n(U_i) \cap V_i \neq \phi$ for $i = 1, 2$. Equivalently, we say that the system is weakly mixing if $\mathbb{F} \times \mathbb{F}$ is transitive. The system is said to be *topologically mixing* if for every pair of non-empty open sets U and V , there exists a natural number K such that $\omega_n(U) \cap V \neq \phi$ for all $n \geq K$. The system is said to be *sensitive* if there exists a $\delta > 0$ such that for each $x \in X$ and each neighborhood U of x , there exists $n \in \mathbb{N}$ such that $\text{diam}(\omega_n(U)) > \delta$. If there exists $K > 0$ such that $\text{diam}(\omega_n(U)) > \delta$, for all $n \geq K$, then the system is *cofinitely sensitive*. A set S is said to be *scrambled* if for any $x, y \in S$, $\limsup_{n \rightarrow \infty} d(\omega_n(x), \omega_n(y)) > 0$, but $\liminf_{n \rightarrow \infty} d(\omega_n(x), \omega_n(y)) = 0$. A system (X, \mathbb{F}) is said to be *Li-Yorke chaotic* if it contains an uncountable scrambled set. In case the f_n 's coincide, the above definitions coincide with the known notions of an autonomous dynamical system. See [4], [5], and [6] for details.

We now define the notion of *topological entropy* for a non-autonomous system (X, \mathbb{F}) .

Let X be a compact space and let \mathcal{U} be an open cover of X . Then \mathcal{U} has a finite subcover. Let \mathcal{L} be the collection of all finite subcovers and let \mathcal{U}^* be the subcover with minimum cardinality, say $N_{\mathcal{U}}$. Define $H(\mathcal{U}) = \log N_{\mathcal{U}}$. Then $H(\mathcal{U})$ is defined as the *entropy* associated with the open cover \mathcal{U} . If \mathcal{U} and \mathcal{V} are two open covers of X , define $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. An open cover β is said to be refinement of an open cover α ; i.e., $\alpha \prec \beta$, if every open set in β is contained in some open set in α . It can be seen that if $\alpha \prec \beta$, then $H(\alpha) \leq H(\beta)$. For a self-map f on X , $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$ is also an open cover of X . Define

$$h_{\mathbb{F}, \mathcal{U}} = \limsup_{k \rightarrow \infty} \frac{H(\mathcal{U} \vee \omega_1^{-1}(\mathcal{U}) \vee \omega_2^{-1}(\mathcal{U}) \vee \dots \vee \omega_{k-1}^{-1}(\mathcal{U}))}{k}.$$

Then $\sup h_{\mathbb{F}, \mathcal{U}}$, where \mathcal{U} runs over all possible open covers of X , is known as the *topological entropy of the system* (X, \mathbb{F}) and is denoted by $h(\mathbb{F})$. In case the maps f_n coincide, the above definition coincides with the known notion of topological entropy. See [4] and [5] for details.

Let (X, d) be a metric space and let $CL(X)$ denote the collection of all non-empty closed subsets of X . For any two closed subsets A and B of X , define

$$d_H(A, B) = \inf\{\epsilon > 0 : A \subseteq S_{\epsilon}(B) \text{ and } B \subseteq S_{\epsilon}(A)\}.$$

It is easily seen that d_H defined above is a metric on $CL(X)$ and is called *Hausdorff metric*. The metric d_H preserves the metric on X ; i.e., $d_H(\{x\}, \{y\}) = d(x, y)$ for all $x, y \in X$. The topology generated by this metric is known as the *Hausdorff metric topology* on $CL(X)$ with respect to the metric d on X (see [3] and [11]). It is known that $\lim_{n \rightarrow \infty} A_n = A$ if and only if A_n converges to A under Hausdorff metric [10].

Many of the systems occurring in nature have been studied using mathematical models. While systems like the logistic model have been used to characterize the population growth, continuous systems like the Lorenz model have been used for weather prediction to great precision. Although various mathematical models exploring such systems have been proposed and long-term behavior of such systems has been studied, most of the mathematical models are autonomous in nature and hence cannot be used to model a general dynamical system. Thus, there is a strong need to study and develop the theory of non-autonomous dynamical systems. The theory of non-autonomous dynamical systems helps to characterize the behavior of various natural phenomena which cannot be modeled by autonomous systems. Some studies in this direction have been made and some results have been obtained. In [8], the authors study the topological entropy of a general non-autonomous dynamical system generated by a family \mathbb{F} . In particular, the authors study the case when the family \mathbb{F} is equicontinuous or uniformly convergent. In [9], the authors discuss minimality conditions for a non-autonomous system on a compact Hausdorff space while focusing on the case when the non-autonomous system is defined on a compact interval of the real line. In [7], the authors prove that if $f_n \rightarrow f$, in general there is no relation between the chaotic behavior of the non-autonomous system generated by f_n and the chaotic behavior of f . In [1], the authors investigate properties like weakly mixing, topological mixing, topological entropy, and Li-Yorke chaos for the non-autonomous system. They prove that the dynamics of a non-autonomous system are very different from those of the autonomous case. They also give a few techniques to study the qualitative behavior of a non-autonomous system.

Although some studies have been made and some useful results have been obtained, a lot of questions in the field are still unanswered and a lot of investigation still needs to be done. In this paper, we study different possible dynamical notions for a non-autonomous dynamical system generated by a family \mathbb{F} . We prove that if $\mathbb{F} = \{f_1, f_2, \dots, f_n\}$ is finite, the non-autonomous system is topological mixing if and only if the autonomous system $(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$ is topological mixing. We also prove that if $(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$ has positive topological

entropy (is Li–Yorke chaotic) then (X, \mathbb{F}) also has positive topological entropy (is Li–Yorke chaotic). We also establish similar results for transitivity/dense periodicity of the non-autonomous system. In addition, if \mathbb{F} is commutative, the non-autonomous system is weakly mixing if and only if $(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$ is weakly mixing. Thus, we prove that if the family \mathbb{F} is finite, under certain assumptions, the study of the non-autonomous dynamical system can be reduced to the autonomous case. We also establish alternate criteria to establish weakly mixing/topological mixing for a general non-autonomous dynamical system. In the end, we study the dynamical behavior of the system with respect to the members of the family \mathbb{F} . We prove that the dynamical behavior of the generating members in general does not carry over to the non-autonomous system generated. While the non-autonomous system can exhibit a certain dynamical notion without any of the generating members exhibiting the same, in some instances, the system might not exhibit certain dynamical behavior even when all the generating members exhibit the same.

2. MAIN RESULTS

Throughout the paper, let (X, d) be a compact metric space and let $\mathbb{F} = \{f_n : n \in \mathbb{N}\}$ be a family of surjective continuous self-maps on X .

We first give some results establishing various dynamical properties of the non-autonomous system when the family $\mathbb{F} = \{f_1, f_2, \dots, f_n\}$ is finite. It is worth mentioning that when the family $\mathbb{F} = \{f_1, f_2, \dots, f_n\}$ is finite, the non-autonomous dynamical system is generated by the relation $x_k = f_k(x_{k-1})$ where $f_k = f_{(1+(k-1) \bmod n)}$.

Lemma 2.1. *$(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$ has a dense set of periodic points $\Rightarrow (X, \mathbb{F})$ has a dense set of periodic points.*

Proof. Let U be any non-empty open subset of X . As $f_n \circ f_{n-1} \circ \dots \circ f_1$ has a dense set of periodic points, there exists $k \in \mathbb{N}$ and $x \in U$ such that $(f_n \circ f_{n-1} \circ \dots \circ f_1)^k(x) = x$. Thus, $\omega_{nk}(x) = x$. Consequently, $\omega_{rnk}(x) = x$ for all $r \geq 1$ and x is also periodic for (X, \mathbb{F}) . Hence, (X, \mathbb{F}) has a dense set of periodic points. \square

Lemma 2.2. *If $(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$ is transitive, then (X, \mathbb{F}) is transitive.*

Proof. Let U and V be any pair of non-empty open subsets of X . Since $f_n \circ f_{n-1} \circ \dots \circ f_1$ is transitive, there exists $k \in \mathbb{N}$ such that $(f_n \circ f_{n-1} \circ \dots \circ f_1)^k(U) \cap V \neq \emptyset$. Consequently, $\omega_{nk}(U) \cap V \neq \emptyset$ and, hence, (X, \mathbb{F}) is transitive. \square

The above result establishes the transitivity of the non-autonomous system when the corresponding autonomous system is transitive. However, the correspondence is one-sided and the converse of the above result is not true. We give an example in support of our statement.

Example 2.3. Let I be the unit interval and let f_1 and f_2 be defined as

$$f_1(x) = \begin{cases} 2x + \frac{1}{2} & \text{for } x \in [0, \frac{1}{4}] \\ -2x + \frac{3}{2} & \text{for } x \in [\frac{1}{4}, \frac{3}{4}] \\ 2x - \frac{3}{2} & \text{for } x \in [\frac{3}{4}, 1] \end{cases}$$

$$f_2(x) = \begin{cases} x + \frac{1}{2} & \text{for } x \in [0, \frac{1}{2}] \\ -4x + 3 & \text{for } x \in [\frac{1}{2}, \frac{3}{4}] \\ 2x - \frac{3}{2} & \text{for } x \in [\frac{3}{4}, 1]. \end{cases}$$

Let $\mathbb{F} = \{f_1, f_2\}$ and (X, \mathbb{F}) be the corresponding non-autonomous dynamical system. As $(X, f_2 \circ f_1)$ has an invariant set $U = [\frac{1}{2}, 1]$, $f_2 \circ f_1$ is not transitive. However, as f_1 expands every open set U in $[0, 1]$ and f_2 expands the right half of the unit interval with $f_2([0, \frac{1}{2}]) = [\frac{1}{2}, 1]$, the non-autonomous system generated by \mathbb{F} is transitive.

Lemma 2.4. *If \mathbb{F} is a commutative family, then $\mathbb{F} \times \mathbb{F}$ is transitive if and only if $\underbrace{\mathbb{F} \times \mathbb{F} \times \dots \times \mathbb{F}}_{n \text{ times}}$ is transitive for all $n \geq 2$.*

Proof. Let $\mathbb{F} \times \mathbb{F}$ be transitive. We prove the forward part with the help of mathematical induction. Let $\underbrace{\mathbb{F} \times \mathbb{F} \times \dots \times \mathbb{F}}_{k \text{ times}}$ be transitive and

let U_1, U_2, \dots, U_{k+1} and V_1, V_2, \dots, V_{k+1} be a pair of $k+1$ non-empty open sets in X . As $\mathbb{F} \times \mathbb{F}$ is transitive, there exists $r > 0$ such that $\omega_r(U_k) \cap U_{k+1} \neq \emptyset$ and $\omega_r(V_k) \cap V_{k+1} \neq \emptyset$. Let $U = U_k \cap \omega_r^{-1}(U_{k+1})$ and $V = V_k \cap \omega_r^{-1}(V_{k+1})$. Then U and V are non-empty open sets in X . Also as $\underbrace{\mathbb{F} \times \mathbb{F} \times \dots \times \mathbb{F}}_{k \text{ times}}$ is transitive, there exists $t > 0$ such that $\omega_t(U_i) \cap V_i \neq \emptyset$ for $i = 1, 2, \dots, k-1$ and $\omega_t(U) \cap V \neq \emptyset$.

As $U \subset U_k$ and $V \subset V_k$, we have $\omega_t(U_k) \cap V_k \neq \emptyset$. Also, $\omega_t(U) \cap V \neq \emptyset$ implies $\omega_r(\omega_t(U)) \cap \omega_r(V) \neq \emptyset$. As f_i commute with each other, we have $\omega_t(\omega_r(U)) \cap \omega_r(V) \neq \emptyset$. As $\omega_r(U) \subseteq U_{k+1}$ and $\omega_r(V) \subset V_{k+1}$, we have $\omega_t(U_{k+1}) \cap V_{k+1} \neq \emptyset$. Consequently, $\omega_t(U_i) \cap V_i \neq \emptyset$ for $i = 1, 2, \dots, k+1$ and, hence, $\underbrace{\mathbb{F} \times \mathbb{F} \times \dots \times \mathbb{F}}_{k+1 \text{ times}}$ is transitive.

Proof of the converse is trivial because if $\underbrace{\mathbb{F} \times \mathbb{F} \times \dots \times \mathbb{F}}_{n \text{ times}}$ is transitive for all $n \geq 2$; in particular, taking $n = 2$ yields that $\mathbb{F} \times \mathbb{F}$ is transitive. \square

Remark 2.5. For autonomous systems, it is known that if $f \times f$ is transitive, then $\underbrace{f \times f \times \dots \times f}_{n \text{ times}}$ is transitive for all $n \geq 2$ [2] and, hence,

the result established above is an analogous extension of the autonomous case. It may be noted that the proof uses the commutative property of the members of the family \mathbb{F} and, hence, is not true for a non-autonomous system generated by any general family \mathbb{F} . However, the proof does not use the finiteness of the family \mathbb{F} and, hence, the result holds even when the generating family \mathbb{F} is infinite.

Lemma 2.6. *If \mathbb{F} is a commutative family, then (X, \mathbb{F}) is weakly mixing if and only if for any finite collection of non-empty open sets $\{U_1, U_2, \dots, U_m\}$, there exists a subsequence (r_n) of positive integers such that $\lim_{n \rightarrow \infty} \omega_{r_n}(U_i) = X$ for all $i = 1, 2, \dots, m$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary and let $\{U_1, U_2, \dots, U_m\}$ be any finite collection of non-empty open sets of X . As X is compact, there exist x_1, x_2, \dots, x_{k_n} such that $X = \bigcup_{i=1}^{k_n} S(x_i, \frac{1}{n})$. As (X, \mathbb{F}) is weakly mixing, by Lemma 2.4, there exists $r_n > 0$ such that $\omega_{r_n}(U_i) \cap S(x_j, \frac{1}{n}) \neq \phi$ for all i and j and, hence, for any i , $d_H(\omega_{r_n}(U_i), X) \leq \frac{1}{n}$. As $n \in \mathbb{N}$ is arbitrary, $\lim_{n \rightarrow \infty} \omega_{r_n}(U_i) = X$ for all i , and the proof for the forward part is complete.

Conversely, let U_1, U_2 and V_1, V_2 be two pairs of non-empty open subsets of X . For $i = 1, 2$, let $v_i \in V_i$ and let $\epsilon > 0$ be such that $S(v_i, \epsilon) \subset V_i$. By the given condition, there exists a subsequence (r_n) of natural numbers such that $\lim_{n \rightarrow \infty} \omega_{r_n}(U_i) = X$ for $i = 1, 2$. Thus, there exists r_k such that $d_H(\omega_{r_k}(U_i), X) < \frac{\epsilon}{2}$, $i = 1, 2$. Consequently, $\omega_{r_k}(U_i) \cap V_i \neq \phi$ and, hence, (X, \mathbb{F}) is weakly mixing. \square

Remark 2.7. It may be noted that the proof of the converse does not need commutativity of the family \mathbb{F} . However, to establish the forward part, we use Lemma 2.4 and, hence, use the commutativity of the family \mathbb{F} . Thus, the result may not hold well when considered for a general non-autonomous system. Also, the result does not use a finiteness condition on \mathbb{F} and hence is valid even when the system is generated by an infinite family \mathbb{F} .

Remark 2.8. It is known that an autonomous system is weakly mixing if and only if for any non-empty open set U , there exists a subsequence

(r_n) of positive integers such that $\lim_{n \rightarrow \infty} f^{r_n}(U) = X$ [10]. Thus, for the non-autonomous case, the result above establishes a stronger extension of the result proved in the autonomous case. However, the above result also holds when the maps f_n coincide and, hence, a stronger version of the result in [10] is true for the autonomous case. For the sake of completeness, we mention the obtained result below.

Corollary 2.9. *A continuous self-map f is weakly mixing if and only if for any finite collection of non-empty open sets $\{U_1, U_2, \dots, U_m\}$, there exists a subsequence (r_n) of positive integers such that $\lim_{n \rightarrow \infty} f^{r_n}(U_i) = X$ for all $i = 1, 2, \dots, m$.*

Lemma 2.10. *(X, \mathbb{F}) is topologically mixing if and only if for each non-empty open set U , $\lim_{n \rightarrow \infty} \omega_n(U) = X$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary and let U be any non-empty open subset of X . As X is compact, there exist x_1, x_2, \dots, x_{k_n} such that $X = \bigcup_{i=1}^{k_n} S(x_i, \frac{1}{n})$. As \mathbb{F} is topologically mixing, there exists M_i , $i = 1, 2, \dots, k_n$ such that $\omega_k(U) \cap S(x_i, \frac{1}{n}) \neq \emptyset$ for all $k \geq M_i$. Let $M = \max\{M_i : 1 \leq i \leq k_n\}$. Then $\omega_k(U) \cap S(x_i, \frac{1}{n}) \neq \emptyset$ for all $k \geq M$. Consequently, $d_H(\omega_k(U), X) < \frac{1}{n}$ for all $k \geq M$. As $n \in \mathbb{N}$ is arbitrary, $\lim_{n \rightarrow \infty} \omega_n(U) = X$, and the proof of the forward part is complete.

Conversely, let U and V be any pair of non-empty open subsets of X . Let $v \in V$ and let $\epsilon > 0$ be such that $S(v, \epsilon) \subset V$. By the given condition, $\lim_{n \rightarrow \infty} \omega_n(U) = X$. Thus, there exists $K > 0$ such that $d_H(\omega_k(U), X) < \frac{\epsilon}{2}$ for all $k \geq K$. Consequently, $\omega_k(U) \cap V \neq \emptyset$ for all $n \geq K$ and, hence, (X, \mathbb{F}) is topologically mixing. \square

Remark 2.11. In [10], the authors establish that an autonomous system (X, f) is topologically mixing if and only if for each non-empty open set U , $\lim_{n \rightarrow \infty} f^n(U) = X$. Once again, we prove that an analogous result does hold when considered for a general non-autonomous system. However, it may be noted that neither commutativity nor finiteness of the family \mathbb{F} was needed to establish the above result and, hence, the result holds for a general non-autonomous dynamical system.

Lemma 2.12. *If $\mathbb{F} = \{f_1, f_2, \dots, f_n\}$ is a finite commutative family, then (X, \mathbb{F}) is weakly mixing if and only if $(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$ is weakly mixing.*

Proof. Let U be a non-empty open subset of X . We will equivalently prove that there exists a sequence (z_k) of natural numbers such that $\lim_{k \rightarrow \infty} (f_n \circ$

$f_{n-1} \circ \dots \circ f_1)^{z_k}(U) = X$. As (X, \mathbb{F}) is weakly mixing, by Lemma 2.6, there exists a sequence (s_k) such that $\lim_{k \rightarrow \infty} \omega_{s_k}(U) = X$. Also, the family \mathbb{F} is finite and, hence, there exists $l \in \{1, 2, \dots, n\}$ and a subsequence (m_k) of (s_k) , $m_k = l + r_k n$ such that $\lim_{k \rightarrow \infty} f_l \circ f_{l-1} \circ \dots \circ f_1 \circ \omega_{r_k n}(U) = X$. As each f_i is surjective, $\lim_{k \rightarrow \infty} \omega_{(r_k+1)n}(U) = X$. Consequently, $\lim_{k \rightarrow \infty} (f_n \circ f_{n-1} \circ \dots \circ f_1)^{r_k+1}(U) = X$ and $(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$ is weakly mixing.

Conversely, let U_1, U_2 and V_1, V_2 be any two pairs of non-empty open subsets of X . As $f_n \circ f_{n-1} \circ \dots \circ f_1$ is weakly mixing, there exists $k \in \mathbb{N}$ such that $(f_n \circ f_{n-1} \circ \dots \circ f_1)^k(U_i) \cap V_i \neq \emptyset$ for $i = 1, 2$. Consequently, $\omega_{nk}(U_i) \cap V_i \neq \emptyset$ for $i = 1, 2$ and, hence, (X, \mathbb{F}) is weakly mixing. \square

Remark 2.13. The result establishes the equivalence of the weakly mixing of the non-autonomous system (X, \mathbb{F}) and the autonomous system $(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$. It may be noted that as the proof uses Lemma 2.6 proved earlier, commutativity of the family \mathbb{F} cannot be relaxed. Thus, the result may not hold if the assumptions in the hypothesis are relaxed.

Remark 2.14. It may be noted that the above result uses the surjectivity of the maps f_i . Thus, if the maps are not surjective, the above result may not hold; i.e., the non-autonomous system may exhibit weakly mixing even if the system $(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$ is not weakly mixing. We now give an example in support of our statement.

Example 2.15. Let I be the unit interval and let f_1 and f_2 be defined as

$$f_1(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ -x + \frac{3}{2} & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

$$f_2(x) = \begin{cases} -2x + \frac{1}{2} & \text{for } x \in [0, \frac{1}{4}] \\ 2x - \frac{1}{2} & \text{for } x \in [\frac{1}{4}, \frac{1}{2}] \\ -2x + \frac{3}{2} & \text{for } x \in [\frac{1}{2}, \frac{3}{4}] \\ 2x - \frac{3}{2} & \text{for } x \in [\frac{3}{4}, 1]. \end{cases}$$

Let \mathbb{F} be a finite family of maps f_1 and f_2 defined above. As $[0, \frac{1}{2}]$ is invariant for $f_2 \circ f_1$, the map $f_2 \circ f_1$ does not exhibit any of the mixing properties. However, for any open set U in $[0, 1]$, there exists $k \in \mathbb{N}$ such that $(f_2 \circ f_1)^k(U) = [0, \frac{1}{2}]$. Consequently, $\omega_{2k+1}(U) = [0, 1]$. As the argument holds for any odd integer greater than k , the non-autonomous system is weakly mixing.

Lemma 2.16. *If $\mathbb{F} = \{f_1, f_2, \dots, f_n\}$ is a finite family, then (X, \mathbb{F}) is topologically mixing if and only if $(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$ is topologically mixing.*

Proof. Let U be a non-empty open subset of X . We will equivalently prove that $\lim_{k \rightarrow \infty} (f_n \circ f_{n-1} \circ \dots \circ f_1)^k(U) = X$. As (X, \mathbb{F}) is topologically mixing, by Lemma 2.10, $\lim_{k \rightarrow \infty} \omega_k(U) = X$. In particular, $\lim_{k \rightarrow \infty} \omega_{nk}(U) = X$ or $\lim_{k \rightarrow \infty} (f_n \circ f_{n-1} \circ \dots \circ f_1)^k(U) = X$ and, hence, $(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$ is topologically mixing.

Conversely, let U be a non-empty open subset of X . We will equivalently prove that $\lim_{k \rightarrow \infty} \omega_k(U) = X$. As $f_n \circ f_{n-1} \circ \dots \circ f_1$ is topologically mixing, $\lim_{k \rightarrow \infty} (f_n \circ f_{n-1} \circ \dots \circ f_1)^k(U) = X$. Consequently, $\lim_{k \rightarrow \infty} \omega_{nk}(U) = X$. As each f_i is surjective, by continuity we have for each $l \in \{1, 2, \dots, n\}$, $f_l \circ f_{l-1} \circ \dots \circ f_1(\lim_{k \rightarrow \infty} \omega_{nk}(U)) = \lim_{k \rightarrow \infty} (f_l \circ f_{l-1} \circ \dots \circ f_1 \circ \omega_{nk}(U)) = X$. Consequently, $\lim_{k \rightarrow \infty} \omega_k(U) = X$ and (X, \mathbb{F}) is topologically mixing. \square

Remark 2.17. The result once again is an analogous extension of the autonomous case. The result proves that the identical conclusion can be made for the non-autonomous case without strengthening the hypothesis. It is worth noting that the result does not use commutativity of \mathbb{F} and, hence, asserts the complex nature of a topological mixing in a general dynamical system.

In [8], the authors prove that if $\mathbb{F} = \{f_1, f_2, \dots, f_n\}$ is a finite family, then $h(\mathbb{F}) = \frac{1}{n} h(f_n \circ f_{n-1} \circ \dots \circ f_1)$. However, as the authors of this paper were not aware of this result while addressing the problem, for the sake of completion, we include the proof here.

Lemma 2.18. *If $\mathbb{F} = \{f_1, f_2, \dots, f_n\}$ is a finite family, then $h(\mathbb{F}) \geq \frac{1}{n} h(f_n \circ f_{n-1} \circ \dots \circ f_1)$. Consequently, if the associated autonomous system has a positive topological entropy, the non-autonomous system also has a positive topological entropy.*

Proof. For any open cover \mathcal{U} of X , the entropy of the system with respect to the open cover \mathcal{U} is defined as

$$h_{\mathbb{F}, \mathcal{U}} = \limsup_{k \rightarrow \infty} \frac{H(\bigvee_{i=0}^{k-1} \omega_i^{-1}(\mathcal{U}))}{k} = \limsup_{k \rightarrow \infty} \frac{H(\bigvee_{i=0}^{nk-1} \omega_i^{-1}(\mathcal{U}))}{nk}$$

where $\bigvee_{i=0}^{k-1} \omega_i^{-1}(\mathcal{U})$ denotes $\mathcal{U} \vee \omega_1^{-1}(\mathcal{U}) \vee \omega_2^{-1}(\mathcal{U}) \vee \dots \vee \omega_{k-1}^{-1}(\mathcal{U})$.

Also, as

$$\bigvee_{i=0}^{k-1} \omega_{in}^{-1}(\mathcal{U}) \prec \bigvee_{i=0}^{nk-1} \omega_i^{-1}(\mathcal{U}),$$

we have

$$H(\mathcal{U} \vee \omega_n^{-1}(\mathcal{U}) \vee \omega_{2n}^{-1}(\mathcal{U}) \vee \dots \vee \omega_{n(k-1)}^{-1}(\mathcal{U})) \leq H(\mathcal{U} \vee \omega_1^{-1}(\mathcal{U}) \vee \omega_2^{-1}(\mathcal{U}) \vee \dots \vee \omega_{nk-1}^{-1}(\mathcal{U}))$$

$$\dots \vee \omega_{nk-1}^{-1}(\mathcal{U}).$$

Therefore,

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \frac{H(\mathcal{U} \vee \omega_n^{-1}(\mathcal{U}) \vee \omega_{2n}^{-1}(\mathcal{U}) \vee \dots \vee \omega_{n(k-1)}^{-1}(\mathcal{U}))}{nk} \\ & \leq \limsup_{k \rightarrow \infty} \frac{H(\mathcal{U} \vee \omega_1^{-1}(\mathcal{U}) \vee \omega_2^{-1}(\mathcal{U}) \vee \dots \vee \omega_{nk-1}^{-1}(\mathcal{U}))}{nk}. \end{aligned}$$

Consequently,

$$\frac{1}{n} \limsup_{k \rightarrow \infty} \frac{H(\bigvee_{i=0}^{k-1} (f_n \circ f_{n-1} \circ \dots \circ f_1)^{-i}(\mathcal{U}))}{k} \leq \limsup_{k \rightarrow \infty} \frac{H(\bigvee_{i=0}^{nk-1} \omega_i^{-1}(\mathcal{U}))}{nk},$$

or $\frac{1}{n} H(f_n \circ f_{n-1} \circ \dots \circ f_1, \mathcal{U}) \leq H(\mathbb{F}, \mathcal{U})$. As \mathcal{U} was arbitrary, $h(\mathbb{F}) \geq \frac{1}{n} h(f_n \circ f_{n-1} \circ \dots \circ f_1)$, and the proof is complete. \square

Lemma 2.19. $(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$ is Li-Yorke chaotic $\Rightarrow (X, \mathbb{F})$ is Li-Yorke chaotic.

Proof. Let $(X, f_n \circ f_{n-1} \circ \dots \circ f_1)$ be Li-Yorke chaotic and let S be an uncountable scrambled set for $g = f_n \circ f_{n-1} \circ \dots \circ f_1$. Consequently, for any $x, y \in S$, there exists a sequence (r_k) and (s_k) of natural numbers such that $\lim_{k \rightarrow \infty} d(g^{r_k}(x), g^{r_k}(y)) > 0$ and $\lim_{k \rightarrow \infty} d(g^{s_k}(x), g^{s_k}(y)) = 0$. Consequently, $\lim_{k \rightarrow \infty} d(\omega_{r_k n}(x), \omega_{r_k n}(y)) > 0$ and $\lim_{k \rightarrow \infty} d(\omega_{s_k n}(x), \omega_{s_k n}(y)) = 0$ and, hence, (X, \mathbb{F}) is Li-Yorke chaotic. \square

In general, investigating the dynamical behavior of a non-autonomous system is difficult. However, if the generating functions f_i are surjective, the above results show that under certain conditions, some of the dynamical properties of the non-autonomous system can be studied using its generating functions. Further, if the generating functions are finite, under certain conditions, some of the dynamical properties of the non-autonomous systems can be studied (and in many cases characterized) using autonomous systems.

We now study the dynamics of the non-autonomous system in terms of its components f_i . We prove that even if the individual maps f_k exhibit certain dynamical behavior, the system (X, \mathbb{F}) may not exhibit similar dynamical behavior.

Example 2.20. Let $\Sigma = \{0, 1\}^{\mathbb{N}}$ be the collection of two-sided sequences of 0 and 1 endowed with the product topology. Let $\sigma_1, \sigma_2 : \Sigma \rightarrow \Sigma$ be defined as $\sigma_1(\dots x_{-2} x_{-1} x_0 x_1 x_2 \dots) = (\dots x_{-2} x_{-1} x_0 x_1 x_2 \dots)$ and $\sigma_2(\dots x_{-2} x_{-1} x_0 x_1 x_2 \dots) = (\dots x_{-2} x_{-1} x_0 x_1 x_2 \dots)$. Then σ_1 and σ_2 are the shift operators and are continuous with respect to the product

topology. Let $\mathbb{F} = \{\sigma_1, \sigma_2\}$ and let (X, \mathbb{F}) be the corresponding non-autonomous system. It can be seen that each σ_i is transitive. However, as $\sigma_1 \circ \sigma_2 = id$, the system generated is not transitive.

Remark 2.21. The above example proves that a non-autonomous dynamical system may not be transitive even if each of its generating systems exhibits the same. It can also be seen that each of the functions is Li-Yorke chaotic. However, as $\sigma_2 \circ \sigma_1 = id$, the system (X, \mathbb{F}) fails to be Li-Yorke chaotic. Thus, the example also shows that the system generated may not exhibit Li-Yorke chaoticity even if each of the generating functions is Li-Yorke chaotic.

Example 2.22. Let I be the unit interval and let $f_1, f_2 : I \rightarrow I$ be defined as

$$f_1(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ \frac{3}{2} - x & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

$$f_2(x) = \begin{cases} \frac{1}{2} - x & \text{if } x \in [0, \frac{1}{2}] \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}.$$

Let $\mathbb{F} = \{f_1, f_2\}$ and let (X, \mathbb{F}) be the corresponding non-autonomous system. As $[\frac{1}{2}, 1]$ and $[0, \frac{1}{2}]$ are invariant for f_1 and f_2 , respectively, none of the f_i are transitive. However, the map

$$f_2 \circ f_1(x) = \begin{cases} \frac{1}{2} - 2x & \text{if } x \in [0, \frac{1}{4}] \\ 4x - 1 & \text{if } x \in [\frac{1}{4}, \frac{1}{2}] \\ 2 - 2x & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

is transitive, and, hence, the non-autonomous system (X, \mathbb{F}) is transitive.

Remark 2.23. Example 2.20 shows that even if each of the maps f_i is transitive, the non-autonomous system generated by $\mathbb{F} = \{f_n : n \in \mathbb{N}\}$ may not be transitive. On the other hand, the above example shows that the non-autonomous system can exhibit transitivity without any of the maps f_i being transitive. Thus, transitivity in general cannot be characterized in terms of transitivity of its generating components f_i .

Example 2.24. Let S^1 be the unit circle and let $\theta \in (0, 1)$ be a rational number. Let $f_n : S^1 \rightarrow S^1$ be defined as $f_n(\phi) = \phi + 2\pi \frac{\theta}{3^n}$ and let $\mathbb{F} = \{f_n : n \in \mathbb{N}\}$ generate the corresponding non-autonomous system. As θ is rational, each map f_n has a dense set of periodic points. However, as $\sum_{n=1}^{\infty} \frac{\theta}{3^n} < 1$, for any $\beta \in S^1$, $\omega_n(\beta) \neq \beta$ for all n . Hence, the non-autonomous system generated by $\mathbb{F} = \{f_n : n \in \mathbb{N}\}$ fails to have any periodic point.

Example 2.25. Let S^1 be the unit circle and let $\theta \in (0, 1)$ be an irrational. Let $f_1, f_2 : S^1 \rightarrow S^1$ be defined as $f_1(\phi) = \phi + 2\pi\theta$ and $f_2(\phi) = \phi - 2\pi\theta$, respectively, and let (X, \mathbb{F}) be the corresponding non-autonomous dynamical system. As each f_i is an irrational rotation, no point is periodic for any f_i . However, as $f_1 \circ f_2 = Id$, the system (S^1, \mathbb{F}) has a dense set of periodic points.

Remark 2.26. Examples 2.24 and 2.25 prove that dense periodicity for a non-autonomous dynamical system cannot be characterized in terms of dense periodicity of the generating functions. While Example 2.25 shows the system may exhibit dense periodicity without any of the generating functions exhibiting the same, Example 2.24 proves that the system may fail to have a dense set of periodic points even when all its generating functions have the same. Also, it may be noted that as θ is irrational, f_1 and f_2 are also minimal. However, as $f_2 \circ f_1 = id$, the system (X, \mathbb{F}) fails to be minimal. Thus, the example also shows that the system generated by a set of minimal systems may not be minimal.

Example 2.27. Let I be the unit interval and let (q_n) be an enumeration of rationals in I . Let $f_n : I \rightarrow I$ be defined as $f_n(x) = q_n$ for all $x \in I$. Then each f_n is a constant map, but the system (X, \mathbb{F}) generated by $\mathbb{F} = \{f_n : n \in \mathbb{N}\}$ is minimal.

Remark 2.28. Once again, Example 2.25 shows that even if each of the maps f_i is minimal, the non-autonomous system generated by \mathbb{F} need not be minimal. On the other hand, Example 2.27 shows that the non-autonomous system can exhibit minimality without any of the maps f_i being minimal. Thus, minimality, in general, cannot be characterized in terms of minimality of its generating functions.

Example 2.29. Let I be the unit interval and let f_1 and f_2 be defined as

$$f_1(x) = \begin{cases} 2x + \frac{1}{2} & \text{for } x \in [0, \frac{1}{4}] \\ -2x + \frac{3}{2} & \text{for } x \in [\frac{1}{4}, \frac{3}{4}] \\ 2x - \frac{3}{2} & \text{for } x \in [\frac{3}{4}, 1] \end{cases}$$

$$f_2(x) = \begin{cases} 2x & \text{for } x \in [0, \frac{1}{2}] \\ -x + \frac{3}{2} & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

Let $\mathbb{F} = \{f_1, f_2\}$ and let (X, \mathbb{F}) be the corresponding non-autonomous system. It can be seen that none of the maps f_i are weakly mixing. However, for any open set U , there exists a natural number n such that

$\omega_n(U) = [0, 1]$. Hence, the non-autonomous system (X, \mathbb{F}) is weakly mixing.

Remark 2.30. The non-autonomous dynamical system generated above also exhibits topological mixing. Thus, the above example also proves that the non-autonomous system generated can be weakly mixing (topologically mixing) without any of its components f_i exhibiting the same. Also, Example 2.20 shows that the non-autonomous system generated need not exhibit weakly mixing (topological mixing) even if each of the generating functions exhibits weakly mixing (topological mixing). This proves that, in general, weakly mixing (topologically mixing) of a non-autonomous system cannot be characterized in terms of weakly mixing (topologically mixing) of its components.

Example 2.31. Let $I \times S^1$ be the unit cylinder. Let $f_1, f_2 : I \times S^1 \rightarrow I \times S^1$ be defined as $f_1((r, \theta)) = (r, \theta + r)$ and $f_2((r, \theta)) = (r, \theta - r)$, respectively. Let $\mathbb{F} = \{f_1, f_2\}$ and let (X, \mathbb{F}) be the corresponding non-autonomous system. As points at different heights of the cylinder are rotating with different speeds, each of the maps f_i is cofinitely sensitive [12]. However, as $f_2 \circ f_1 = Id$, the system $(I \times S^1, \mathbb{F})$ is not sensitive.

Remark 2.32. Example 2.31 shows that even if each of the maps f_i is sensitive, the non-autonomous system generated need not be sensitive. Also, Example 2.22 proves that the non-autonomous system can exhibit sensitivity without any of the maps f_i being sensitive. Thus, sensitivity of the non-autonomous system also, in general, cannot be characterized in terms of sensitivity of its generating functions.

Example 2.33. Let $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f_1(x) = |x|$ and $f_2(x) = 2x - 1$. Let $\mathbb{F} = \{f_1, f_2\}$ and let (X, \mathbb{F}) be the corresponding non-autonomous system. Then f_1 and f_2 fail to be Li-Yorke chaotic. However, as $f_2(f_1(-\frac{7}{9})) = \frac{5}{9}$, $f_2(f_1(\frac{5}{9})) = \frac{1}{9}$, and $f_2(f_1(\frac{1}{9})) = -\frac{7}{9}$, the map $f_2 \circ f_1(x) : \mathbb{R} \rightarrow \mathbb{R}$ possesses a period 3 point, and hence is Li-Yorke chaotic. Consequently, (X, \mathbb{F}) is Li-Yorke chaotic.

Remark 2.34. The above example shows that the non-autonomous system may be Li-Yorke chaotic without the generating members being Li-Yorke chaotic. Also, Example 2.20 shows that the non-autonomous system may not be Li-Yorke chaotic even when all the generating functions are Li-Yorke chaotic. Thus, Li-Yorke chaoticity of a non-autonomous system cannot be characterized in terms of Li-Yorke chaoticity of its generating functions.

3. CONCLUSION

In this paper, the dynamical behavior of the non-autonomous system generated by a family \mathbb{F} of continuous self-maps on a compact metric space are discussed. Properties like dense periodicity, transitivity, various forms of sensitivity, weakly mixing, topologically mixing, Li–Yorke chaoticity, and topological entropy are studied and investigated. For a commutative finite family, we establish that some of the stronger notions of mixing for the non-autonomous system can be studied using autonomous systems. We also establish that the characterization of properties like weakly mixing also holds analogously in the non-autonomous case, if the generating family is commutative. A similar characterization is proved for topological mixing for a general non-autonomous dynamical system asserting the complex behavior of a non-autonomous topologically mixing system. It is also observed that the dynamics of the non-autonomous system generated by the family \mathbb{F} cannot be characterized in terms of the dynamics of the generating functions. While the non-autonomous system can exhibit a certain dynamical behavior without any of the generating functions exhibiting the same, a non-autonomous system may fail to exhibit a dynamical behavior even if all the generating functions exhibit the same.

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