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PRESERVATION OF A NEIGHBORHOOD BASE OF A SET BY CCC FORCINGS

AKIRA IWASA

Abstract. Let $\langle X, \tau \rangle$ be a topological space and let A be a subset of X. We investigate under what circumstances a neighborhood base of A remains a neighborhood base of A in countable chain condition (ccc) forcing extensions.

We prove that if $\langle X, \tau \rangle$ is a metrizable space and $A \subseteq X$, then the following are equivalent (Corollary 2.9):

(1) Every forcing preserves a neighborhood base of A.
(2) Every ccc forcing preserves a neighborhood base of A.
(3) If B is the boundary of A, then $B \cap A$ is scattered and compact.

1. Introduction

Let $\mathbf{V}$ be a ground model and let $\mathbb{P}$ be a forcing. $\mathbf{V}^{\mathbb{P}}$ denotes the forcing extension of $\mathbf{V}$ by the forcing $\mathbb{P}$. For a topological space $\langle X, \tau \rangle$ in $\mathbf{V}$, we define a topological space $\langle X, \tau^{\mathbb{P}} \rangle$ in $\mathbf{V}^{\mathbb{P}}$ such that

$$\tau^{\mathbb{P}} \text{ is the topology on } X \text{ generated by } \tau \text{ in } \mathbf{V}^{\mathbb{P}}.$$

Observe that, in general, $\tau \subsetneqq \tau^{\mathbb{P}}$ (because the forcing $\mathbb{P}$ introduces new open sets) and that τ serves as a base for $\tau^{\mathbb{P}}$ by definition.

We say that a topological property φ is preserved by forcing if, whenever a space $\langle X, \tau \rangle$ satisfies φ, $\langle X, \tau^{\mathbb{P}} \rangle$ satisfies φ for any forcing $\mathbb{P}$. Topological properties such as Hausdorffness, regularity, and complete regularity are preserved by forcing ([2, Lemma 22]), but normality may not be preserved by forcing ([10, Theorem 1.8]). Renata Grunberg, Lúcia

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R. Junqueira, and Franklin D. Tall in [6] and William G. Fleissner, Tim LaBerge, and Adrienne Stanley in [5] studied under what circumstances Cohen forcing preserves (or destroys) normality. In this note, instead of topological properties, we consider a neighborhood base of a set in a topological space. We are interested in whether countable chain condition (ccc) forcings preserve being a neighborhood base of the set. Let us explain this in detail below.

Definition 1.1. Let $\langle X, \tau \rangle$ be a topological space and let $A \subseteq X$. We denote the set of all neighborhoods of A by

$$\mathcal{N}_\tau(A) := \{H \subseteq X : \exists U \in \tau(A \subseteq U \subseteq H)\}.$$

If $A = \{x\}$, then we write $\mathcal{N}_\tau(x)$ for $\mathcal{N}_\tau(\{x\})$.

Definition 1.2. Let $\langle X, \tau \rangle$ be a topological space and let $A \subseteq X$. We say that a family $\mathcal{B}$ of subsets of X is a neighborhood base of A if for each $B \in \mathcal{B}$, the interior of B contains A, and for every open set U containing A, there is $B \in \mathcal{B}$ such that $B \subseteq U$.

Definition 1.3. Let $\langle X, \tau \rangle$ be a topological space and let $A \subseteq X$. We say that a forcing $\mathbb{P}$ preserves a neighborhood base of A if for some neighborhood base $\mathcal{B}$ of A, $\mathcal{B}$ remains a neighborhood base of A in the space $\langle X, \tau^{\mathbb{P}} \rangle$. Or equivalently, we say that a forcing $\mathbb{P}$ preserves a neighborhood base of A if $\mathcal{N}_\tau(A)$ remains a neighborhood base of A in the space $\langle X, \tau^{\mathbb{P}} \rangle$; that is, we say that a forcing $\mathbb{P}$ preserves a neighborhood base of A if

$$(\forall\, W \in \tau^{\mathbb{P}} \text{ with } A \subseteq W)(\exists\, H \in \mathcal{N}_\tau(A))(H \subseteq W).$$

If a forcing $\mathbb{P}$ does not preserve a neighborhood base of A, then we say that $\mathbb{P}$ destroys a neighborhood base of A.

It is not difficult to prove the proposition below.

Proposition 1.4. *Let $\langle X, \tau \rangle$ be a topological space and let $A \subseteq X$. The following are equivalent:*

(1) *A forcing $\mathbb{P}$ preserves a neighborhood base of A.*
(2) *Every neighborhood base of A remains a neighborhood base of A in the space $\langle X, \tau^{\mathbb{P}} \rangle$.*

The purpose of this note is to investigate under what circumstances a neighborhood base of a set remains a neighborhood base of the set in ccc forcing extensions. The main results are Theorem 2.4 and Corollary 2.9. In Theorem 2.4, we establish a necessary condition for a neighborhood base of a set to be preserved by ccc forcings. In Corollary 2.9, we establish a necessary and sufficient condition for a neighborhood base of a subset of a metrizable space to be preserved by ccc forcings.

The proposition below says that if a set A is a singleton, then $\mathcal{N}_\tau(A)$ remains a neighborhood base of A in any forcing extension.

Proposition 1.5. *Let $\langle X, \tau\rangle$ be a topological space and let $x \in X$. Every forcing preserves a neighborhood base of x.*

Proof. Let $\mathbb{P}$ be a forcing. In $\mathbf{V}^{\mathbb{P}}$, take $W \in \tau^{\mathbb{P}}$ such that $x \in W$. By the definition of $\tau^{\mathbb{P}}$, there is $U \in \tau$ such that $x \in U \subseteq W$, and we have $U \in \mathcal{N}_\tau(x)$. Thus, $\mathcal{N}_\tau(x)$ is a neighborhood base of x in the space $\langle X, \tau^{\mathbb{P}}\rangle$. □

We use the following forcings.

Fact 1.6. (1) $\mathbb{C}$ *is the Cohen forcing that adds a Cohen real.* ([9, VII, Definition 5.1]*;* $\mathbb{C} = Fn(\omega, 2)$.)

(2) $\mathbb{B}(\mathcal{F})$ *is the Booth forcing for a free filter* $\mathcal{F} \subseteq [\omega]^\omega$*.* $\mathbb{B}(\mathcal{F})$ *is σ-centered and so it has the ccc. Forcing with* $\mathbb{B}(\mathcal{F})$ *adds an infinite set $E \subseteq \omega$ such that $E \setminus F$ is finite for all $F \in \mathcal{F}$.* ([3]*; see also* [7, Definition 2.5])

(3) $\mathbb{D}$ *is the dominating forcing.* $\mathbb{D}$ *has the ccc, and forcing with* $\mathbb{D}$ *adds a dominating function $g \in {}^\omega\omega$ over* $\mathbf{V}$*; that is, for every $f \in {}^\omega\omega \cap \mathbf{V}$, $f(n) < g(n)$ for all but finitely many $n \in \omega$.* ([1, Definition 3.1.9.])

We finish this section illustrating an example where a neighborhood base of a set is destroyed by a ccc forcing. We will use the idea of this example later to prove the main theorem (Theorem 2.4).

Example 1.7. *There are a topological space $\langle X, \tau\rangle$, a subset A of X, and a ccc forcing $\mathbb{P}$ such that $\mathbb{P}$ destroys a neighborhood base of A.*

Proof. Let

- $X = \omega \times (\omega + 1)$ and
- $A = \omega \times \{\omega\}$.

Let the topology τ on X be the usual product topology, and we consider the order topology on ω and $\omega + 1$. For each $f \in {}^\omega\omega$, define

$$U_f := A \cup \{(i, j) \in \omega \times \omega : i \in \omega \text{ and } j > f(i)\}.$$

Then $\mathcal{N}_\tau(A) = \{H \subseteq X : (\exists f \in {}^\omega\omega)(U_f \subseteq H)\}$. We consider the forcing $\mathbb{D}$ in Fact 1.6(3) and take a dominating function $g \in {}^\omega\omega$. Let $E = \{(n, g(n)) : n \in \omega\}$; then E is a closed subset of X, $E \cap A = \emptyset$, and $E \cap U_f \neq \emptyset$ for all $f \in {}^\omega\omega \cap \mathbf{V}$. Let $W = X \setminus E$; then $W \in \tau^{\mathbb{P}}$, $A \subseteq W$, and $U_f \nsubseteq W$ for all $f \in {}^\omega\omega \cap \mathbf{V}$. This implies that for every $H \in \mathcal{N}_\tau(A)$, we have $H \nsubseteq W$. Thus, $\mathcal{N}_\tau(A)$ is not a neighborhood base of A in the space $\langle X, \tau^{\mathbb{P}}\rangle$. □

2. Preserving a Neighborhood Base of a Set

A subset S of a space X is called *scattered* if for every nonempty subset S' of S, S' contains an isolated point in the relative topology of S'. Note that the empty set is scattered. We use the following notation.

Definition 2.1. For a subset A of a topological space X, let $bd(A)$ denote the boundary of A and define

$$bd^*(A) := bd(A) \cap A.$$

The theorem below gives a sufficient condition for a neighborhood base of a set to be preserved by any forcing.

Theorem 2.2. *Let $\langle X, \tau \rangle$ be a topological space and let $A \subseteq X$. Suppose that*

(1) *$bd^*(A)$ is scattered and*
(2) *$bd^*(A)$ is compact.*

Then every forcing preserves a neighborhood base of A.

Proof. Let $\mathbb{P}$ be a forcing and take $W \in \tau^{\mathbb{P}}$ such that $A \subseteq W$. We will find $H \in \mathcal{N}_\tau(A)$ such that $H \subseteq W$. If $bd^*(A) = \emptyset$, then A is an open set, and $\{A\}$ is a neighborhood base of A. Clearly, $\{A\}$ remains a neighborhood base of A in any forcing extension. So assume that $bd^*(A) \neq \emptyset$. Scattered and compact spaces remain compact in any forcing extension by [8, Lemma 7]. Therefore, in $\mathbf{V}^{\mathbb{P}}$, $bd^*(A)$ is compact. In $\mathbf{V}^{\mathbb{P}}$, using Proposition 1.5, take $U_x \in \tau$ for each $x \in bd^*(A)$ such that $x \in U_x \subseteq W$. In $\mathbf{V}^{\mathbb{P}}$, $\{U_x : x \in bd^*(A)\}$ is an open cover of the compact set $bd^*(A)$, so it has a finite subcover, say $\{U_{x_i} : i < n\}$. Note that the finite subcover $\{U_{x_i} : i < n\}$ of $bd^*(A)$ is in $\mathbf{V}$. Let $H = A \cup \bigcup\{U_{x_i} : i < n\}$; then $H \in \mathcal{N}_\tau(A)$ and $H \subseteq W$. □

We will show in Proposition 3.3 that the converse of Theorem 2.2 is actually true, but if we require forcings to preserve ω_1, then the converse of Theorem 2.2 would not hold (see Example 3.4). We use the following notation.

Definition 2.3. For a subset A of a topological space X, define

$$bd_c(A) := \{x \in \overline{A} : x \in \overline{C} \text{ for some countable set } C \subseteq X \setminus \overline{A}\}$$

and

$$bd_c^*(A) := bd_c(A) \cap A.$$

Note that $bd_c(A) \subseteq bd(A)$. Now we are ready to prove the main theorem. We use $bd_c^*(A)$, instead of $bd^*(A)$, to characterize a set A whose neighborhood base is preserved by ccc forcings. The idea of the proof is similar to that of Example 1.7 (see Remark 2.5). We assume that a space is regular (see Example 2.6 for the reason for this assumption).

Theorem 2.4. *Let $\langle X, \tau \rangle$ be a regular space and let $A \subseteq X$. Suppose that every ccc forcing preserves a neighborhood base of A. Then*

(1) *$bd^*_c(A)$ is scattered,*
(2) *$bd^*_c(A)$ is countably compact, and*
(3) *if $bd^*_c(A)$ is not compact, then it is not separable.*

Proof. We prove the contrapositive by considering the following three cases:

Case 1. $bd^*_c(A)$ is not scattered.
Case 2. $bd^*_c(A)$ is not countably compact.
Case 3. $bd^*_c(A)$ is not compact and is separable.

In each case, we construct a ccc forcing which destroys a neighborhood base of A.

Proof of Case 1. Suppose that $bd^*_c(A)$ is not scattered.

Claim 1. In $\mathbf{V}^{\mathbb{C}}$, $bd^*_c(A)$ is not countably compact, where $\mathbb{C}$ is as in Fact 1.6(1).

Proof. By [4, Problem 1.7.10], $bd^*_c(A)$ can be uniquely represented as $bd^*_c(A) = P \cup S$, where P is a perfect set, S is a scattered set, and $P \cap S = \emptyset$. Further, P is a closed subset of $bd^*_c(A)$. Since $bd^*_c(A)$ is not scattered, P is not empty. Adjoining a real makes a perfect set non-countably compact by [7, Proposition 2.10], so P is not countably compact in $\mathbf{V}^{\mathbb{C}}$. Since every closed subspace of a countably compact space is countably compact ([4, Theorem 3.10.4]) and P is a closed subspace of $bd^*_c(A)$, we can conclude that $bd^*_c(A)$ is not countably compact in $\mathbf{V}^{\mathbb{C}}$. $\dashv$ (Claim)

By Claim 1, in $\mathbf{V}^{\mathbb{C}}$, there is a closed discrete subset

$$D = \{d_n : n \in \omega\}$$

of $bd^*_c(A)$ ([4, Theorem 3.10.3]). Since $d_n \in bd^*_c(A)$ for each $n \in \omega$, there is a countable set $C_n \subseteq X \setminus \overline{A}$ such that $d_n \in \overline{C_n}$. For each $n \in \omega$, let

$$\mathcal{F}_n = \{C_n \cap U : U \in \mathcal{N}_\tau(d_n)\};$$

then $\mathcal{F}_n$ is a free filter on C_n. Consider the Booth forcing $\mathbb{B}(\mathcal{F}_n)$ for each $n \in \omega$ as in Fact 1.6(2). Let $\mathbb{B}$ be the product of the family $\{\mathbb{B}(\mathcal{F}_n) : n \in \omega\}$ with finite support:

$$\mathbb{B} = \left\{ p \in \prod_{n \in \omega} \mathbb{B}(\mathcal{F}_n) : \text{ the support of } p \text{ is finite} \right\}.$$

Since each $\mathbb{B}(\mathcal{F}_n)$ is σ-centered, it is not difficult to prove that $\mathbb{B}$ is still σ-centered, and so it has the ccc. For each $n \in \omega$, $\mathbb{B}(\mathcal{F}_n)$ adds an infinite set $E_n \subseteq C_n$ such that $E_n \setminus F$ is finite for all $F \in \mathcal{F}_n$ by Fact 1.6(2), so $E_n \setminus U$ is finite for all $U \in \mathcal{N}_\tau(d_n)$. This means that a sequence formed by any enumeration of E_n converges to d_n. For each $n \in \omega$, enumerate

$$E_n = \{e_{n,i} : i \in \omega\}.$$

Let $\dot{\mathbb{B}}$ be a $\mathbb{C}$-name for $\mathbb{B}$. Let $\dot{\mathbb{D}}$ be a $\mathbb{C} * \dot{\mathbb{B}}$-name for the dominating forcing $\mathbb{D}$ in Fact 1.6(3). Let

$$\mathbb{P} = \mathbb{C} * \dot{\mathbb{B}} * \dot{\mathbb{D}}.$$

Then $\mathbb{P}$ has the ccc ([9, VIII, Lemma 5.7]). In $\mathbf{V}^{\mathbb{P}}$, take a dominating function g over $\mathbf{V}^{\mathbb{C}*\dot{\mathbb{B}}}$, and let

$$E^g = \{e_{n,g(n)} : n \in \omega\}.$$

We prove three claims which involve the set E^g.

Claim 2. $\overline{E^g} \cap D = \emptyset$.

Proof. Assume to the contrary that there is $d_k \in D$ such that $d_k \in \overline{E^g}$. Since D is a discrete subset of $bd^*_c(A)$ and X is regular, we can find $U \in \tau$ such that $d_k \in U$ and $\overline{U} \cap (D \setminus \{d_k\}) = \emptyset$. Note that for each $n \in \omega$ with $n \neq k$, $E_n \cap U$ is finite because E_n converges to $d_n \in X \setminus \overline{U}$. Define a function $f \in {}^\omega\omega \cap \mathbf{V}^{\mathbb{C}*\dot{\mathbb{B}}}$ such that for n with $n \neq \kappa$ and $E_n \cap U \neq \emptyset$,

$$f(n) = \max\{i \in \omega : e_{n,i} \in U\};$$

if $n = k$ or $E_n \cap U = \emptyset$, then let $f(n) = 0$. Since $E^g \cap U$ is an infinite set, we have that $f(n) \geq g(n)$ for infinitely many $n \in \omega$. This contradicts the fact that g is a dominating function over $\mathbf{V}^{\mathbb{C}*\dot{\mathbb{B}}}$. $\dashv$ (Claim)

Claim 3. $\overline{E^g} \setminus E^g \subseteq \overline{D}$.

Proof. Assume that $x \notin \overline{D}$; we will show that $x \notin \overline{E_g} \setminus E_g$. Using the regularity of X, take $U \in \tau$ such that $x \in U$ and $\overline{U} \cap \overline{D} = \emptyset$. Define a function $f \in {}^\omega\omega \cap \mathbf{V}^{\mathbb{C}*\dot{\mathbb{B}}}$ such that

$$f(n) = \min\{i \in \omega : (\forall j \geq i)(e_{n,j} \in X \setminus \overline{U})\}.$$

Note that f is well defined because for each $n \in \omega$, $E_n = \{e_{n,i} : i \in \omega\}$ converges to $d_n \in D \subseteq X \setminus \overline{U}$. Since $f(n) < g(n)$ for all but finitely many $n \in \omega$, $e_{n,g(n)} \in X \setminus \overline{U}$ for all but finitely many $n \in \omega$. Therefore, $\overline{E^g} \setminus E^g \subseteq \overline{X \setminus \overline{U}}$. Since $\overline{X \setminus \overline{U}} = X \setminus U$, we have $(\overline{E^g} \setminus E^g) \cap U = \emptyset$, and thus $x \notin \overline{E^g} \setminus E^g$. $\dashv$ (Claim)

Claim 4. $\overline{E^g} \cap A = \emptyset$.

Proof. First recall that $E^g \subseteq \bigcup\{E_n : n \in \omega\}$ and that $E_n \subseteq C_n$ and $C_n \cap \overline{A} = \emptyset$ for each $n \in \omega$. Assume to the contrary that there is $y \in \overline{E^g} \cap A$. Since $E^g \cap A = \emptyset$, we have $y \in \overline{E^g} \setminus E^g$. By Claim 3, we have $y \in \overline{D}$. Since $E^g \subseteq \bigcup\{C_n : n \in \omega\}$, we have $y \in \overline{\bigcup\{C_n : n \in \omega\}}$, and so $y \in bd_c(A)$. Since $y \in A$, we have $y \in bd_c(A) \cap A = bd_c^*(A)$. Consequently, we have $y \in \overline{D} \cap bd_c^*(A)$. Since D is a closed subset of $bd_c^*(A)$, $D = \overline{D}^{bd_c^*(A)} = \overline{D} \cap bd_c^*(A)$. This would imply that $y \in D$, contradicting Claim 2. $\dashv$ (Claim)

Let $W = X \setminus \overline{E^g}$; then $W \in \tau^{\mathbb{P}}$ and, by Claim 4, we have $A \subseteq W$. To show that W witnesses the fact that $\mathbb{P}$ destroys a neighborhood base of A, take $H \in \mathcal{N}_\tau(A)$; we will show that $H \not\subseteq W$. Define a function $h \in {}^\omega\omega \cap \mathbf{V}^{\mathbb{C}*\dot{\mathbb{B}}}$ such that for each $n \in \omega$,

$$h(n) = \min\{i \in \omega : (\forall\, j \geq i)(e_{n,j} \in H)\}.$$

Since $E_n = \{e_{n,i} : i \in \omega\}$ converges to $d_n \in D \subseteq H$ for each $n \in \omega$, h is well defined. If $H \subseteq W$, then $H \cap E^g = \emptyset$, and so $h(n) > g(n)$ for all $n \in \omega$, contradicting the fact that g is a dominating function over $\mathbf{V}^{\mathbb{C}*\dot{\mathbb{B}}}$. Therefore, we must have $H \not\subseteq W$, and we can conclude that $\mathbb{P}$ destroys a neighborhood base of A. This finishes the proof of Case 1.

Proof of Case 2. Suppose that $bd_c^*(A)$ is not countably compact. Skip the forcing $\mathbb{C}$ in the proof of Case 1 and in $\mathbf{V}$ take a closed discrete subset $\{d_n : n \in \omega\}$ of $bd_c^*(A)$. Using the set $\{d_n : n \in \omega\}$, construct the forcing $\mathbb{B}$ as in the proof of Case 1 and let

$$\mathbb{P} = \mathbb{B} * \dot{\mathbb{D}},$$

where $\dot{\mathbb{D}}$ is a $\mathbb{B}$-name for the dominating forcing $\mathbb{D}$ in Fact 1.6(3). Then $\mathbb{P}$ is a ccc forcing, and in $\mathbf{V}^{\mathbb{P}}$, $\mathcal{N}_\tau(A)$ is no longer a neighborhood base of A as in Case 1.

Proof of Case 3. Suppose that $bd_c^*(A)$ is not compact and is separable. Because of Case 2, we may assume that $bd_c^*(A)$ is countably compact. By [7, Corollary 2.8], there is a Booth forcing $\mathbb{B}(\mathcal{F})$ for some filter $\mathcal{F}$ such that in $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$, $bd_c^*(A)$ is not countably compact. Following Case 2, let

$$\mathbb{P} = \mathbb{B}(\mathcal{F}) * \dot{\mathbb{B}} * \dot{\mathbb{D}}.$$

Then $\mathbb{P}$ has the ccc and destroys a neighborhood base of A.

This concludes the proof of the theorem. $\square$

Remark 2.5. The proof of Theorem 2.4 is similar to that of Example 1.7: The discrete set $D = \{d_n : n \in \omega\}$ corresponds to $A = \omega \times \{\omega\}$; the sequence E_n converging to d_n corresponds to the column $\{n\} \times \omega$ converging to (n, ω); the set $E^g = \{e_{n,g(n)} : n \in \omega\}$ corresponds to $E = \{(n, g(n)) : n \in \omega\}$.

We give an example which shows that the regularity of $\langle X, \tau\rangle$ in Theorem 2.4 cannot be weakened to Hausdorffness.

Example 2.6. *There are a Hausdorff (non-regular) space $\langle X, \tau\rangle$ and a subset A of X such that*

(1) *$bd_c^*(A)$ is not countably compact and*
(2) *every forcing preserves a neighborhood base of A.*

Proof. Let $X = \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers. Let us define a topology τ on X. Each point in $X \setminus \{0\}$ has the usual (Euclidean) neighborhoods. An open neighborhood U of 0 has the form

$$U = V \setminus \{1/n : n \in \mathbb{N}\},$$

where V is a Euclidean open set containing 0 and $\mathbb{N}$ is the set of all natural numbers. The space $\langle X, \tau\rangle$ is not regular because the point 0 and the closed set $\{1/n : n \in \mathbb{N}\}$ cannot be separated by disjoint open sets. Let

$$A = \{0\} \cup \{1/n : n \in \mathbb{N}\}.$$

Then A is a closed discrete subset of X and is not countably compact. It is easy to see that $A = bd_c^*(A)$. Let $\mathbb{P}$ be any forcing. To show that $\mathbb{P}$ preserves a neighborhood base of A, take $W \in \tau^{\mathbb{P}}$ such that $A \subseteq W$; we will find $H \in \mathcal{N}_\tau(A)$ such that $H \subseteq W$. Take $U \in \tau$ such that $0 \in U \subseteq W$. Then $U = V \setminus \{1/n : n \in \mathbb{N}\}$ for some Euclidean open set V in the ground model which contains 0. Since $\{1/n : n \in \mathbb{N}\} \subseteq A \subseteq W$, we have $V \subseteq W$. Take $k \in \mathbb{N}$ such that for each $i \geq k$, $1/i \in V$. For each $i < k$, take a Euclidean open set O_i in the ground model such that $1/i \in O_i \subseteq W$. Let

$$H = V \cup \{O_i : i < \kappa\}.$$

Then $H \in \mathcal{N}_\tau(A)$ and $H \subseteq W$. Thus, $\mathbb{P}$ preserves a neighborhood base of A. □

A space X is called *countably tight* if, whenever $B \subseteq X$ and $x \in \overline{B} \setminus B$, there is a countable set $C \subseteq B$ such that $x \in \overline{C}$. A space X is called *locally separable* if every point in X is contained in a separable open set. It is not difficult to prove that if a space X is countably tight or locally separable, then $bd_c(A) = bd(A)$ for every subset A of X and so $bd_c^*(A) = bd^*(A)$ for every $A \subseteq X$. Consequently, we have the following corollary to Theorem 2.4.

Corollary 2.7. *Let $\langle X, \tau\rangle$ be a countably tight regular space or a locally separable regular space, and let $A \subseteq X$. Suppose that every ccc forcing preserves a neighborhood base of A. Then*

(1) *$bd^*(A)$ is scattered,*
(2) *$bd^*(A)$ is countably compact, and*
(3) *if $bd^*(A)$ is not compact, then it is not separable.*

We give an example of a subset A of a separable regular space such that $bd^*(A)$ is scattered, countably compact, and non-separable, yet a ccc forcing destroys a neighborhood base of A. If CH holds, then the space is first countable (so it is countably tight). This example shows that the converse of Corollary 2.7 does not hold.

Example 2.8. *There are a separable regular space $\langle X, \tau \rangle$, a subset A of X, and a ccc forcing $\mathbb{P}$ such that*

(1) *$bd^*(A)$ is scattered,*
(2) *$bd^*(A)$ is countably compact,*
(3) *$bd^*(A)$ is not separable, and*
(4) *$\mathbb{P}$ destroys a neighborhood base of A.*

If CH is assumed, then $\langle X, \tau \rangle$ is first countable.

Proof. Let $\gamma'\mathbb{N}$ be the space in [4, Problem 3.12.17(d)]. $\gamma'\mathbb{N}$ is a compactification of the set $\mathbb{N}$ of all natural numbers such that $\gamma'\mathbb{N} \setminus \mathbb{N}$ is homeomorphic to $\delta + 1$ for some ordinal δ of uncountable cofinality. Let

- $X = \gamma'\mathbb{N} \setminus \{\delta\}$ and
- $A = X \setminus \mathbb{N}$.

The subspace A is homeomorphic to the ordinal δ, and so $A = bd^*(A)$ satisfies the conditions (1), (2), and (3).

Claim. For every $H \in \mathcal{N}_\tau(A)$, $X \setminus H$ is finite.

Proof. Assume to the contrary that $X \setminus H$ is infinite for some $H \in \mathcal{N}_\tau(A)$. Since $X \setminus H \subseteq \mathbb{N}$, $X \setminus H$ consists of isolated points of $\gamma'\mathbb{N}$. Since $\gamma'\mathbb{N}$ is compact, $X \setminus H$ has an accumulation point in $\gamma'\mathbb{N}$. The only non-isolated point in $\gamma'\mathbb{N} \setminus H$ is δ, and so a sequence formed by an enumeration of $X \setminus H$ must converge to δ. However, as explained in [4, Problem 3.12.17(d)], no sequence of points in $\mathbb{N}$ converges to δ in the space $\gamma'\mathbb{N}$. Thus, $X \setminus H$ must be finite. $\dashv$ (Claim)

To prove condition (4), first note that X is a separable non-compact regular space. By applying [7, Lemma 2.7], we can find a free filter $\mathcal{F} \subseteq [\mathbb{N}]^\omega$ such that in $\mathbf{V}^{\mathbb{B}(\mathcal{F})}$ there is an infinite set $E \subseteq \mathbb{N}$ such that E is a closed subset of X, where $\mathbb{B}(\mathcal{F})$ is as in Fact 1.6(2). Note that $\mathbb{B}(\mathcal{F})$ has the ccc. Let $W = X \setminus E$; then $W \in \tau^{\mathbb{B}(\mathcal{F})}$ and $A \subseteq W$. By the claim, for every $H \in \mathcal{N}_\tau(A)$, $X \setminus H$ is finite and so $H \nsubseteq W$. Thus, $\mathcal{N}_\tau(A)$ is not a neighborhood base of A in the space $\langle X, \tau^{\mathbb{B}(\mathcal{F})} \rangle$. If CH holds, then X is first countable by [4, Problem 3.12.17(e)]. $\square$

If we assume that a space is metrizable, then we have the following.

Corollary 2.9. *Let $\langle X, \tau \rangle$ be a metrizable space and let $A \subseteq X$. Then the following are equivalent:*

(1) *Every forcing preserves a neighborhood base of* A.
(2) *Every ccc forcing preserves a neighborhood base of* A.
(3) $bd^*(A)$ *is scattered and compact.*

Proof. (3) $\Longrightarrow$ (1) is proved in Theorem 2.2.

(1) $\Longrightarrow$ (2) is obvious.

(2) $\Longrightarrow$ (3) Note that metrizable spaces are countably tight, so by Corollary 2.7, $bd^*(A)$ is scattered and countably compact. In metrizable spaces, countably compact sets are compact ([4, Theorem 4.1.17]). □

3. Collapsing ω_1

In this section, we consider forcings that collapse ω_1 and obtain the converse of Theorem 2.2. To do so, we use the following two lemmas, which say that "not scattered" and "not compact" are preserved by any forcing.

Lemma 3.1. *Let* $\langle X, \tau\rangle$ *be a topological space and let* $\mathbb{P}$ *be a forcing. If* $\langle X, \tau\rangle$ *is not scattered, then* $\langle X, \tau^{\mathbb{P}}\rangle$ *is not scattered either.*

Proof. Suppose that $\langle X, \tau\rangle$ is not scattered. Then there is a non-empty subset $B \subseteq X$ such that B does not contain an isolated point in the relative topology of B; that is, for any $b \in B$ and any $U \in \tau$, $U \cap B \neq \{b\}$. By absoluteness, this property of the set B holds in $\mathbf{V}^{\mathbb{P}}$ as well. Therefore, $\langle X, \tau^{\mathbb{P}}\rangle$ is not scattered. □

Lemma 3.2. *Let* $\langle X, \tau\rangle$ *be a topological space and let* $\mathbb{P}$ *be a forcing. If* $\langle X, \tau\rangle$ *is not compact, then* $\langle X, \tau^{\mathbb{P}}\rangle$ *is not compact either.*

Proof. Assume that $\langle X, \tau\rangle$ is not compact. Let $\mathcal{U}$ be an open cover of $\langle X, \tau\rangle$ with no finite subcover. Then $\mathcal{U}$ is also an open cover of $\langle X, \tau^{\mathbb{P}}\rangle$ with no finite subcover. Thus, $\langle X, \tau^{\mathbb{P}}\rangle$ is not compact. □

Now let us prove the converse of Theorem 2.2.

Proposition 3.3. *Let* $\langle X, \tau\rangle$ *be a regular space and let* $A \subseteq X$. *Then the following are equivalent:*

(1) *Every forcing preserves a neighborhood base of* A.
(2) $bd^*(A)$ *is scattered and compact.*

Proof. (2) $\Longrightarrow$ (1) is proved in Theorem 2.2.

Let us prove (1) $\Longrightarrow$ (2). Suppose that $|X| = \kappa$. Let $\mathbb{P} = Fn(\omega, \kappa)$ as in [9, VII, Definition 5.1]. If κ is an uncountable cardinal, then $\mathbb{P}$ collapses κ to a countable ordinal ([9, VII, Lemma 5.2]). Therefore, X is a countable space in $\mathbf{V}^{\mathbb{P}}$, and so $bd^*_c(A) = bd^*(A)$ in $\mathbf{V}^{\mathbb{P}}$. By the assumption, every forcing preserves a neighborhood base of A. Therefore, for a $\mathbb{P}$-name for any forcing $\dot{\mathbb{Q}}$, $\mathcal{N}_\tau(A)$ remains a neighborhood base of A in $\mathbf{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$. By Theorem 2.4 applied in $\mathbf{V}^{\mathbb{P}}$, $bd^*(A)$ is scattered and countably compact in $\mathbf{V}^{\mathbb{P}}$. Since $bd^*(A)$ is countable in $\mathbf{V}^{\mathbb{P}}$, it is actually compact. Thus, $bd^*(A)$ is scattered and compact in $\mathbf{V}^{\mathbb{P}}$. By Lemma 3.1, every forcing preserves the property of being not scattered, and by Lemma 3.2, every forcing preserves the property of being not compact. Therefore, $bd^*(A)$ must be scattered and compact in $\mathbf{V}$. $\square$

We give an example which shows that if we restrict ourselves to forcings which preserve ω_1, then (1) $\Longrightarrow$ (2) in the proof of Proposition 3.3 would not hold.

Example 3.4. *There are a topological space $\langle X, \tau\rangle$ and a subset A of X such that*

(1) *any forcing that preserves ω_1 preserves a neighborhood base of A, and*
(2) *$bd^*(A)$ is not compact.*

Proof. Let

- $X = \omega_1 \times (\omega + 1)$ and
- $A = \omega_1 \times \{\omega\}$.

Let the topology τ on X be the usual product topology, and we consider the order topology on ω_1 and $\omega + 1$. Clearly, A is not compact and $A = bd^*(A)$. Let $\mathbb{P}$ be a forcing which preserves ω_1. To show that $\mathbb{P}$ preserves a neighborhood base of A, take $W \in \tau^{\mathbb{P}}$ such that $A \subseteq W$; we will find $H \in \mathcal{N}_\tau(A)$ such that $H \subseteq W$.

Claim. The set $\{n \in \omega : (\exists\, \xi \in \omega_1)[(\xi, n) \in X \setminus W]\}$ is finite.

Proof. Let $J = \{n \in \omega : (\exists\, \xi \in \omega_1)[(\xi, n) \in X \setminus W]\}$. Assume to the contrary that J is infinite. For each $n \in J$, pick ξ_n such that $(\xi_n, n) \in X \setminus W$. Since $\mathbb{P}$ preserves ω_1, there is $\gamma < \omega_1$ such that $\{\xi_n : n \in J\} \subseteq \gamma$. The product $(\gamma + 1) \times (\omega + 1)$ is a scattered compact space in $\mathbf{V}$, and so it remains compact in $\mathbf{V}^{\mathbb{P}}$ ([8, Lemma 7]). Therefore, $\{(\xi_n, n) : n \in J\}$ has an accumulation point in $(\gamma + 1) \times (\omega + 1)$, and the accumulation point must be in A. This contradicts the fact that $X \setminus W$ is a closed set containing $\{(\xi_n, n) : n \in J\}$ and missing A. $\dashv$ (Claim)

By the claim, there is $k \in \omega$ such that $X \setminus W \subseteq \{(\xi, n) : \xi \in \omega_1,\ n \leq k\}$. Let $H = A \cup \{(\xi, n) : \xi \in \omega_1,\ n > k\}$. Then $H \in \mathcal{N}_\tau(A)$ and $H \subseteq W$. $\square$

Before concluding this note, let us ask a question. Proposition 3.3 says that a necessary and sufficient condition for a neighborhood base of a set A to be preserved by any forcing is that $bd^*(A)$ is scattered and compact. We do not know such a condition for ccc forcings. (Note that the converse of Theorem 2.4 does not hold because of Example 2.8.)

Question 3.5. What is a necessary and sufficient condition for a neighborhood base of a subset of a regular space to be preserved by ccc forcings?

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