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by

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SECOND COUNTABLE UC METRIC SPACES ARE LEBESGUE IN ZF

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Abstract. We show in the Zermelo-Fraenkel set theory ZF that

(i) if $\mathbf{X}$ is second countable, then $\mathbf{X}$ is Lebesgue if and only if it is *UC space* (: every continuous real valued function on $\mathbf{X}$ is uniformly continuous) if and only if it is *normal* (: the distance of every two disjoint, non-empty, closed subsets of $\mathbf{X}$ is strictly positive);

(ii) $\mathbf{X}$ is Lebesgue if and only if every open cover $\mathcal{U}$ of $\mathbf{X}$ has a refinement $\mathcal{V} = \{B(x,\delta) : x \in K\}$ for some $\delta > 0$ and some $K \subseteq X$ if and only if for every open cover $\mathcal{U}$ of $\mathbf{X}$ consisting of open balls there exists $\varepsilon > 0$ and a subcover $\mathcal{V}$ of $\mathcal{U}$ such that for every $V \in \mathcal{V}$, $\delta(V) > 2\varepsilon$;

(iii) $\mathbf{X}$ is complete if and only if it is almost complete;

(iv) $\mathbf{X}$ is almost Cauchy if and only if every sequence in $\mathbf{X}$ admits an almost Cauchy subsequence;

(v) every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbf{X}$ has a Cauchy subsequence if and only if each countable subspace of $\mathbf{X}$ is almost compact if and only if each countable subspace of $\mathbf{X}$ is totally bounded;

(vi) $\mathbf{X}$ is sequentially compact if and only if each countable subspace of $\mathbf{X}$ is almost compact and almost complete;

(vii) if each countable subspace of $\mathbf{X}$ is totally bounded, then $\mathbf{X}$ is sequentially bounded.

1. Notation and Terminology

Let $\mathbf{X} = (X, d)$ be a metric space, $x \in X$, and $\varepsilon > 0$. $B(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ denotes the open ball in $\mathbf{X}$ with center x and radius

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ε. Given $B \subseteq X$, $\delta(B) = \sup\{d(x,y) : x, y \in B\} \in [0,\infty) \cup \{+\infty\}$ will denote the *diameter* of B.

X is said to be *bounded* if and only if $\delta(X) < +\infty$.

Let $\mathcal{U}$ be an open cover of **X**. We say that $\mathcal{U}$ has a *Lebesgue number* $\delta > 0$ if and only if, for every $A \subseteq X$ with $\delta(A) < \delta$, there exists $U \in \mathcal{U}$ with $A \subseteq U$.

Given $\varepsilon > 0$, a subset O of X is called ε*-open* if and only if, for every $x \in O$, there is $y \in O$ such that $x \in B(y, \varepsilon) \subseteq O$. The complement of an ε-open set is called ε*-closed*. A subset D of X is called ε*-dense* if and only if, for every $x \in X$, $B(x,\varepsilon) \cap D \neq \varnothing$. Equivalently, $X = \bigcup\{B(d,\varepsilon) : d \in D\}$. A finite ε-dense set D of **X** is called ε*-net*. A *refinement* of a cover $\mathcal{U}$ of **X** is another cover $\mathcal{V}$ such that for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ with $V \subseteq U$.

X is said to be *Heine-Borel compact* or *compact* if and only if every open cover $\mathcal{U}$ of **X** has a finite subcover $\mathcal{V}$.

X is said to be *countably compact* if and only if every countable open cover $\mathcal{U}$ of **X** has a finite subcover $\mathcal{V}$.

X is said to be *Lebesgue* (*countably Lebesgue*, respectively) if and only if every open (every countably open, respectively) cover of **X** has a Lebesgue number.

Given $\varepsilon > 0$, **X** is said to be ε*-compact* if and only if every open cover $\mathcal{U}$ of **X** consisting of open balls of radius ε has a finite subcover $\mathcal{V}$.

X is called *almost compact* if and only if for every $\varepsilon > 0$, **X** is ε-compact.

X is said to be *almost Lebesgue* if and only if, for every open cover $\mathcal{U}$ of **X**, there exists a $\delta > 0$ such that $\mathcal{V} = \{B(x,\delta) : x \in K\}$ is a refinement of $\mathcal{U}$ for some δ-dense subset K of **X**.

X is *sequentially compact* if and only if every sequence has a convergent subsequence.

X is said to be *totally bounded* if and only if, for every $\varepsilon > 0$, there exists an ε-net of **X**. Evidently, each totally bounded metric space is bounded, but the converse is not true in general. For example, every infinite set equipped with the discrete metric is bounded but not totally bounded.

X is said to be *sequentially bounded* if and only if every sequence in **X** admits a Cauchy subsequence.

X is said to be *complete* or *Frechét complete* if and only if every Cauchy sequence of points of X converges to some element of X.

A sequence $(x_n)_{n\in\mathbb{N}}$ in **X** satisfying for each $\varepsilon > 0$ there exists $F \in [\mathbb{N}]^{<\omega}$ such that for every $n \in \mathbb{N}$, there is $m \in F$ with $x_n \in B(x_m, \varepsilon)$ is called *almost Cauchy*.

X is called *almost Cauchy* (*almost complete*, respectively) if and only if each sequence in **X** is almost Cauchy (each almost Cauchy sequence in **X** has a limit point, respectively).

An infinite set X is said to be

- *Dedekind-infinite,* denoted by $\mathbf{DI}(X)$, if and only if X contains a countably infinite subset. Otherwise, it is said to be *Dedekind-finite.*
- *Weakly Dedekind-infinite*, denoted by $\mathbf{WDI}(X)$, if and only if $\mathcal{P}(X)$ contains a countably infinite set. Otherwise, it is said to be *weakly Dedekind-finite.*

By universal quantifying over X, $\mathbf{DI}(X)$ gives rise to the choice principle **IDI**: For all $X(X \text{ infinite} \to \mathbf{DI}(X))$, that is, "every infinite set is Dedekind-infinite" ([6, Form 9]). One defines **IWDI** ([6, Form 82]) similarly.

Below we list some of the weak forms of the axiom of choice we shall deal with in the sequel.

- CAC ([6, Form 8]): For every countable family $\mathcal{A}$ of non-empty sets there exists a function f such that for all $x \in \mathcal{A}$, $f(x) \in x$.
- CAC_{fin} ([6, Form 10]): CAC restricted to countable families of non-empty finite sets. Equivalently, see [6, Form [10 O]], every infinite well-ordered family $\mathcal{A}$ of non empty finite sets has a partial choice set, i.e., some infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ with a choice set.
- $\mathsf{PKW}(\aleph_0, \geq 2, \infty)$ Partial Kinna-Wagner Principle ([6, Form 167]): Every disjoint family $\mathcal{A} = \{A_i : i \in \omega\}$ such that for all $i \in \omega, |A_i| \geq 2$ has a partial Kinna-Wagner choice; i.e., there exists an infinite subfamily $\mathcal{B} = \{A_{k_i} : i \in \omega\}$ of $\mathcal{A}$ and a family $\mathcal{F} = \{F_i : i \in \omega\}$ of non-empty sets such that for all $i \in \omega$, $F_i \subsetneq A_{k_i}$.

2. Introduction and Some Preliminary and Known Results

In this paper, the intended context for reasoning and statements of theorems will be the Zermelo-Fraenkel set theory ZF unless otherwise noted. In order to stress the fact that a result is proved in ZF (ZFC (= ZF+AC), respectively), we shall write in the beginning of the statements of the theorems and propositions (ZF) ((ZFC), respectively).

Some of the very basic notions in a course of metric spaces one encounters when compactness is studied are those of countable compactness, sequential compactness, total boundedness, completeness, and the Lebesgue number of an open cover. One invariably learns the following.

Proposition 2.1 ([12]). *Let $\mathbf{X} = (X, d)$ be a metric space.*
(a) (ZFC) *The following are equivalent:*

(i) $\mathbf{X}$ *is compact.*

(ii) $\mathbf{X}$ *is sequentially compact.*
(iii) $\mathbf{X}$ *is complete and totally bounded.*
(iv) $\mathbf{X}$ *is complete and sequentially bounded.*
(v) $\mathbf{X}$ *is countably compact.*

(b) (ZF) $\mathbf{X}$ *is sequentially compact if and only if it is complete and sequentially bounded.*
(c) (ZF) $\mathbf{X}$ *is compact if and only if it is totally bounded and Lebesgue.*

Evidently, (i) of Proposition 2.1(a) implies each of the conditions (ii)–(v). In addition, if one backtracks to the proofs of the equivalence of each of (ii)–(v) with (i), then one will realize that some portion of AC is needed to carry out the proofs. Likewise, a portion of AC is needed for some of the implications $p \to q$, where p and q denote the properties given in (ii)–(v). The following diagram lists the implications which hold in ZF.

Compact

$\updownarrow$

totally bounded
and Lebesgue

$\downarrow$

complete and
totally bounded

$\downarrow$

complete and
sequentially bounded

$\updownarrow$

sequentially compact

Diagram 1

The given implications in Diagram 1 cannot be reversed in ZF. Counterexamples are supplied in [9], [7], and [8].

The most useful of all characterizations of compactness given in Proposition 2.1 has proven to be sequential compactness. We stress the fact, see e.g., Diagram 1, that sequential compactness is the weakest equivalent of all. However, the real equivalent of compactness is the conjunction of totally bounded with Lebesgue in the sense that it is choice free. In view of the latter equivalence, one may ask if there are other similar properties equivalent to compactness in ZF. In the next theorem we observe that the conjunction almost compact and almost Lebesgue is a pair of such properties. This justifies the initiation of these two notions. First, we give a characterization of almost compactness which points to compactness if we read "open cover" as "ε-dense" and "subcover" as "finite ε-dense subset." We call attention here to the fact that almost compactness in this project refers solely to metric spaces and should not be confused with the notion of almost compact (= every open cover has a finite subset with a dense union in the space) in general topology.

Proposition 2.2 (ZF). *A metric space $\mathbf{X} = (X, d)$ is almost compact if and only if for every $\varepsilon > 0$, every ε-dense subset of $\mathbf{X}$ has a finite ε-dense subset.*

Proof. ($\rightarrow$) Fix $\varepsilon > 0$ and let K be an ε-dense subset K of $\mathbf{X}$. Then $\mathcal{U} = \{B(k, \varepsilon) : k \in K\}$ is an open cover of $\mathbf{X}$. By our hypothesis, $\mathcal{U}$ has a finite subcover; i.e., there exists $k_i \in K, i \leq n$ such that $X = \bigcup\{B(k_i, \varepsilon) : i \leq n\}$. It follows that $S = \{k_i : i \leq n\} \subseteq K$ is an ε-dense subset of $\mathbf{X}$.

($\leftarrow$) Fix $\varepsilon > 0$ and let $\mathcal{U} = \{B(x, \varepsilon) : x \in K\}$ be a cover of X. Clearly, for every $y \in X \backslash K$ there exists $x \in K$ with $y \in B(x, \varepsilon)$. Hence, $d(x, y) < \varepsilon$ and K is an ε-dense subset of $\mathbf{X}$. By our hypothesis, there exists a finite ε-dense set $S = \{k_i : i \leq n\} \subseteq K$ of $\mathbf{X}$. Since $X = \bigcup\{B(k_i, \varepsilon) : i \leq n\}$, it follows that $\{B(k_i, \varepsilon) : i \leq n\}$ is a finite subcover of $\mathcal{U}$ and $\mathbf{X}$ is ε-compact as required. □

Theorem 2.3 (ZF). *A metric space $\mathbf{X} = (X, d)$ is compact if and only if it is almost compact and almost Lebesgue.*

Proof. ($\rightarrow$) This is obvious.

($\leftarrow$) Fix $\mathcal{U}$ an open cover of $\mathbf{X}$ and let, by our hypothesis, $\mathcal{V} = \{B(x, \delta) : x \in K\}$ be a refinement of $\mathcal{U}$ for some $\delta > 0$ and some $K \subseteq X$. By the almost compactness of $\mathbf{X}$, it follows that $\mathcal{V}$ has a finite subcover $\mathcal{W} = \{B(x_i, \delta) : i \leq n\}$. For every $i \leq n$, fix $U_i \in \mathcal{U}$ such that $B(x_i, \delta) \subseteq U_i$. Clearly, $\{U_i : i \leq n\}$ is a finite subcover of $\mathcal{U}$ and $\mathbf{X}$ is compact as required. □

Compact metric spaces possess a kind of "bridge" which allows the passage from "infinite" to "finite." This bridge seems to be inconspicuous in the definition of sequential compactness, but it is apparent in the proposition that "every sequence is almost Cauchy." This is the reason for the introduction of the notion of the almost Cauchy sequence which we shall study in §5. We show in Theorem 5.2 that a metric space is almost Cauchy if and only if each of its sequences admits an almost Cauchy subsequence.

The following two propositions list some results from [8] and [9].

Proposition 2.4 ([8]). (i) *Proposition 2.1(a) holds true in (*ZF + CAC*);*

(ii) *the statement "every sequentially compact metric space is compact" implies* **IDI**.

Proposition 2.5 ([9]). (i) (ZF) *Every countably compact metric space is Lebesgue;*

(ii) (ZF) *every totally bounded metric space is sequentially bounded;*

(iii) (ZF + CAC) *every sequentially bounded metric space is totally bounded and separable;*

(iv) *the statement "every sequentially bounded metric space is totally bounded" implies* **IDI**.

The notion of countably Lebesgue was initiated in [9] where the following result has been established.

Theorem 2.6 ([9]). (a) (ZF) *Let* $\mathbf{X} = (X, d)$ *be a metric space. Each one of the following statements implies the one beneath it.*

(i) $\mathbf{X}$ *is Lebesgue.*

(ii) $\mathbf{X}$ *is countably Lebesgue.*

(iii) *Every continuous real valued function on* $\mathbf{X}$ *is uniformly continuous.*

(iv) *The distance of every two disjoint, non-empty closed subsets of* $\mathbf{X}$ *is strictly positive.*

(b) $(\mathsf{ZF}+\mathsf{CAC})$ (i)–(iv) *of* (a) *are equivalent.*

(c) *The statement, "every countably Lebesgue metric space is Lebesgue" implies* $\mathsf{PKW}(\aleph_0, \geq 2, \infty)$.

Spaces satisfying Theorem 2.6(iii) were initiated by Masahiko Atsuji in [1]. In the survey article [10] and in [2] and [3], they are called Atsuji spaces, but in [4], they are called *UC* spaces. Spaces satisfying Theorem 2.6(iv) were initiated in [11] where they are called normal metric spaces, but in [13] and [15], they are called Lebesgue. Theorem 2.6(b) indicates that all these notions are equivalent in ZF + CAC, and Theorem 2.6(c) shows that the equivalencies cannot be proved in ZF. In Theorem 4.5 we show that if we restrict to the class of second countable metric spaces and, in particular, to subspaces of the real line $\mathbb{R}$, then CAC is not needed. In [9], it is asked if the existence of a normal metric space $\mathbf{X} = (X, d)$, which fails to be a *UC* space, is consistent with ZF . Hence, if the answer is in the affirmative, then the space $\mathbf{X}$ cannot be second countable.

We give the following theorem here for future reference.

Theorem 2.7 ([14]). (ZF + CAC) *Let* $\mathbf{X} = (X, d)$ *be a totally bounded metric space. Then* $\mathbf{X}$ *is second countable and separable.*

The rest of the paper is organized as follows. In §3 we scrutinize the properties of almost compact metric spaces and study the almost complete metric spaces. We show in Corollary 3.5 that a metric space is complete if and only if it is almost complete. In §4 we study the almost Lebesgue metric spaces. We show in Theorem 4.2 that a metric space is Lebesgue if and only if it is almost Lebesgue. Finally, in §5 we study sequentially bounded metric spaces. We show in Theorem 5.2 that a metric space is sequentially bounded if and only if it is almost Cauchy and conclude in Corollary 5.3 that a metric space is sequentially compact if and only if it is almost Cauchy and almost complete.

3. Almost Compact and Almost Complete Metric Spaces

The notions of precompact and almost compact metric spaces look alike and one may be deceived by this similarity and think that they are equivalent. In the first result in this section we summarize some properties of almost compact metric spaces and show that precompact $\neq$ almost compact.

Proposition 3.1. *Let* $\mathbf{X} = (X, d)$ *be a metric space. Then*

(i) *if* $\mathbf{X}$ *is* ε*-compact, then every* ε *-closed subset of* $\mathbf{X}$ *is* ε*-compact, but closed subspaces of* ε*-compact need not be* ε*-compact;*
(ii) $\mathbf{X}$ *is* ε*-compact if and only if every* ε*-dense subset of* $\mathbf{X}$ *has a finite* ε*-dense subset;*
(iii) *if* $\mathbf{X}$ *is almost compact, then* $\mathbf{X}$ *is totally bounded. The converse fails;*
(iv) *if* $\mathbf{X}$ *is almost compact, then* $\mathbf{X}$ *is almost Cauchy;*
(v) $\mathbf{X}$ *is precompact if and only if* $\mathbf{X}$ *is totally bounded;*
(vi) CAC *implies every almost compact metric space is separable;*
(vii) *if* $\mathbf{X}$ *is compact, then* $\mathbf{X}$ *is almost compact. The converse fails;*
(viii) *almost compact subsets of* $\mathbf{X}$ *need not be closed;*
(ix) *subspaces of compact metric spaces need not be almost compact;*
(x) *precompact metric spaces are almost Cauchy;*
(xi) *almost Cauchy metric spaces need not be almost compact;*
(xii) *almost Cauchy metric spaces need not be sequentially compact.*

Proof. (i) The first part is straightforward and it is left as a warm up exercise for the reader.

To see the second part, let $X = \{0\} \cup (1/6, 1]$ carry the usual metric and $\varepsilon = 1/5$. We claim that $\mathbf{X}$ is ε-compact. Indeed, let $\mathcal{U} = \{B(k, \varepsilon) : k \in K \subseteq X\}$ be a cover of X. Then $0 \in B(k_1, \varepsilon)$ for some $k_1 \in K$. Clearly, $B(k_1, \varepsilon) \cap (1/6, 1] \neq \varnothing$ and $B = (1/6, 1] \backslash B(k_1, \varepsilon)$ is some closed subset of $(1/6, 1]$. Since B is compact and $\mathcal{U}$ covers B, it follows that there exist $k_2, k_3, ..., k_n \in K$ and $n \in \mathbb{N}$ such that $B \subseteq \bigcup\{B(k_i, \varepsilon) : i = 2, ..., n\}$.

Hence, $X = \bigcup\{B(k_i, \varepsilon) : i = 1, ..., n\}$ and $\mathbf{X}$ is ε-compact as required. On the other hand, the closed subspace $(1/6, 1]$ of $\mathbf{X}$ is not ε-compact as the open cover $\mathcal{V} = \{B(k, \varepsilon) : k \in (11/30, 1]\}$ has no finite subcover.

(ii) This follows at once from Proposition 2.2.

(iii) The first part is obvious. For the second assertion, let $(0, 1]$ be the subspace of $\mathbb{R}$ taken with the usual metric. Clearly, $(0, 1]$ is totally bounded. However, for every $\varepsilon > 0$,

$$\mathcal{U}_\varepsilon = \{B(x, \varepsilon) : x \in (\varepsilon, 1)\}$$

is an open cover of $(0, 1]$ consisting of open balls of radius ε without a finite subcover. Thus, $(0, 1]$ is totally bounded but not almost compact.

(iv) Fix $(x_n)_{n\in\mathbb{N}}$ a sequence in $\mathbf{X}$ and let $\varepsilon > 0$. By our hypothesis, there exists an $\varepsilon/2$-dense subset $A = \{a_i : i \leq k\}$ of $\mathbf{X}$. Without loss of generality, we may assume that for every $i \leq k$, $B(a_i, \varepsilon/2)$ contains at least one term of $(x_n)_{n\in\mathbb{N}}$. Fix for every $i \leq k$, $x_{n_i} \in B(a_i, \varepsilon/2)$. It is straightforward to verify that for every $n \in \mathbb{N}$, there is $i \leq k$ such that $x_n \in B(x_{n_i}, \varepsilon)$, meaning that $(x_n)_{n\in\mathbb{N}}$ is almost Cauchy.

(v) This is easy.

(vi) Combine (iii) with Theorem 2.7.

(vii) The first part is straightforward. To see the second part, let $Y = \{1/n : n \in \mathbb{N}\}$ carry the usual metric. Clearly, $\mathbf{Y}$ is not complete. However, the sequence $(1/n)_{n\in\mathbb{N}}$ being convergent is Cauchy. Hence, for every $k \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $1/n \in B(1/n_0, 1/k)$. Thus, there are only finitely many members of Y left out of $B(1/n_0, 1/k)$. So, we need finitely many open balls of radius $1/k$ to cover Y. Thus, $\mathbf{Y}$ is almost compact but not complete, hence not compact also.

(viii) Observe that the subspace $Y = \{1/n : n \in \mathbb{N}\}$ of $\mathbb{R}$ with the usual metric is almost compact and almost Cauchy but not closed and sequentially compact.

(ix) and (xii) In view of the proof of (iii), the subspace $(0, 1]$ of the compact space $[0, 1]$ with the usual metric fails to be almost compact.

(x) This can be proved as in (iii).

(xi) By (ix), the subspace $(0, 1]$ of $\mathbb{R}$ is not almost compact. However, in view of (x), being totally bounded, it is almost Cauchy. $\square$

Remark 3.2. (i) We remark here that the open cover $\mathcal{U} = \{\{1/n\} : n \in \mathbb{N}\}$ of the metric space $\mathbf{Y}$ given in the proof of Proposition 3.1(vii) has no Lebesgue number. To see this, fix $\delta > 0$ and let $n_0 \in \mathbb{N}$ satisfy $2/n_0 < \delta$. Clearly, $B(1/n_0, 1/n_0) = \{1/n : n \geq n_0\}$ has diameter less that δ, but it is included in no member of $\mathcal{U}$. Thus, $\mathcal{U}$ has no Lebesgue number. Hence,

Y is almost compact but not Lebesgue. Clearly, **Y** can be replaced by the range of any non-convergent, one-to-one Cauchy sequence.

(ii) A Lebesgue metric space need not be totally bounded. Indeed, any infinite set with the discrete metric is trivially Lebesgue but not totally bounded.

(iii) A totally bounded metric space need not be countably Lebesgue. Indeed, the subspace $(0,1)$ of $\mathbb{R}$ is totally bounded but not countably Lebesgue. Indeed, $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ where, for every $n \in \mathbb{N}$, $U_n = (1/n, 1)$ is an open cover of $(0,1)$ without a Lebesgue number. (For every $\delta > 0, (0,\delta)$ is a set of diameter δ, and it is not included in some member of $\mathcal{U}$.)

Our next result shows that a subsequence of an almost Cauchy sequence is almost Cauchy.

Proposition 3.3. *Let $\mathbf{X} = (X,d)$ be a metric space and $(x_n)_{n\in\mathbb{N}}$ be an almost Cauchy sequence in $\mathbf{X}$. Then every subsequence of $(x_n)_{n\in\mathbb{N}}$ is almost Cauchy.*

Proof. Fix an almost Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbf{X}$ and let $(x_{k_n})_{n\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$. Since $(x_n)_{n\in\mathbb{N}}$ is almost Cauchy, it follows that for every $\varepsilon > 0$ there exists $G \in [\mathbb{N}]^{<\omega}$ such that for all $n \in \mathbb{N}$, $x_n \in \bigcup\{B(x_v, \varepsilon/2) : v \in G\}$. For every $v \in G$ with $B(x_v, \varepsilon/2)$ containing terms of $(x_{k_n})_{n\in\mathbb{N}}$, fix $n_v \in \mathbb{N}$ such that $x_{k_{n_v}} \in B(x_v, \varepsilon/2)$. It is straightforward to see that for all $n \in \mathbb{N}$, $x_{k_n} \in \bigcup\{B(x_{k_{n_v}}, \varepsilon) : v \in G\}$, meaning that $(x_{k_n})_{n\in\mathbb{N}}$ is almost Cauchy. □

Theorem 3.4. *Let $\mathbf{X} = (X,d)$ be a metric space and $(x_n)_{n\in\mathbb{N}}$ be an almost Cauchy sequence in $\mathbf{X}$. Then $(x_n)_{n\in\mathbb{N}}$ has a Cauchy subsequence.*

Proof. Fix an almost Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of points of $\mathbf{X}$. If some term of $(x_n)_{n\in\mathbb{N}}$ repeats infinitely often, then $(x_n)_{n\in\mathbb{N}}$ has a constant subsequence and the conclusion follows. So, without loss of generality, we may assume that $(x_n)_{n\in\mathbb{N}}$ is one-to-one. For our convenience, for the rest of the proof we shall identify subsequences of $(x_n)_{n\in\mathbb{N}}$ with their range. Let $Y_0 = (x_n)_{n\in\mathbb{N}}$. Clearly, the set $W = [Y_0]^{<\omega}$ of all finite subsets of Y_0 is well orderable. Via a straightforward induction, we construct a Cauchy subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a family of almost Cauchy subsequences $\{Y_n : n \in \mathbb{N}\}$ of $(x_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, Y_n is a subsequence of Y_{n-1}, the diameter of Y_n is less than $1/n$, and $x_{k_n} \in Y_n$ as follows:

For $n = 1$, we observe, in view of our hypothesis, that Y_0 can be covered by finitely many open balls of radius $1/2$. Let F_1 be the first element of W such that

$$\bigcup\{B(x, 1/2) : x \in F_1\} \supseteq Y_0.$$

Since Y_0 is infinite, it follows that for some $m \in \mathbb{N}$, $x_m \in F_1$, and infinitely many terms of $(x_n)_{n\in\mathbb{N}}$ are included in $B(x_m, 1/2)$. Let k_1 be the first such $m \in \mathbb{N}$ and put $Y_1 = B(x_{k_1}, 1) \cap Y_0$. By Proposition 3.3, Y_1 is an almost Cauchy subsequence of Y_0.

Assume that $k_1 < k_2 < ... < k_n$ have been constructed subject to our induction hypotheses. Since Y_n is an almost Cauchy sequence, it follows that $U_n = Y_n \backslash \{x_i : i \leq k_n\}$ is also an almost Cauchy sequence. Thus, U_n can be covered by finitely many open balls of radius $1/2(n+1)$. Let F_{n+1} be the first element of W such that

$$\bigcup \{B(x, 1/2(n+1)) : x \in F_{n+1}\} \supseteq U_n.$$

Since U_n is infinite, it follows that for some $m \in \mathbb{N}$, $x_m \in F$, and $B(x_m, 1/2(n+1))$ includes infinitely many terms of U_n. Let k_{n+1} be the first such $m \in \mathbb{N}$ and put $Y_{n+1} = B(x_{k_{n+1}}, 1/2(n+1)) \cap Y_n$. Clearly, $k_{n+1} > k_n$, and by Proposition 3.3, Y_{n+1} is an almost Cauchy sequence in $\mathbf{X}$ having diameter less than $1/n$, terminating the induction.

To see that $(x_{k_n})_{n\in\mathbb{N}}$ is a Cauchy sequence, fix $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ satisfy $1/n_0 < \varepsilon$. Clearly, for every $n, m \geq n_0$, $x_{k_n}, x_{k_m} \in Y_{n_0}$, meaning that $d(x_{k_n}, x_{k_m}) < 1/n_0 < \varepsilon$. □

Corollary 3.5. *A metric space $\mathbf{X} = (X, d)$ is complete if and only if it is almost complete.*

Proof. ($\rightarrow$) Fix an almost Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbf{X}$. By Theorem 3.4, $(x_n)_{n\in\mathbb{N}}$ has a Cauchy subsequence $(x_{k_n})_{n\in\mathbb{N}}$, and by the completeness of $\mathbf{X}$, $(x_{k_n})_{n\in\mathbb{N}}$ converges to some point $x \in \mathbf{X}$. Thus, x is a limit point of $(x_n)_{n\in\mathbb{N}}$ and $\mathbf{X}$ is almost complete as required.

($\leftarrow$) Fix a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbf{X}$. Clearly, $(x_n)_{n\in\mathbb{N}}$ is almost Cauchy. Hence, by our hypothesis, some subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ converges to some point $x \in \mathbf{X}$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, it follows that $(x_n)_{n\in\mathbb{N}}$ converges to x. Thus, $\mathbf{X}$ is complete as required. □

By propositions 3.1 and 2.1, it follows in ZFC that

- every complete and almost compact metric space is compact.

Similarly, in ZFC,

- every sequentially compact metric space is almost compact and,
- every countably compact metric space is almost compact.

We show next that none of the above mentioned propositions holds in ZF.

Theorem 3.6. (i) *The statement, "Every complete and almost compact metric space is countably compact" implies CAC_{fin}.*

(ii) *The statement, "Every sequentially compact metric space is almost compact" implies* **IDI**.

(iii) *The statement, "Every countably compact metric space is almost compact" implies* **IWDI**.

Proof. (i) Assume the contrary and let $\mathcal{A} = \{X_n : n \in \mathbb{N}\}$ be a disjoint family of finite non-empty sets such that no infinite subfamily of $\{X_n : n \in \mathbb{N}\}$ has a choice set. Consider the following metric d on $Y = \bigcup\{X_n : n \in \mathbb{N}\}$ given by

$$d(x,y) = \begin{cases} 0, \text{ if } x = y, \\ \max\{1/n, 1/m\}, \text{ if } x \in X_n \text{ and } y \in X_m. \end{cases} \tag{3.1}$$

Claim 1. $\mathbf{Y}$ is complete.

Proof. Fix $(x_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $\mathbf{Y}$. Since $\mathcal{A}$ has no partial choice, it follows that $\{n \in \mathbb{N} : x_m \in X_n \text{ for some } m \in \mathbb{N}\}$ is finite (otherwise, a partial choice set for $\mathcal{A}$ can be defined). Hence, there exists $n_0 \in \mathbb{N}$ such that X_{n_0} includes infinitely many terms of $(x_n)_{n\in\mathbb{N}}$. Since X_{n_0} is finite, it follows that for some $x \in X_{n_0}$, infinitely many terms of $(x_n)_{n\in\mathbb{N}}$ equal x. Hence, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence to x. Thus, $lim_{n\to\infty} x_n = x$ ($(x_n)_{n\in\mathbb{N}}$ is Cauchy) and $\mathbf{Y}$ is complete, finishing the proof of Claim 1.

Claim 2. $\mathbf{Y}$ is almost compact.

Proof. Fix $n \in \mathbb{N}$ and let $\mathcal{U} = \{B(x, 1/n) : x \in X \subseteq Y\}$ be an open cover of $\mathbf{Y}$. Let $k \geq n$ be such that $X_k \cap X \neq \varnothing$ (if $X \subseteq \bigcup\{X_i : i \in n\}$, then $\bigcup\mathcal{U} \subseteq \bigcup\{X_i : i \in n\}$ and $\mathcal{U}$ is not a cover of Y) and fix $x \in X_k \cap X$. Clearly, $\bigcup\{X_i : i > k\} \subseteq B(x, 1/n)$. Since $\bigcup\{X_i : i \leq k\}$ is finite, it follows that finitely many members of $\mathcal{U}$ are needed to cover $\bigcup\{X_i : i \leq k\}$. Hence, $\mathcal{U}$ has a finite subcover and $\mathbf{Y}$ is weakly bounded, finishing the proof of Claim 2.

By our hypothesis and claims 1 and 2, $\mathbf{Y}$ is countably compact. However, since the metric d produces the discrete topology on Y (for every $n \in \mathbb{N}$ and $x \in X_n$, $B(x, 1/n+1) = \{x\}$), it follows that $\mathcal{A}$ is a countable open cover of $\mathbf{Y}$ having no finite subcover. Thus, $\mathbf{Y}$ is not countably compact and this leads to a contradiction.

(ii) and (iii) Assume the contrary and fix a Dedekind-finite set X (weakly Dedekind-finite set X, respectively). Clearly, X with the discrete metric is sequentially compact (countably compact, respectively). Since the open cover $\mathcal{U} = \{B(x, 1) : x \in X\} = \{\{x\} : x \in X\}$ of $\mathbf{X}$ has no finite subcover, it follows that $\mathbf{X}$ is not almost compact and this leads to a contradiction. $\square$

4. Lebesgue Metric Spaces

Theorem 4.1 (ZF). *Every countably Lebesgue metric space is complete.*

Proof. Let $\mathbf{X} = (X, d)$ be a countably Lebesgue metric space. We show that $\mathbf{X}$ is complete. Assume the contrary and fix a non-convergent Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbf{X}$. For every $n \in \mathbb{N}$, let $A_n = \{x_m : m \geq n\}$. We observe that each A_n is a closed subset of $\mathbf{X}$ and $\bigcap\{A_n : n \in \mathbb{N}\} = \varnothing$ because otherwise $(x_n)_{n\in\mathbb{N}}$ will converge to some point in $\mathbf{X}$. It follows that $\mathcal{U} = \{A_n^c : n \in \mathbb{N}\}$ is a countable open cover of $\mathbf{X}$. Thus, by our hypothesis, $\mathcal{U}$ has a Lebesgue number $\delta > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, it follows that there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $d(x_n, x_m) < \delta/2$. Hence, $\delta(A_{n_0}) < \delta$ and, consequently, $A_{n_0} \subseteq A_k^c = \{x_m : m < k\}$ for some $k \in \mathbb{N}$. This is a contradiction, finishing the proof of the theorem. □

Next we show that Lebesgue = almost Lebesgue in ZF.

Theorem 4.2 (ZF). *Let $\mathbf{X} = (X, d)$ be a metric space. Then $\mathbf{X}$ is Lebesgue if and only if it is almost Lebesgue.*

Proof. (i) ($\rightarrow$) Fix an open cover $\mathcal{U}$ of $\mathbf{X}$, and by our hypothesis, let $2\delta > 0$ be a Lebesgue number for $\mathcal{U}$. Then $\mathcal{V} = \{B(x, \delta) : x \in X\}$ is a refinement of $\mathcal{U}$ and $\mathbf{X}$ is almost Lebesgue.

($\leftarrow$) Fix an open cover $\mathcal{U}$ of $\mathbf{X}$, and let $\mathcal{W} = \{B(x, \varepsilon_x/6) : x \in X\}$, where for every $x \in X$,

$$\varepsilon_x = \sup\{t > 0 : B(x, t) \subseteq U \text{ for some } U \in \mathcal{U}\}. \tag{4.1}$$

By replacing $\mathcal{U}$ if necessary with an open refinement of sets of diameter < 1, we may assume that for every $x \in X$, $\varepsilon_x \in \mathbb{R}$. Clearly, $\mathcal{W}$ is an open cover of $\mathbf{X}$. By our hypothesis, let $\mathcal{V} = \{B(x, \delta) : x \in K \subseteq X\}$ be a refinement of $\mathcal{W}$ for some $\delta > 0$. We claim that δ is a Lebesgue number for $\mathcal{U}$. To see this, fix $y \in X$. We show that $B(y, \delta) \subseteq U$ for some $U \in \mathcal{U}$. Fix $x \in K \cap B(y, \delta)$. Since $\mathcal{V}$ is a refinement of $\mathcal{W}$, it follows that $B(x, \delta) \subseteq B(s, \varepsilon_s/6) \subseteq B(s, \varepsilon_s) \subseteq U$ for some $s \in X$ and $U \in \mathcal{U}$. Fix $p \in B(y, \delta)$. We have

$$\begin{aligned} d(p, s) &\leq d(p, y) + d(y, x) + d(x, s) < 2\delta + \varepsilon_s/6 \\ &\leq 2\varepsilon_s/6 + \varepsilon_s/6 = \varepsilon_s/2 < \varepsilon_s. \end{aligned} \tag{4.2}$$

Hence, $B(y, \delta) \subseteq B(s, \varepsilon_s) \subseteq U$ and δ is a Lebesgue number for $\mathcal{U}$ as required. □

Clearly, a metric space $\mathbf{X} = (X, d)$ is compact if and only if for every basic open cover $\mathcal{U} = \{B(x_i, \varepsilon_i) : i \in I, x_i \in X, \varepsilon_i > 0\}$ of $\mathbf{X}$ there exists $i_0, i_1, ..., i_n \in I$ such that $\mathcal{V} = \{B(x_{i_j}, \varepsilon_{i_j}) : j \leq n\}$ is a cover of

X. Clearly, $\varepsilon = \min\{\varepsilon_{i_j} : j \leq n\} > 0$. Hence, the proof of the next proposition is complete.

Proposition 4.3. *A metric space $\mathbf{X} = (X, d)$ is compact if and only if for every basic open cover $\mathcal{U}$ of $\mathbf{X}$ there exists $\varepsilon > 0$ and a finite subcover $\mathcal{V}$ of $\mathcal{U}$ such that for every $V \in \mathcal{V}$, $\delta(V) > 2\varepsilon$.*

Next we show that if in Proposition 4.3 we drop the requirement that the subcover $\mathcal{V}$ of $\mathcal{U}$ be finite, then the resulting theorem characterizes the Lebesgue metric spaces.

Theorem 4.4. *A metric space $\mathbf{X} = (X, d)$ is Lebesgue if and only if for every basic open cover $\mathcal{U}$ of $\mathbf{X}$, there exists $\varepsilon > 0$ and a subcover $\mathcal{V}$ of $\mathcal{U}$ such that for every $V \in \mathcal{V}$, $\delta(V) > 2\varepsilon$.*

Proof. ($\rightarrow$) Fix an open cover $\mathcal{U}$ of $\mathbf{X}$ consisting of open balls. By our hypothesis and Theorem 4.2, there exists $\delta > 0$ and a δ-dense subset K of $\mathbf{X}$ such that $\mathcal{W} = \{B(x, \delta) : x \in K\}$ is a refinement of $\mathcal{U}$. Clearly, $\varepsilon = \delta$ and $\mathcal{V} = \{U \in \mathcal{U} : B(x, \delta) \subseteq U$ for some $x \in K\}$ satisfy for every $V \in \mathcal{V}$, $\delta(V) > 2\varepsilon$.

($\leftarrow$) Fix an open cover $\mathcal{U}$ of $\mathbf{X}$, and let $\mathcal{W} = \{B(x, \varepsilon_x/6) : x \in X\}$, where for every $x \in X$, ε_x is given by (4.1). Let $\delta > 0$, and let $\mathcal{V} \subseteq \mathcal{W}$ be a subcover of $\mathcal{U}$ such that for every $V \in \mathcal{V}$, $\delta(V) > 2\delta$. We claim that δ is a Lebesgue number for the cover $\mathcal{U}$. To see this, fix $y \in X$. We show that $B(y, \delta) \subseteq U$ for some $U \in \mathcal{U}$. Fix $V \in \mathcal{V}$ with $y \in V = B(x, \varepsilon_x/6)$ for some $x \in X$. Clearly, there exists $U \in \mathcal{U}$ such that

$$V \subseteq B(x, 5\varepsilon_x/6) \subseteq U. \tag{4.3}$$

For every $p \in B(y, \delta)$, we have

$$d(p, x) \leq d(p, y) + d(y, x) < \delta + \varepsilon_x/6. \tag{4.4}$$

Since $\varepsilon_x/3 = \delta(V) > 2\delta$, we get that $\delta < \varepsilon_x/6$. Hence, from the latter inequality and (4.4), it follows that $d(p, x) < \varepsilon_x/3$. Therefore, $B(y, \delta) \subseteq B(x, \varepsilon_x/3)$ and, in view of (4.3), it follows that $B(y, \delta) \subseteq B(x, \varepsilon_x/3) \subseteq B(x, 5\varepsilon_x/6) \subseteq U$. Thus, δ is a Lebesgue number for $\mathcal{U}$ as required. $\square$

Our next result shows that if we restrict to second countable metric spaces, then Lebesgue = almost Lebesgue = countably Lebesgue = UC = normal. In particular, the latter notions are equivalent for subspaces of the real line.

Theorem 4.5 (ZF). *Let $\mathbf{X} = (X, d)$ be a second countable metric space. Then the following are equivalent:*

(i) **X** *is Lebesgue;*
(ii) **X** *is countably Lebesgue;*

(iii) *every countable open cover* $\mathcal{U}$ *of* $\mathbf{X}$ *has a refinement* $\mathcal{V} = \{B(x, \delta) : x \in K \subseteq X\}$ *for some* $\delta > 0$*;*

(iv) *every continuous real valued function on* $\mathbf{X}$ *is uniformly continuous;*

(v) $\mathbf{X}$ *is normal.*

Proof. Fix a countable base $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ for $\mathbf{X}$.

(i) $\to$ (ii) and (ii) $\to$ (iii) These are straightforward.

(ii) $\to$ (i) Fix an open cover $\mathcal{U}$ of $\mathbf{X}$. Clearly, $\mathcal{W} = \{B \in \mathcal{B} : B \subseteq U$ for some $U \in \mathcal{U}\}$ is a countable open refinement of $\mathcal{U}$. Hence, by our hypothesis, $\mathcal{W}$ has a Lebesgue number $\delta > 0$. It is straightforward to see that δ is a Lebesgue number for $\mathcal{U}$ and $\mathbf{X}$ is Lebesgue as required.

(iii) $\to$ (i) Fix an open cover $\mathcal{U}$ of $\mathbf{X}$. For every $x \in X$, let ε_x be given by (4.1) and let B_x be the first member of $\mathcal{B}$ such that $x \in B_x \subseteq B(x, \varepsilon_x/6)$. Clearly, $\mathcal{W} = \{B_x : x \in X\}$ is a countable open cover of $\mathbf{X}$. By our hypothesis, let $\mathcal{V} = \{B(x, \delta) : x \in K \subseteq X\}$ be a refinement for the cover $\mathcal{W}$ for some $\delta > 0$. We show that for every $y \in X$, there is a $U \in \mathcal{U}$ such that $B(y, \delta) \subseteq U$. Fix $x \in K \cap B(y, \delta)$. Since $\mathcal{V}$ is a refinement of $\mathcal{W}$, it follows that there exists $s \in X$ with $B(x, \delta) \subseteq B_s \subseteq B(s, \varepsilon_s/6) \subseteq B(s, \varepsilon_s) \subseteq U$ for some $U \in \mathcal{U}$. Using (4.2) we can show that $B(y, \delta) \subseteq U$. Thus, δ is a Lebesgue number of $\mathcal{U}$ and $\mathbf{X}$ is Lebesgue.

(i) $\to$ (iv) $\to$ (v) This follows from Theorem 2.6.

(v) $\to$ (i) Assume, aiming for a contradiction, that $\mathbf{X}$ is not Lebesgue and let $\mathcal{U}$ be an open cover of $\mathbf{X}$ without a Lebesgue number. Clearly, for every $n \in \mathbb{N}$, the set

$$A_n = \{x \in X : \forall U \in \mathcal{U}, B(x, 1/n) \nsubseteq U\}$$

is non-empty. Let $K = \bigcap\{\overline{A_n} : n \in \mathbb{N}\}$.

Claim. $K = \varnothing$.

Proof. Assume the contrary and fix $x \in K$. Since $\mathcal{U}$ is a cover of $\mathbf{X}$, it follows that $x \in U$ for some $U \in \mathcal{U}$. Since U is open, we see that $B(x, 1/n) \subseteq U$ for some $n \in \mathbb{N}$. Since $x \in \overline{A_{2n}}$, it follows that $B(x, 1/2n) \cap A_{2n} \neq \varnothing$. Fix $z \in B(x, 1/2n) \cap A_{2n}$. We claim that $B(z, 1/2n) \subseteq B(x, 1/n) \subseteq U$. Indeed, if $y \in B(z, 1/2n)$, then by the triangle inequality, we have $d(x, y) \leq d(x, z) + d(z, y) < 1/2n + 1/2n = 1/n$, meaning that $y \in B(x, 1/n)$. Thus, $z \notin A_{2n}$, and we have arrived at a contradiction.

By the claim, the family $\mathcal{A} = \{\overline{A_n} : n \in \mathbb{N}\}$ has an infinite strictly decreasing subfamily $\mathcal{C}$. Without loss of generality, we may assume that $\mathcal{A} = \mathcal{C}$. For every $n \in \mathbb{N}$, put $Y_n = \overline{A_n} \backslash \overline{A_{n+1}}$. It is easy to see that for every $n \in \mathbb{N}$,

$$A_n \backslash \overline{A_{n+1}} \neq \varnothing. \tag{4.5}$$

Clearly, for every subspace $\mathbf{Y}$ of $\mathbf{X}$, the restriction $\mathcal{B}_Y$ of the base $\mathcal{B}$ to $\mathbf{Y}$, i.e., $\mathcal{B}_Y = \{B_n \cap Y : n \in \mathbb{N}\}$, is a countable base for $\mathbf{Y}$. Hence, the well ordering of $\mathcal{B}$ can be used to define a well ordering on $\mathcal{B}_Y$. In the sequel we shall assume that whenever a subspace $\mathbf{Y}$ of $\mathbf{X}$ is given, a well ordering of $\mathcal{B}_Y$ is also given. Since $\{\{x\} : x \in \mathrm{Iso}(Y)\} \subseteq \mathcal{B}_Y$, we can assume that a well ordering of $\mathrm{Iso}(Y)$ is also given. We consider the following two cases.

Case 1. $\forall^\infty n \in \mathbb{N}$ (for all but finitely many $n \in \mathbb{N}$), $\mathbf{Y}_n$ is discrete. For our convenience, we assume that for all $n \in \mathbb{N}$, $\mathbf{Y}_n$ is discrete. In this case, for every $n \in \mathbb{N}$, $Y_n = \mathrm{Iso}(Y_n)$. Hence, $\overline{A_1} = \bigcup\{Y_n : n \in \mathbb{N}\}$, being a countable union of countable sets is countable. Fix an enumeration $\{a_i : i \in \omega\}$ of $\overline{A_1}$ and for every $n \in \mathbb{N}$, let, by (4.5), $x_n \in A_n \setminus \overline{A_{n+1}}$ and

$$L_n = B(x_n, 1/n) \cap \overline{A_1} \setminus \{x_n\}.$$

We consider the following subcases.

Case 1.1. $\exists^\infty n \in \mathbb{N}$ (there are infinitely many $n \in \mathbb{N}$), such that $L_n \neq \varnothing$. Let $k_1 = \min\{m \in \mathbb{N} : L_m \neq \varnothing\}$, and for $n = v+1$, let $k_n = \min\{m \in \mathbb{N} : m > k_v \text{ and } L_m \neq \varnothing\}$. For our convenience, assume that for every $n \in \mathbb{N}$, $k_n = n$. For every $n \in \mathbb{N}$, let y_n be the first member of $\overline{A_1}$ which lies in L_n. We claim that $(y_n)_{n\in\mathbb{N}}$ has no limit point. Indeed, if y is a limit point of $(y_n)_{n\in\mathbb{N}}$, then it can be easily seen that y is also a limit point of $(x_n)_{n\in\mathbb{N}}$. Hence, $y \in K \neq \varnothing$, and this is a contradiction.

We construct, via an easy induction, two disjoint closed sets $H = \{x_{k_n} : n \in \mathbb{N}\}$ and $F = \{y_{k_n} : n \in \mathbb{N}\}$ of $\mathbf{X}$ with $d(H, F) = 0$ contradicting the fact that $\mathbf{X}$ is normal.

For $n = 1$, let $k_1 = 1$.

For $n = v + 1$, we let

$$k_n = min\{m \in \mathbb{N} : y_m \notin \{y_{k_i}, x_{k_i} : i \leq v\}\}.$$

Since no term of $(y_n)_{n\in\mathbb{N}}$ repeats infinitely often and $\{y_{k_i}, x_{k_i} : i \leq v\}$ is a finite set, it follows that k_n is well defined, terminating the induction.

It is straightforward to verify H and F are closed and disjoint. Since for every $n \in \mathbb{N}$, $lim_{n\to\infty} d(y_{k_n}, x_{k_n}) = 0$, it follows that $d(H, F) = 0$. Hence, H and F are as required.

Case 1.2. $\forall^\infty n \in \mathbb{N}$, $L_n = \varnothing$. For our convenience, we assume that for all $n \in \mathbb{N}$, $L_n = \varnothing$. We construct inductively a disjointed sequence $(C_n)_{n\in\mathbb{N}}$ of closed sets of $\mathbf{X}$ such that for every $n \in \mathbb{N}$, $0 < d(x_n, C_n) < 2/n$ and C_n is contained in $\overline{A_1}^c$.

For $n = 1$, we observe that $B(x_1, 1) \cap \overline{A_1}^c \neq \varnothing$. If not, then $B(x_1, 1) = \{x_1\}$, and since $\mathcal{U}$ is an open cover of $\mathbf{X}$, $x_1 \in U$ for some $U \in \mathcal{U}$, meaning that $x_1 \notin A_1$, and this leads to a contradiction. Let B be the first member of $\mathcal{B}$ such that $\overline{B} \subseteq B(x_1, 1) \cap \overline{A_1}^c$, $B \subseteq B(x, r)$ for some $x \in B$, and $0 < r < 1/2$, and put $C_1 = \overline{B}$.

For $n = v+1$, as in the $n = 1$ case, $B(x_n, 1/n) \cap \overline{A_1}^c \neq \varnothing$. Let B be the first member of $\mathcal{B}$ such that $\overline{B} \subseteq B(x_n, 1/n) \cap \overline{A_1}^c, B \subseteq B(x, r)$ for some $x \in B$, and $0 < r < 1/2n$, and put $C_n = \overline{B}$. Clearly, $0 < d(x_n, C_n) < 2/n$, terminating the induction.

We claim that the disjoint sets $H = \{x_n : n \in \mathbb{N}\}$ and $F = \bigcup\{C_n : n \in \mathbb{N}\}$ are closed. That H is closed is evident. (If x is a limit point of H, then $x \in K$, contradicting the fact that $K = \varnothing$). To see that F is closed, we assume the contrary and fix $x \in \overline{F} \backslash F$. Clearly, every neighborhood V of x meets infinitely many members of $\{C_n : n \in \mathbb{N}\}$. If not, then $x \in C_n \subseteq F$, and this leads to a contradiction.

Let $\varepsilon > 0$ and pick $n_0 \in \mathbb{N}$ with $1/n_0 < \varepsilon/8$. Let $m > n_0$ such that $C_m \cap B(x, \varepsilon/4) \neq \varnothing$. Fix $y \in C_m \cap B(x, \varepsilon/4)$ and $t \in C_m$ with $d(x_m, t) < 2/m + \varepsilon/8$. We have

$$\begin{aligned} d(x_m, x) &< d(x_m, t) + d(t, y) + d(x, y) < 2/m + \varepsilon/8 + 1/m + \varepsilon/4 \\ &= 3/m + 3\varepsilon/8 < 6\varepsilon/8 < \varepsilon. \end{aligned}$$

Hence, $x_m \in B(x, \varepsilon)$ for all $m > n_0$ such that $C_m \cap B(x, \varepsilon/4) \neq \varnothing$. Since there are infinitely many such $m \in \mathbb{N}$, it follows that x is a limit point of H and this leads to a contradiction. Hence, F is closed as required.

Since F and H are non-empty closed disjoint subsets of $\mathbf{X}$ with $d(F, H) = 0$, it follows that $\mathbf{X}$ is not normal and this leads to a contradiction. Hence, Case 1.2 and, consequently, Case 1, cannot be the case.

Case 2. $\exists^\infty n \in \mathbb{N}$, $\mathbf{Y}_n$ has a limit point. Fix a strictly increasing sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$ such that $\forall^\infty n \in \mathbb{N}$, $\mathbf{Y}_{k_n}$ has a limit point. Fix for every $n \in \mathbb{N}$ a limit point a_n of $\mathbf{Y}_{k_n}$ and a point $b_n \in B(a_n, 1/k_n) \cap Y_{k_n} \backslash \{a_n\}$. Since $Y_{k_n} \cap Y_{k_m} = \varnothing$ for all $m, n \in \mathbb{N}$ where $m \neq n$, it follows that the sets $H = \{a_n : n \in \mathbb{N}\}$ and $F = \{b_n : n \in \mathbb{N}\}$ are non-empty and disjoint. Arguing as in Case 1, we can show that they are also closed subsets of $\mathbf{X}$. Clearly, $d(H, F) = 0$, meaning that $\mathbf{X}$ is not normal, and this leads to a contradiction. So, Case 2 cannot be the case.

Cases 1 and 2 lead to a contradiction. Thus, $\mathbf{X}$ is Lebesgue and the proof of the theorem is complete. $\square$

Remark 4.6. (i) We remark here that if $\mathbf{X} = (X, d)$ is a separable metric space, then $\mathbf{X}$ is second countable. Hence, the conclusion of Theorem 4.5 holds true for separable metric spaces.

(ii) Clearly, the assumption that $\mathbf{X}$ is second countable is not needed for the proof of (i) $\to$ (ii) in Theorem 4.5. However, in view of Theorem 2.6, second countability is needed for the proof of the converse. Also, second countability is not needed for the proof of (ii) $\to$ (iii). We do not know whether, in ZF or in ZFC, second countability is needed for the proof of (iii) $\to$ (ii) of Theorem 4.5.

(iii) We observe, in view of Theorem 2.6(a) and Theorem 4.5, that if $\mathbf{X}$ is countably Lebesgue but not Lebesgue, then $\mathbf{X}$ cannot be second countable, hence separable also.

Theorem 4.7 (ZF). *Let $\mathbf{X} = (X, d)$ be a separable metric space. Then, for every countable open cover $\mathcal{U}$ of $\mathbf{X}$, there exists $\delta > 0$ and a subset K of X such that $\mathcal{V} = \{B(x, \delta) : x \in K\}$ is a refinement of $\mathcal{U}$ if and only if for every countable open cover $\mathcal{U}$ of $\mathbf{X}$, there exists $\delta > 0$ and a countable subset H of X such that $\mathcal{V} = \{B(x, \delta) : x \in H\}$ is a refinement of $\mathcal{U}$.*

Proof. Fix $D = \{d_n : n \in \mathbb{N}\}$ a countable dense subset of $\mathbf{X}$.

($\leftarrow$) This is straightforward.

($\rightarrow$) Fix $\mathcal{U}$ a countable open cover of $\mathbf{X}$. By our hypothesis and Theorem 4.5, $\mathcal{U}$ has a Lebesgue number 4δ for some $\delta > 0$. Clearly, $\mathcal{V} = \{B(x, 2\delta) : x \in X\}$ is a refinement of $\mathcal{U}$.

For every $x \in X$, let $n_x = \min\{n \in \mathbb{N} : d_n \in B(x, \delta)\}$. It is easy to see that $x \in B(d_{n_x}, \delta) \subseteq B(x, 2\delta) \subseteq U$ for some $U \in \mathcal{U}$. Hence, $\mathcal{W} = \{B(d_{n_x}, \delta) : x \in X\}$ is a countable refinement of $\mathcal{U}$. □

Corollary 4.8 (ZF). *Let $\mathbf{X} = (X, d)$ be a metric space. The following are equivalent.*

(i) $\mathbf{X}$ *is compact.*
(ii) $\mathbf{X}$ *is Lebesgue and totally bounded.*
(iii) $\mathbf{X}$ *is almost compact and countably compact.*
(iv) $\mathbf{X}$ *is almost compact and almost Lebesgue.*

Proof. (i) $\leftrightarrow$ (ii) Proposition 2.1.

(i) $\rightarrow$ (iii), (iv) $\rightarrow$ (ii), and (i) $\rightarrow$ (iv) are straightforward.

(iii) $\rightarrow$ (ii) This follows from Theorem 4.2 and Proposition 3.1. □

5. Sequentially Bounded and Almost Cauchy Metric Spaces

Our first result in this section is straightforward and the proof is left for the reader as an easy exercise.

Theorem 5.1 (ZF). *Let $\mathbf{X} = (X, d)$ be a metric space. The following are equivalent.*

(i) $\mathbf{X}$ *is almost Cauchy.*
(ii) *Each countable subspace of* $\mathbf{X}$ *is almost compact.*
(iii) *Each countable subspace of* $\mathbf{X}$ *is totally bounded.*

Next we give a characterization of the property of almost Cauchy which points to compactness if we read "every open cover" as "every sequence" and "finite subcover" as "almost Cauchy subsequence." We also show that each statement in Theorem 5.1 is equivalent to the proposition that $\mathbf{X}$ is sequentially bounded.

Theorem 5.2 (ZF)**.** *Let $\mathbf{X} = (X, d)$ be a metric space. The following are equivalent.*

(i) *$\mathbf{X}$ is sequentially bounded.*
(ii) *For every sequence $(x_n)_{n\in\mathbb{N}}$ of $\mathbf{X}$ and for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for infinitely many $n \in \mathbb{N}$, $x_n \in B(x_{n_\varepsilon}, \varepsilon)$.*
(iii) *$\mathbf{X}$ is almost Cauchy.*
(iv) *Every sequence in $\mathbf{X}$ admits an almost Cauchy subsequence.*

Proof. (i) $\to$ (ii) This is straightforward.

(ii) $\to$ (iii) If the sequence $(x_n)_{n\in\mathbb{N}}$ is not almost Cauchy, then there is an $\varepsilon_0 > 0$ such that the cover $\{B(x_n, 0) : n \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$ does not have a finite subcover. Define the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ recursively by $n_1 = 1$ and $n_{k+1} =$ the least $j \in \mathbb{N}$ such that $j > n_k$ and $x_{n_j} \notin B(x_{n_1}, \varepsilon_0) \cup B(x_{n_1}, \varepsilon_0) \cup ... \cup B(x_{n_k}, \varepsilon_0)$. Then $(x_{n_k})_{k\in\mathbb{N}}$ and ε_0 provide a counterexample for (ii).

(iii) $\to$ (iv) This is straightforward.

(iv) $\to$ (i) Fix a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $\mathbf{X}$. By our hypothesis, $(x_n)_{n\in\mathbb{N}}$ has an almost Cauchy subsequence $(x_{k_n})_{n\in\mathbb{N}}$. Hence, by Theorem 3.4, $(x_{k_n})_{n\in\mathbb{N}}$, and consequently, $(x_n)_{n\in\mathbb{N}}$ have a Cauchy subsequence. Thus, $\mathbf{X}$ is totally bounded as required. $\square$

Corollary 5.3. *A metric space is sequentially compact if and only if it is almost Cauchy and almost complete.*

Proof. Fix a metric space $\mathbf{X} = (X, d)$. By Proposition 3.1, $\mathbf{X}$ is sequentially compact if and only if it is complete and sequentially bounded if and only if $\mathbf{X}$ is, by Theorem 5.2, almost complete and almost Cauchy. $\square$

Corollary 5.4. (i) (ZF + CAC) *A metric space is totally bounded if and only if each of its countable subspaces is totally bounded.*
(ii) *The statement, "Every metric space $\mathbf{X} = (X, d)$ such that each of its countable subspaces is totally bounded, is totally bounded" implies* **IDI**.

Proof. (i) ($\to$) This is straightforward and CAC is not needed.

($\leftarrow$) Fix $\mathbf{X} = (X, d)$ a metric space such that each of its countable subspaces is totally bounded. By theorems 5.1 and 5.2, $\mathbf{X}$ is sequentially bounded. Hence, by Proposition 2.5, $\mathbf{X}$ is totally bounded.

(ii) This follows at once from theorems 5.1 and 5.2 and Proposition 2.5. However, a direct proof is easier. If X is a Dedekind-finite set and

d is the discrete metric on X, then each of the countable subspaces of $\mathbf{X}$ being finite is totally bounded. Hence, by our hypothesis, $\mathbf{X}$ is totally bounded, and this leads to a contradiction. $\square$

Remark 5.5. (i) Metric spaces $\mathbf{X} = (X, d)$ satisfying Theorem 5.2(ii), i.e., for every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbf{X}$ and for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for infinitely many $n \in \mathbb{N}$, $x_n \in B(x_{n_\varepsilon}, \varepsilon)$ are called *cofinally Cauchy* in [9]. It has been proved in [9] that the classes of cofinally Cauchy and sequentially bounded metric spaces coincide in ZF with the class of all pseudo Cauchy metric spaces, i.e., spaces in which every sequence $(x_n)_{n\in\mathbb{N}}$ satisfies

- for each $\varepsilon > 0$ and for each $n_0 \in \mathbb{N}$, there exist $n, m \in \mathbb{N}$, $n, m \geq n_0$, and $n \neq m$ with $d(x_n, x_m) < \varepsilon$.

Hence, in view of Theorem 5.2,

- sequentially bounded = cofinally Cauchy = almost Cauchy = pseudo Cauchy.

Earlier, Gerald Beer [5] proved that Theorem 5.2(ii) was equivalent in ZFC to total boundedness.

(ii) We observe that

(1) Complete $\nrightarrow$ sequentially bounded and sequentially bounded $\nrightarrow$ complete. ($\mathbb{R}$ with the usual metric is complete but not sequentially bounded, and the subspace $(0, 1)$ of $\mathbb{R}$ is sequentially bounded but not complete.)

(2) Lebesgue $\nrightarrow$ totally bounded and totally bounded $\nrightarrow$ Lebesgue. (Any infinite set with the discrete metric is Lebesgue but not totally bounded, and the subspace $(0, 1)$ of $\mathbb{R}$ is totally bounded but not Lebesgue.)

(3) Countably Lebesgue $\nrightarrow$ sequentially bounded and sequentially bounded $\nrightarrow$ countably Lebesgue. ($\mathbb{N}$ with the discrete metric is countably Lebesgue but not sequentially bounded, and the subspace $(0, 1)$ of $\mathbb{R}$ is sequentially bounded but not countably Lebesgue.) Since sequentially compact $\rightarrow$ sequentially bounded, it follows that countably Lebesgue $\nrightarrow$ sequentially compact also.

(4) Sequentially compact $\nrightarrow$ countably Lebesgue in ZF. In the basic Cohen model $\mathcal{M}1$ in [6], the set A of all added Cohen reals is a Dedekind finite, dense subset of $\mathbb{R}$ avoiding $\mathbb{Q}$. Hence, the subspace $\mathbf{Y} = (0, 1) \cap A$ is sequentially compact as every sequence of A has finite range. However, $\mathbf{Y}$ is not countably Lebesgue because the open cover $\mathcal{U} = \{(1/n + 1, 1/n) \cap A : n \in \mathbb{N}\}$ has no Lebesgue number.

(5) In view of Proposition 2.5 and Theorem 5.2, it is consistent with ZF that the existence of a non-totally bounded metric space $\mathbf{X} = (X, d)$ such that each countable subspace of $\mathbf{X}$ is totally bounded.
(6) The conclusion of Corollary 5.4(i) does not hold for almost compact metric spaces. Indeed, $\mathbf{X} = [0, 1]$ as a subspace of $\mathbb{R}$ with the usual metric is almost compact but, in view of the proof of Proposition 3.1(iii), $Y = (0, 1] \cap \mathbb{Q}$ is a non-almost compact countable subspace of $\mathbf{X}$.

The following diagram summarizes the ZF implications and nonimplications of the properties of metric spaces studied in this paper.

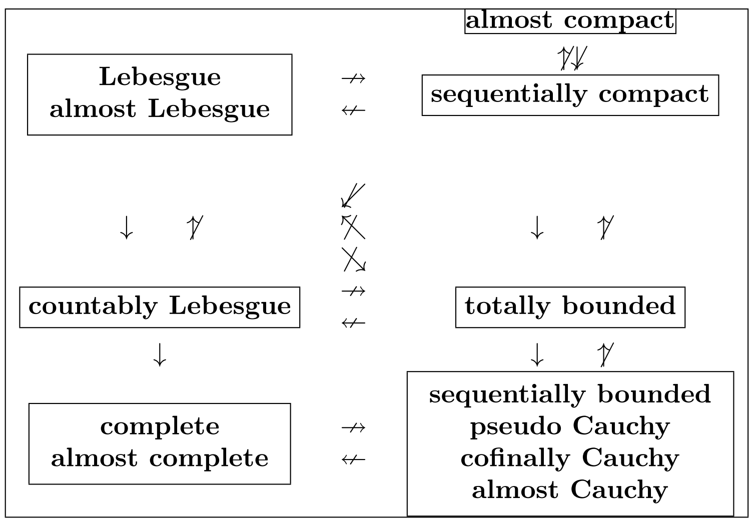

Diagram 2.

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