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A MONOTONICALLY RETRACTABLE REALCOMPACT SPACE WHICH IS NOT LINDELÖF

MASAMI SAKAI

Abstract. We construct a monotonically retractable realcompact space which is not Lindelöf. This answers a question posed by R. Rojas-Hernández in *Function spaces and D-property* [Topology Proc. **43** (2014)].

1. Introduction

Throughout this paper, all spaces are assumed to be Tychonoff. For a set S, $[S]^{\leq\omega}$ stands for the set of countable subsets in S. A space having a countable network is said to be *cosmic*, where a family $\mathcal{N}$ of subsets of a space X is said to be a *network* for X if for any $x \in X$ and any neighborhood U of x, there exists some $N \in \mathcal{N}$ such that $x \in N \subset U$.

For a space X, let $C_p(X)$ be the space of all real-valued continuous functions of X with the topology of pointwise convergence. For each $n \in \mathbb{N}$, let $C_{p,n}(X)$ be the n-times iterated function space of X. G. A. Sokolov ([9], [10]) proved that $C_{p,n}(K)$ of a Corson compact space K is Lindelöf for each $n \in \mathbb{N}$. Motivated by Sokolov's result, Vladimir V. Tkachuk introduced the following.

Definition 1.1 ([11])**.** A space X is *Sokolov* if for any sequence $\{F_n : n \in \mathbb{N}\}$ with F_n closed in X^n, there exists a continuous map $f : X \to X$ such that $f(X)$ is cosmic and $f^n(F_n) \subset F_n$ for each $n \in \mathbb{N}$.

A Corson compact space is Sokolov, and all the spaces $C_{p,n}(X)$ are Lindelöf for a Sokolov space X with an additional condition [11, Theorem 2.1]. A Sokolov space is collectionwise normal, ω-monolithic (i.e., the

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closure of each countable subset is cosmic), and the cardinality of each closed discrete subspace of a Sokolov space is countable [11, Proposition 2.2]. And Tkachuk asked the following question.

Question 1.2 ([11, Problem 3.9])**.** Must every Sokolov realcompact space be Lindelöf?

To study the D-property of function spaces $C_p(X)$, R. Rojas-Hernández introduced the class of monotonically retractable spaces.

Definition 1.3 ([5])**.** A space X is *monotonically retractable* if we can assign to any $A \in [X]^{\leq\omega}$ a set $K(A) \subset X$, a continuous retraction $r_A : X \to K(A)$, and a countable family $\mathcal{N}(A)$ of subsets of X such that

(r1) $A \subset K(A)$;
(r2) if W is an open subset in $K(A)$, then $r_A^{-1}(W) = \bigcup \mathcal{N}$ for some $\mathcal{N} \subset \mathcal{N}(A)$;
(r3) if $A, B \in [X]^{\leq\omega}$ and $A \subset B$, then $\mathcal{N}(A) \subset \mathcal{N}(B)$;
(r4) if $A_n \in [X]^{\leq\omega}$ for each $n \in \omega$, $A_n \subset A_{n+1}$, and $A = \bigcup\{A_n : n \in \omega\}$, then $\mathcal{N}(A) = \bigcup\{\mathcal{N}(A_n) : n \in \omega\}$.

Obviously each cosmic space is monotonically retractable, in particular, so is each countable space. Rojas-Hernández [5] proved that $C_p(X)$ of a monotonically retractable space X is a Lindelöf space with the D-property. Moreover, a monotonically retractable space is Sokolov [6, Corollary 4.9]. Hence, Rojas-Hernández asked the next question.

Question 1.4 ([5, Question 4.4])**.** Suppose that X is a monotonically retractable realcompact space. Must X be Lindelöf?

We show that the answer to this question is in the negative.

2. An Example

Lemma 2.1 ([1, Theorem 8.17, Corollary 8.15])**.** *The following statements hold.*

(1) *If a space Y is hereditarily realcompact and there exists a continuous map $\tau : X \to Y$ such that $\tau^{-1}(y)$ is compact for each $y \in Y$, then X is realcompact.*
(2) *If a space X is realcompact and each point of X is a G_δ-set, then X is hereditarily realcompact.*

Our example considered here is the same as in [8].

Proposition 2.2. *There exists a monotonically retractable, hereditarily realcompact space X which is not Lindelöf.*

Proof. Fix any second countable space Y with $|Y| = \omega_1$ and let $Z = Y \times \omega_1$. For each $\alpha < \omega_1$, we define Z_α, r_α, $\mathcal{B}_\alpha$, and $\mathcal{N}_\alpha$. Let $Z_\alpha = Y \times [0, \alpha]$. We define a map $r_\alpha : Z \to Z_\alpha$ as follows: for each $(y, \beta) \in Z$, $r_\alpha((y, \beta)) = (y, \beta)$ if $\beta \le \alpha$; $r_\alpha((y, \beta)) = (y, \alpha)$ if $\beta > \alpha$. Then r_α is a continuous retraction. Let $\mathcal{B}_Y$ be a countable base for Y, and let

$$\mathcal{B}_\alpha = \{B \times (\beta, \gamma] : B \in \mathcal{B}_Y, \beta < \gamma \le \alpha\}.$$

Then $\mathcal{B}_\alpha$ is a countable base for Z_α and $\alpha < \alpha'$ implies $\mathcal{B}_\alpha \subset \mathcal{B}_{\alpha'}$. Let

$$\mathcal{N}_\alpha = \{r_\alpha^{-1}(B) : B \in \mathcal{B}_\alpha\}.$$

By the definition of $\mathcal{B}_\alpha$,

$$\mathcal{N}_\alpha = \{B \times (\beta, \gamma] : B \in \mathcal{B}_Y, \beta < \gamma < \alpha\} \cup \{B \times (\beta, \omega_1) : B \in \mathcal{B}_Y, \beta < \alpha\}.$$

Then $\mathcal{N}_\alpha$ is a countable open cover of Z and satisfies the following:

(a) if W is an open set in Z_α, then $r_\alpha^{-1}(W) = \bigcup \mathcal{N}$ for some $\mathcal{N} \subset \mathcal{N}_\alpha$;
(b) if $\alpha \le \alpha'$, then $\mathcal{N}_\alpha \subset \mathcal{N}_{\alpha'}$;
(c) if $\alpha_n < \omega_1$ for each $n \in \omega$, $\alpha_n \le \alpha_{n+1}$, and $\alpha = \sup\{\alpha_n : n \in \omega\}$, then $\mathcal{N}_\alpha = \bigcup\{\mathcal{N}_{\alpha_n} : n \in \omega\}$.

Conditions (a) and (b) can be easily checked. We observe (c). If $\alpha = \alpha_n$ for some $n \in \omega$, then the conclusion obviously holds. Assume $\alpha_n < \alpha$ for each $n \in \omega$. Let $N \in \mathcal{N}_\alpha$. If N is of the form $N = B \times (\beta, \gamma]$, where $B \in \mathcal{B}_Y$ and $\beta < \gamma < \alpha$, take an $n \in \omega$ with $\gamma < \alpha_n < \alpha$, then we have $N \in \mathcal{N}_{\alpha_n}$. If N is of the form $N = B \times (\beta, \omega_1)$, where $B \in \mathcal{B}_Y$ and $\beta < \alpha$, take an $n \in \omega$ with $\beta < \alpha_n < \alpha$, then we have $N \in \mathcal{N}_{\alpha_n}$.

Now fix an onto map $\varphi : Y \to \omega_1$. Let

$$X = \{(y, \alpha) \in Z : \alpha \le \varphi(y)\}.$$

By Lemma 2.1, X is hereditarily realcompact. However, it is not Lindelöf because ω_1 is a continuous image of X. We see that X is monotonically retractable. Let $A \in [X]^{\le\omega}$, and if $A = \{(y_n, \alpha_n) \in X : n \in \omega\}$, we put $\alpha(A) = \sup\{\alpha_n : n \in \omega\}$. We define $K(A)$, r_A, and $\mathcal{N}(A)$ naturally. Let $K(A) = X \cap Z_{\alpha(A)}$. Obviously, $A \subset K(A)$ (condition (r1)) holds. Recall the retraction $r_{\alpha(A)} : Z \to Z_{\alpha(A)}$. By the definitions of $r_{\alpha(A)}$ and X, the inclusion $r_{\alpha(A)}(X) \subset K(A)$ can be easily checked. Hence, the restricted map $r_A = r_{\alpha(A)} \restriction X : X \to K(A)$ is a retraction. Let

$$\mathcal{N}(A) = \{X \cap N : N \in \mathcal{N}_{\alpha(A)}\}.$$

This family is a countable open cover of X. We examine condition (r2). Let W be an open set in $K(A)$, and take an open set W' in $Z_{\alpha(A)}$ such that $W = K(A) \cap W'$. By (a), $r_{\alpha(A)}^{-1}(W') = \bigcup \mathcal{N}$ for some $\mathcal{N} \subset \mathcal{N}_{\alpha(A)}$. Hence,

$$r_A^{-1}(W) = X \cap r_{\alpha(A)}^{-1}(W') = X \cap \left(\bigcup \mathcal{N}\right) = \bigcup\{X \cap N : N \in \mathcal{N}\}.$$

Condition (r3) easily follows from (b). Finally, we examine condition (r4). Suppose $A_n \in [X]^{\leq\omega}$ for each $n \in \omega$, $A_n \subset A_{n+1}$, and $A = \bigcup\{A_n : n \in \omega\}$. Then $\alpha(A) = \sup\{\alpha(A_n) : n \in \omega\}$ holds. By (c), we have $\mathcal{N}_{\alpha(A)} = \bigcup\{\mathcal{N}_{\alpha(A_n)} : n \in \omega\}$. This implies $\mathcal{N}(A) = \bigcup\{\mathcal{N}(A_n) : n \in \omega\}$. Thus, X is monotonically retractable. □

Tkachuk [11, Corollary 2.13] proved that every compact Sokolov space X is Fréchet-Urysohn and $C_p(X)$ is Lindelöf. We give a slight improvement of Tkachuk's result. For a point $x \in X$, a family $\mathcal{P}$ of subsets of X is said to be a *π-network at x* if every neighborhood of x contains some member of $\mathcal{P}$. According to Gary Gruenhage and Paul J. Szeptycki [2], a space X is said to be *Fréchet-Urysohn for finite sets* if for every point $x \in X$ and a π-network $\mathcal{P}$ at x consisting of non-empty finite subsets of X, there exists a sequence $\{P_n : n \in \omega\} \subset \mathcal{P}$ converging to x (i.e., every neighborhood of x contains P_n for all but finitely many $n \in \omega$). This notion was first studied systematically by E. A. Reznichenko and O. V. Sipacheva [4]. A space X is said to have *countable supertightness* [3] if for every point $x \in X$ and a π-network $\mathcal{P}$ at x consisting of non-empty finite subsets of X, there exists a countable subfamily $\mathcal{Q} \subset \mathcal{P}$ such that $\mathcal{Q}$ is a π-network at x.

Proposition 2.3. *Every compact Sokolov space is Fréchet-Urysohn for finite sets.*

Proof. Let K be a compact Sokolov space, and let $\mathcal{P}$ be a π-network at $x \in X$ consisting of non-empty finite subsets of X. Since $C_p(X)$ is Lindelöf [11, Corollary 2.13], X has countable supertightness [7]. Take a countable subfamily $\mathcal{Q} \subset \mathcal{P}$ such that $\mathcal{Q}$ is a π-network at x. Since K is ω-monolithic [11, Proposition 2.2], $\overline{\bigcup \mathcal{Q}}$ is a compact cosmic space, so $\overline{\bigcup \mathcal{Q}}$ is metrizable. Hence, $\mathcal{Q}$ contains a sequence converging to x. □

Remark 2.4. Tkachuk asked in [11, Problem 3.15]: Is it true that any Sokolov space has a point-countable π-base? The answer to this question is in the negative. Let G be a countable topological group which is not metrizable. Since G is countable, it is monotonically retractable (hence, Sokolov). If G has a point-countable π-base $\mathcal{B}$, then $\mathcal{B}$ is countable and $\{B \cdot B^{-1} : B \in \mathcal{B}\}$ is a countable neighborhood base at the identity $e \in G$, so G must be metrizable.

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