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by

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REOPENING SEVERAL QUESTIONS CONCERNING $\mathcal{P}$-CLOSED SPACES

R. M. STEPHENSON, JR.

Abstract. Examples are given to point out gaps or errors in certain recently published assertions, proofs, and answers to questions concerning $\mathcal{P}$-closed spaces. In particular, examples are given to show that it is still unknown if a Urysohn (regular) space in which every closed subset is Urysohn-closed (regular-closed) must be compact.

1. Introduction and Terminology

All hypothesized spaces are Hausdorff. A space in which any two distinct points have disjoint closed neighborhoods is called a *Urysohn space*. We recall that for a topological property $\mathcal{P}$, a $\mathcal{P}$-space which is a closed subspace of any $\mathcal{P}$-space in which it can be embedded is called $\mathcal{P}$-closed. In [13] and [15] proofs were given that every space in which every closed subset is Hausdorff-closed is compact. Other researchers later raised analogous questions by asking if every space in which every closed subset is Urysohn-closed must be compact [2] and if every space in which every closed subset is regular-closed must be compact [1].

In the recent articles, James E. Joseph and Bhamini M. P. Nayar [12] and Terrence A. Edwards, et al. [6] present proofs that for $\mathcal{P}$ = Hausdorff, Urysohn, or regular, a $\mathcal{P}$-space in which every closed subset is $\mathcal{P}$-closed must be compact. In this note we point out, however, that certain spaces and filter bases provide counterexamples to some of their assertions and proofs, and, consequently, the latter two questions above are not settled in their articles. We also review several other assertions, proofs, and

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questions given or considered in [6] and indicate why several of these questions are still open and a number of the results claimed in [6] are not true or are in need of clarification or correction.

Some terms used below are the following. For a space X, point $p \in X$, and filter base $\mathcal{F}$ on X, p is called

- an *adherent point* (a *θ-adherent point*) of $\mathcal{F}$ if every (closed) neighborhood of p and every set in $\mathcal{F}$ have nonempty intersection;
- a *u-adherent point* of $\mathcal{F}$ provided that for every open set U containing the closure of some open set containing p and every set $F \in \mathcal{F}$, $F \cap \overline{U} \neq \emptyset$;
- an *s-adherent point* of $\mathcal{F}$ provided that for every shrinkable family $\mathcal{V}$ of open neighborhoods of p there exists $V \in \mathcal{V}$ such that $F \cap V \neq \emptyset$ for every set $F \in \mathcal{F}$, where the word *shrinkable* means that for every $V \in \mathcal{V}$ there exists $W \in \mathcal{V}$ with $\overline{V} \subset W$;
- or an *sw-adherent point* of $\mathcal{F}$ provided that for every shrinkable family $\mathcal{V}$ of open neighborhoods of p and every set $F \in \mathcal{F}$ there exists $V \in \mathcal{V}$ such that $F \cap V \neq \emptyset$.

For clarification purposes, throughout this note we use the term "s-adherent point" as defined above to mean "s-adherent point" as defined by Larry L. Herrington [9], and we use the term "sw-adherent point" as defined above to mean "s-adherent point" as defined in [12] and [6] since the authors of [9] and [12] assigned different, nonequivalent meanings to the same term "s-adherent point." We will also use this convention for similar terms, such as "s-adherence" or "sw-adherence" of a filter base and when stating assertions in [12] or [6]. Since the authors of [12] and [6] stated that the concept of sw-adherence (which they called "s-adherence") was due to Herrington, and they were using some of his results concerning s-adherence, they may not have intended to change the meaning in his definition (by reordering some of the quantified expressions in it).

Previously published articles note that for $\mathcal{P}$ = Hausdorff, Urysohn, or regular, a $\mathcal{P}$-space X is $\mathcal{P}$-closed if and only if every filter base $\mathcal{F}$ on X has a θ-, a u-, or an s-adherent point, respectively. These characterizations, the last two of which were given in [8] and [9], reformulate, in interesting ways, analogous ones obtained earlier, e.g., in [1], [3], and [10]. We also recall and mention that for the properties considered here, $\mathcal{P}$-minimal implies $\mathcal{P}$-closed. Our terminology generally agrees with that in such articles as [2] and [3].

2. The Examples

The examples below, which may be of some independent interest, provide further information about properties of different types of adherent points in $\mathcal{P}$-closed spaces. In sections 3 and 4 we will note why they illustrate gaps or errors in some of the assertions and proofs in [12], [11], and [6].

Example 2.1. *There exist a minimal Urysohn space H and an (open) ultrafilter on H which has two u-adherent points, but no convergent point.*

Proof. In [10, Example 4] Horst Herrlich modified a subspace of a space due to A. Tychonoff [16] in order to show that not every minimal Urysohn space is regular. For convenience, so that we can later use some statements in [3] and not re-prove them here, we slightly change the notation in Herrlich's example to the following. Let T denote the Tychonoff plank described as $T = \{(\alpha, \beta) : \alpha \leq \omega_1 \text{ and } \beta \leq \omega_0\} \setminus \{(\omega_1, \omega_0)\}$, where ω_0 (ω_1) denotes the first infinite (uncountable) ordinal. Let A denote the quotient space obtained by identifying, in the product space $T \times \{-1, 0, 1\}$, where T has its usual topology, each point $(\omega_1, \beta, -1)$ with $(\omega_1, \beta, 0)$ and each point $(\alpha, \omega_0, 0)$ with $(\alpha, \omega_0, 1)$. Finally, let $H = A \cup \{p, q\}$, where p and q are two points not in A, and a subset V of H is defined to be open if and only if

- $V \cap A$ is open in A, and
- if $p \in V$, then $\{(\alpha, \beta, -1) : \alpha_0 < \alpha < \omega_1 \text{ and } \beta_0 < \beta \leq \omega_0\} \subset V$ for some $\alpha_0 < \omega_1$ and $\beta_0 < \omega_0$, and
- if $q \in V$, then $\{(\alpha, \beta, 1) : \alpha_0 < \alpha \leq \omega_1 \text{ and } \beta_0 < \beta < \omega_0\} \subset V$ for some $\alpha_0 < \omega_1$ and $\beta_0 < \omega_0$.

Next, define $\mathcal{F}$ to be the filter base $\{F(\alpha_0, \beta_0) : \alpha_0 < \omega_1 \text{ and } \beta_0 < \omega_0\}$, where for each $\alpha_0 < \omega_1$ and $\beta_0 < \omega_0$, $F(\alpha_0, \beta_0) = \{(\alpha, \beta, 0) : \alpha_0 < \alpha < \omega_1, \beta_0 < \beta < \omega_0, \text{ and } \alpha \text{ is a nonlimit ordinal}\}$. Finally, let $\mathcal{U}$ be any filter on the set $F(0,0)$ such that $\mathcal{F} \subset \mathcal{U}$. We observe that $\mathcal{U}$ is an open filter base on H, and if $\mathcal{U}$ is an ultrafilter on $F(0,0)$, then it is a base for an ultrafilter on H that is also an open ultrafilter on H since the points in $F(0,0)$ are all isolated points of H.

We wish to show that $\mathcal{U}$ has both p and q as u-adherent points, but neither as an adherent point. Using the well-known properties of T, we note the following.

Let U, V, and W be arbitrary open sets such that $p \in V \subset \overline{V} \subset W$ and $U \in \mathcal{U}$. Then $\{(\alpha, \beta, -1) : \alpha_0 < \alpha < \omega_1 \text{ and } \beta_0 < \beta \leq \omega_0\} \subset V$ for some $\alpha_0 < \omega_1$ and $\beta_0 < \omega_0$. Hence, $\{(\omega_1, \beta, 0) \equiv (\omega_1, \beta, -1) : \beta_0 < \beta < \omega_0\} \subset \overline{V}$, which implies that for some $\alpha_1 < \omega_1$, $\{(\alpha, \beta, 0) : \alpha_1 < \alpha \text{ and } \beta_0 < \beta < \omega_0\} \subset W$. As the latter shows that W contains a member of $\mathcal{F}$, namely $F(\alpha_1, \beta_0)$, and $\mathcal{F} \subset \mathcal{U}$, it follows that $U \cap W \neq \emptyset$. Thus, p is a u-adherent point of $\mathcal{U}$.

Let U, V, and W be arbitrary open sets such that $q \in V \subset \overline{V} \subset W$ and $U \in \mathcal{U}$. Then $\{(\alpha, \beta, 1) : \alpha_0 < \alpha \leq \omega_1 \text{ and } \beta_0 < \beta < \omega_0\} \subset V$ for some

$\alpha_0 < \omega_1$ and $\beta_0 < \omega_0$. Thus, $\{(\alpha, \omega_0, 0) \equiv (\alpha, \omega_0, 1) : \alpha_0 < \alpha < \omega_1\} \subset \overline{V} \subset W$. Since $\mathcal{F} \subset \mathcal{U}$, then $U \cap F(\alpha, \beta) \neq \emptyset$ for all $\alpha < \omega_1$ and $\beta < \omega_0$. Hence, there exist strictly increasing sequences of ordinal numbers $\{\alpha_n\}$ and $\{\beta_n\}$, where each $\alpha_0 < \alpha_n < \omega_1$, $\beta_n < \omega_0$, and $(\alpha_n, \beta_n, 0) \in U$. Let $\lim_{n\to\infty} \alpha_n = \gamma < \omega_1$. Then $(\gamma, \omega_0, 0) \in W \cap \overline{U}$, and so the open sets U and W must satisfy $U \cap W \neq \emptyset$. Thus, q is a u-adherent point of $\mathcal{U}$.

Since p and q have neighborhoods which do not intersect the set $F(0, 0)$, neither point is an adherent point of $\mathcal{F}$, and $\mathcal{F}$ has no adherent point in $H \setminus \{p, q\}$, so the same is true of $\mathcal{U}$. □

Example 2.2. *There exist a Urysohn-closed space S and an (open) ultrafilter on S which has a unique u-adherent, but no convergent point.*

Proof. It follows from some of the above statements and properties of the Tychonoff plank that the subspace $S = \{q\} \cup \{(\alpha, \beta, n) : n = 0 \text{ or } 1\}$ of the space H and the filter base $\mathcal{F}$ can be used to illustrate Example 2.2. □

Example 2.3. *There exist a minimal regular space B and an (open) ultrafilter on B which has two s-adherent points, but no convergent point.*

Proof. In [3], Manuel P. Berri and R. H. Sorgenfrey modified a space due to Tychonoff [16] in order to show that not every minimal regular space is compact. We slightly modify their notation. Let T be as in Example 2.1, and let A denote the quotient space obtained by identifying in the product space $T \times \mathbb{Z}$ (where T has its usual topology and $\mathbb{Z}$ has the discrete topology) each point $(\omega_1, \beta, 2n-1)$ with $(\omega_1, \beta, 2n)$ and each point $(\alpha, \omega_0, 2n)$ with $(\alpha, \omega_0, 2n+1)$, for each $n \in \mathbb{Z}$. Finally, let $H = A \cup \{p, q\}$, where p and q are two points not in A, and a subset V of H is defined to be open if and only if

- $V \cap A$ is open in A, and
- if $p \in V$, then for some $n \in \mathbb{Z}$, $T \times \{m \in \mathbb{Z} : m \leq n\} \subset V$, and
- if $q \in V$, then for some $n \in \mathbb{Z}$, $T \times \{m \in \mathbb{Z} : m \geq n\} \subset V$.

Next, with respect to this space, let $\mathcal{F}$ and $\mathcal{U}$ be defined as in Example 2.1, and note that as was the case for the space H, if $\mathcal{U}$ is an ultrafilter on $F(0, 0)$, then it is a base for an ultrafilter on B which is also an open ultrafilter on B.

In order to show that $\mathcal{U}$ has both p and q as s-adherent points, but neither as an adherent point, we will use the terminology given and remarks established in statement 2 on page 456 and statement 3 on page 457 in [3], re-stated for B as follows.

We will say that a set V *gets into the n-corner* if whenever $\alpha_0 < \omega_1$ and $\beta_0 < \omega_0$, there is a point $(\alpha, \beta, n) \in V$ for some $\alpha_0 < \alpha$ and $\beta_0 < \beta$. (a) If the open set V gets into the n-corner, then there is an infinite sequence $\{\beta_i\}$ of distinct ordinals $< \omega_0$ such that each $(\omega_1, \beta_i, n) \in \overline{V}$. (b) If V_1, V_2, and V_3 are open sets such that V_1 gets into the n-corner and $V_1 \subset \overline{V}_1 \subset V_2 \subset \overline{V}_2 \subset V_3$, then V_3 gets into the $(n-1)$- and $(n+1)$-corners.

Suppose $\mathcal{V}$ is an arbitrary shrinkable family of open neighborhoods of p. We wish to show that for some $V \in \mathcal{V}, V \cap U \neq \emptyset$ for every set $U \in \mathcal{U}$. Choose $V_1 \in \mathcal{V}$. Then for some integer k, $\bigcup\{T \times \{n\} : n \leq k\} \subset V_1$, so V_1 gets into the n-corner for all $n \leq k$. If $k \geq 0$, then $V_1 \supset T \times \{0\}$, and hence $V_1 \cap U \neq \emptyset$ for every set $U \in \mathcal{U}$. Suppose $k < 0$. Since $\mathcal{V}$ is shrinkable, there exist $2(-k)+2$ sets $V_i \in \mathcal{V}$ where $2 \leq i \leq 2(-k)+3$ such that for each i, $i = 1, 2, \ldots, 2(-k)+2$, $\overline{V}_i \subset V_{i+1}$. By applying statement (b) $-k$ times, it follows from (a) that there is an infinite sequence $\{\beta_i\}$ of distinct ordinals $< \omega_0$ such that each $(\omega_1, \beta_i, 0) \in \overline{V}_{2(-k)+1}$. Let $U \in \mathcal{U}$ be arbitrary. Then reasoning as in the penultimate paragraph of the proof of Example 2.1, one can show that $V_{2(-k)+3} \cap U \neq \emptyset$. Therefore, p is an s-adherent point of $\mathcal{U}$.

The proof that q is an s-adherent point of $\mathcal{U}$ is similar.

Since p and q have neighborhoods which miss $T \times \{0\}$, neither is an adherent point of $\mathcal{F}$, and no other point of B is an adherent point of $\mathcal{F}$, so $\mathcal{U}$ cannot have an adherent point in the space B. □

Example 2.4. *There exist a regular-closed space E and an (open) ultrafilter $\mathcal{U}$ on E such that $\mathcal{U}$ has no convergent point, but has a unique s-adherent point q which is also its only sw-adherent point.*

Proof. The subspace $E = \{q\} \cup \{(\alpha, \beta, n) : n \geq 0\}$ of the space B and the filter base $\mathcal{F}$ of Example 2.3 can be used to illustrate Example 2.4. In [10, Example 2], Herrlich modified an example of Tychonoff [16] to obtain E, and he used it to provide a negative answer to Berri and Sorgenfrey's question in [3]: "Is every regular-closed space minimal regular?" Since $\mathcal{F}$ is free, and every point of E other than q has a neighborhood base consisting of clopen sets, it follows from statements similar to those justifying the properties of Example 2.3 that E has the stated (open) ultrafilter properties. □

While it is immediate from the definitions recorded in the previous section that s-adherence implies sw-adherence, we briefly describe an example showing that the converse is not true.

Example 2.5. *There exist a regular space X and a regular filter base $\mathcal{F}$ on X which has an sw-adherent point p, but no s-adherent point. (An open filter base $\mathcal{F}$ is called* regular *if each set in $\mathcal{F}$ contains the closure of some set in $\mathcal{F}$.)*

Proof. Let X be the subspace $B \setminus \{q\}$ of the space B in Example 2.3, and let $\mathcal{F} = \{F_n : n \in \mathbb{Z},\ n \geq 0\}$, where each $F_n = (T \times \{m \in \mathbb{Z} : m \geq n+1\}) \cup \{(\alpha, \beta, n) : \alpha < \omega_1 \text{ and } \beta < \omega_0\}$. Note that each F_n is an open set and $F_n \supset \overline{F_{n+1}}$, so $\mathcal{F}$ is a regular filter base.

To show that p is an sw-adherent point of $\mathcal{F}$, let $\mathcal{V}$ be any shrinkable family of open neighborhoods of p in X, and consider any $F_n \in \mathcal{F}$. By slightly modifying the paragraph three of the proof in Example 2.3, one can show that there is a set $V \in \mathcal{V}$ such that $V \cap F_n \neq \emptyset$.

To show that p is not an s-adherent point of $\mathcal{F}$, define $\mathcal{V} = \{X \setminus \overline{F_n} : F_n \in \mathcal{F}\}$. Then $\mathcal{V}$ is a shrinkable family of open neighborhoods of p, and obviously for any $V \in \mathcal{V}$ there exists $F \in \mathcal{F}$ such that $V \cap F = \emptyset$. Since $\mathcal{F}$ is free, and all the other points of X have neighborhood bases consisting of clopen sets, no other point of X is an s-adherent point of $\mathcal{F}$. (For an alternate proof of this, one can just notice that $\mathcal{F}$ has no adherent point in X, and then appeal to the theorem in [9] that in a regular space any s-adherent point of a regular filter base is also an adherent point of that filter base.) □

The next remark will be used in the following sections.

Remark 2.6. On any space which is not regular there exists an ultrafilter having a unique θ-adherent point, but no convergent point, and on any space which is not Urysohn there exists an ultrafilter which has two θ-adherent points, but no convergent point.

3. Why It Is Still Unknown If a Space in Which Every Closed Subset Is Urysohn-Closed (Regular-Closed) Must Be Compact

In [12, lemmas 3.2 and 3.3], Joseph and Nayar state that in a Urysohn-closed (regular-closed) space every open ultrafilter has a single u-adherent (sw-adherent) point, and they use these lemmas to provide parts of their proofs of theorems 3.4 and 3.5, which state that a space in which every closed subset is Urysohn-closed (regular-closed) must be compact. Our examples 2.1 and 2.3 illustrate that lemmas 3.2 and 3.3 are not true. In addition, in their proofs of theorems 3.4 and 3.5, they seem to be asserting without proof that in a Urysohn (regular) space if an ultrafilter has a unique u-adherent (sw-adherent) point, then it is convergent. Examples 2.2 and 2.4 above illustrate that, at least in a Urysohn-closed (regular-closed) space, those assertions need not hold. Remark 2.6 points out a gap in their proof of Lemma 3.1 in [12], the known theorem that a space in which every closed subset is Hausdorff-closed must be compact. In the last line of their proof they seem to be asserting, without stating what hypothesis they are invoking, that any ultrafilter on the space must have a unique θ-adherent point and therefore must be convergent. Furthermore, in all three of the proofs of Lemma 3.1, Theorem 3.4, and Theorem 3.5, their conclusion that a certain open ultrafilter has an adherent (u-adherent) [sw-adherent] point does not require or need to use the hypothesis that besides the entire space, every closed subspace of the space be $\mathcal{P}$-closed for the appropriate value of $\mathcal{P}$. Another omission is that in [12] the authors do not mention that the theorem in Lemma 3.1 was obtained previously by M. Katětov [13] and M. Stone [15], or that in [11] they had provided a virtually identical proof of this same theorem, although in [11] they did state that they were providing a new proof of the theorem and that Katětov and Stone had obtained it and provided different proofs in their respective papers.

4. Additional Related Assertions Needing Clarification Or Correction

In response to an e-mail message I sent to the authors of [12] and [11], one of them sent me a copy of [6]. (This occurred after I had submitted the original version of this note, i.e., the original version of the preceding three sections, to *Topology Proceedings*, and it had been accepted for publication.) In [6] the authors state that they are giving different proofs of the answers to the three questions presented in [12] (and in [11] for the first of the three questions), and they present a number of other assertions concerning $\mathcal{P}$-closed spaces. Unfortunately, these new proofs also contain gaps or errors, as do some of the assertions. We base this on the details discussed below.

The two main assertions in [6] used to justify several of the new proofs in that article are in lemmas 3.1 and 3.2. Lemma 3.1 of [6] states that in a Hausdorff-closed (Urysohn-closed) [regular-closed] space every open ultrafilter has a single point of adherence (u-adherence) [sw-adherence]. The second and third statements of this lemma restate [12, lemmas 3.2 and 3.3], which we have shown are not true. Lemma 3.2 of [6] states that if $\mathcal{W}$ is an ultrafilter on any space X and $\mathcal{O} = \{V : V \text{ is open in } X \text{ and } V \supset W \text{ for some } W \in \mathcal{W}\}$, then $\mathcal{O}$ is an open ultrafilter, and $adh_\theta \mathcal{W} = adh \mathcal{O}$ ($adh_u \mathcal{W} = adh_u \mathcal{O}$) [$adh_{sw} \mathcal{W} = adh_{sw} \mathcal{O}$], where each prefix adh, adh_θ, ... denotes the set of adherent, θ-adherent,... points of the designated filter or filter base. But the following example illustrates that its conclusion, "$\mathcal{O}$ is an open ultrafilter," is not always true.

Example 4.1. *To continue Remark 2.6 above, let X be any space containing two points p and q which do not have disjoint closed neighborhoods (such as the well-known noncompact, minimal Hausdorff space of Urysohn described, e.g., in* [4] *and* [2, Example 3.14]*), let $\mathcal{W}$ be any ultrafilter on X*

which contains the family of all closed neighborhoods of p *and the family of all closed neighborhoods of* q*, and let* $\mathcal{O}$ *be the open filter base defined as in* [6, Lemma 3.2]*. Then* $\mathcal{O}$ *has at least two adherent points,* p *and* q*, and hence it is not a base for any open ultrafilter on* X*. Moreover,* $\mathcal{W}$ *has no convergent point.*

Proof. For any $O \in \mathcal{O}$ and open neighborhood N of p, $\overline{N} \in \mathcal{W}$ and $O \in \mathcal{W}$, so $\overline{N} \cap O \neq \emptyset$, and thus $N \cap O \neq \emptyset$, which shows that p is an adherent point of $\mathcal{O}$. Similarly, one notes that q is an adherent point of $\mathcal{O}$. Since X is Hausdorff, $\mathcal{W}$ has no adherent point. $\square$

We consider next some specific instances in [6] of the authors' use of lemmas 3.1 and 3.2 in providing answers to questions or proofs of theorems.

Their Theorem 3.3 states that if every closed subset of a Hausdorff (Urysohn) [regular] space X is Hausdorff-closed (Urysohn-closed) [regular-closed], then $adh_\theta\mathcal{W}$ $(adh_u\mathcal{W})$ $[adh_{sw}\mathcal{W}]$ is a singleton for any ultrafilter $\mathcal{W}$ on X, and thus X is compact. But their proof of Theorem 3.3 begins with the statements

> Let $\mathcal{W}$ be an ultrafilter on X and let $\mathcal{O}$ be the open ultrafilter from Lemma 3.2. Then from lemmas 3.1 and 3.2, $adh_\theta\mathcal{W}$ $(adh_u\mathcal{W})$ $[adh_{sw}\mathcal{W}] = adh\mathcal{O}$ $(adh_u\mathcal{O})$ $[adh_{sw}\mathcal{O}] = \{x\}$ for some $x \in X$.

However, our examples 4.1, 2.1, and 2.3 show that $\mathcal{O}$ may have more than one adherent (u-adherent) [sw-adherent] point, and we also note that in their application of lemmas 3.1 and 3.2, the authors did not need to use the hypothesis that besides X, every other closed subset of X is Hausdorff-closed (Urysohn-closed) [regular-closed]. Then they conclude their proof by asserting, without giving any justification, "Therefore $\mathcal{W}$ converges to x, and X is compact."

The authors also begin the proofs of theorems 3.5 and 3.14 with the same use of lemmas 3.1 and 3.2 that was made of them for the proof of Theorem 3.3, and they assert that Theorem 3.5 improves Theorem 3.3, and Theorem 3.14 affirmatively answers the question of Mike Girou [7] and J. Vermeer [18]: If every Hausdorff-closed subspace of a Hausdorff-closed space is minimal Hausdorff, must the space be compact? So this question is still open. But in the case of Theorem 3.5, it is difficult for the reader to interpret the statements in it, for the reasons noted next.

Theorem 3.5 states that if every closed subset of a Hausdorff (Urysohn) [regular] space X is an H-set (a U-set) [an R-set], then $adh_\theta\mathcal{W}$ $(adh_u\mathcal{W})$ $[adh_{sw}\mathcal{W}]$ is a singleton for any ultrafilter $\mathcal{W}$ on X, and thus X is compact. In Definition 3.4 the authors state

> A set A is called an H-set (U-set) [R-set] if every open filter base Ω on A satisfies $A \cap adh\Omega$ ($A \cap adh_u\Omega$) [$A \cap adh_{sw}\Omega$] $\neq \emptyset$.

In this definition there is no reference to a containing space, so either they left it out by mistake, or they want the reader to interpret these concepts to be absolute rather than relative. If the latter is the case, then as a result of previously derived characterizations by others of $\mathcal{P}$-closed spaces (referred to in the first section of this note), Theorem 3.5 has the same meaning as Theorem 3.3 for the cases H-set or U-set and is ambiguous for the case R-set (since Herrington's characterization of regular-closed spaces does not involve sw-adherence). However, later in [6], the authors state that they are giving a new proof of a result by Vermeer in [18], which would suggest to the reader that they intend for an *H-set* to have the same meaning as the following one assigned to it by Vermeer and other authors, such as N. V. Velicko [17], R. F. Dickman, Jr., and Jack R. Porter [5], or Porter and John Thomas [14] (who used the term *H-closed relative to* X): A subset A of a space X is said to be an *H-set in* X (or, more briefly, an *H-set*) provided that every cover of A by sets open in X has a finite subfamily whose closures in X cover A. (It is known, e.g., see [18], that this definition is equivalent to the condition that if Ω is any open filter base on a space X such that $O \cap A \neq \emptyset$ for every $O \in \Omega$, then there is an adherent point x of Ω in the space X such that $x \in A$.) Under this usual meaning of H-set, the first statement in [6, Theorem 3.5] is not true, as shown by the following.

Example 4.2. *Dickman and Porter* [5, p. 410] *note that the requirement that every closed subset of a space be an H-set is the same as the requirement that the space be C-compact in the sense of* [19]*, and they remark that there are noncompact, C-compact spaces, e.g., as first proved by Giovanni Viglino in* [19].

The authors assert in Corollary 3.15, a corollary to Theorem 3.14, that they are providing a new proof of Vermeer's result in [18]: A Hausdorff-closed space in which every H-set is minimal Hausdorff is compact; however, as noted above, the proofs of theorems 3.3, 3.5, and 3.14 begin similarly.

We next state two more theorems, [6, theorems 3.7 and 3.8], and give examples to show that they are not always true. First, some additional notation given in [6] is needed. For a subset A of a space X, $cl_u(A)$ denotes the set of all points $x \in X$ such that for every open set U containing a closed neighborhood of x, $\overline{U} \cap A \neq \emptyset$, and $cl_{sw}(A)$ denotes the set of all points $x \in X$ such that for every shrinkable family $\mathcal{S}$ of open neighborhoods of x and every $S \in \mathcal{S}$, $S \cap A \neq \emptyset$. (Since $\{V, X\}$ is a shrinkable family of open neighborhoods of x for any open neighborhood V of x, the set $cl_{sw}(A)$ could instead have been defined simply to be $\overline{A}$.) Theorems 3.7 and 3.8 (combined) state that if V is any open set in a Urysohn-closed (regular-closed) space X, then $cl_u(V)$ $(cl_{sw}(V))$ is Urysohn-closed (regular-closed).

Example 4.3. *Let S (E) be the Urysohn-closed (regular-closed) space in our Example 2.2 (Example 2.4), and $V = F(0,0)$. Then V is an open subset of S (E), $cl_u(V)$ $(cl_{sw}(V))$ equals the set $W_u = \{q\} \cup (T \times \{0\})$ $(W_{sw} = T \times \{0\})$, and since $T \times \{0\}$ is obviously a clopen, Tychonoff, non-compact subspace of W_u (W_{sw}), the space $cl_u(V)$ $(cl_{sw}(V))$ is not Urysohn-closed (regular-closed).*

The authors use theorems 3.7, 3.8, and 3.9 to form part of the proof of Theorem 3.10; they assert that the proof of Theorem 3.10 provides a new proof of a theorem of Girou in [7] and a similar result for Urysohn-closed spaces and regular-closed spaces, and they use theorems 3.3 and 3.9 to obtain Theorem 3.13. Unfortunately, they do not provide correct proofs of these theorems.

Finally, we note that each statement in [6, Theorem 3.9] is either not true or not new. According to this theorem,

> A Hausdorff (Urysohn) [regular] space [X] is Hausdorff-closed (Urysohn-closed) [regular-closed] if and only if for every open cover Λ of X there is a finite $\Lambda^* \subset \Lambda$ such that $\{cl(W) : W \in \Lambda^*\}$ $(\{cl_u(W) : W \in \Lambda^*\})$ $[\{cl_{sw}(W) : W \in \Lambda^*\}]$ covers X.

The first statement is due to Alexandroff and Urysohn, e.g., see [2] or Engelking's *General Topology*. Since $cl_{sw}(W) = \overline{W}$ for every subset W of X, the third statement is false, as the stated condition in it is equivalent to the requirement that the space X be Hausdorff-closed. The example below shows that the second statement is also false.

Example 4.4. *Let S be the Urysohn-closed space in our Example 2.2. There exists an open cover $\mathcal{W}$ of S such that for every finite $\mathcal{F} \subset \mathcal{W}$, $\{cl_u(W) : W \in \mathcal{F}\}$ does not cover S.*

Proof. Herrlich remarks in [10, Example 4] that the space B as in our Example 2.1 is not Hausdorff-closed, and as noted in examples 2.1 and 2.2, neither is S. Since S is not Hausdorff-closed, there is an open cover $\mathcal{W}$ of S such that for every finite $\mathcal{F} \subset \mathcal{W}$, $\{\overline{W} : W \in \mathcal{F}\}$ does not cover S. Hence, for every finite $\mathcal{F} \subset \mathcal{W}$, infinitely many points of S are not covered by $\{\overline{W} : W \in \mathcal{F}\}$. Because each point $x \in S \setminus \{q\}$ has a neighborhood base of clopen subsets of S, it follows that for every subset

W of S, $cl_u(W) \subset \{q\} \cup \overline{W}$. Thus, for every finite $\mathcal{F} \subset \mathcal{W}$, infinitely many points of S are not covered by $\{cl_u(W) : W \in \mathcal{F}\}$. $\square$

5. Concluding Remark

It is important to any research community to have correct information about the status of unsolved problems. Our purpose in writing this note is to try to provide correct information to the topological community about the status of such long standing open problems as some of the ones considered in the articles [12], [6], and [11], and to point out that in those articles some of the assertions, proofs, and claimed new proofs of known results contain gaps or errors.

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