Geodesic Curves on Shimura Surfaces

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Electronically published on September 18, 2017

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Dedicated to the memory of Colin Maclachlan

ABSTRACT. We parametrize the commensurability classes of curves on Shimura surfaces that are totally geodesic, i.e., the commensurability classes of so-called $\mathbb{C}$-Fuchsian subgroups. In particular, if a Shimura surface contains one commensurability class of totally geodesic curves, it contains infinitely many.

1. INTRODUCTION

A Shimura surface is the quotient of either the product $\mathbf{H}^2 \times \mathbf{H}^2$ of two hyperbolic planes or the unit ball $\mathbf{H}^2_{\mathbb{C}}$ in $\mathbb{C}^2$ by an irreducible arithmetic lattice. Examples include the normal quasiprojective varieties associated with the Hilbert and Picard modular groups, along with the solutions to many moduli problems for principally polarized abelian varieties. Special amongst the immersed projective algebraic curves on these surfaces are those which are geodesic for the metric descending from the universal covering. In this paper, we completely classify the geodesic curves on Shimura surfaces up to commensurability. A consequence of this classification is the following.

Theorem 1.1. *Let S be a Shimura surface. If S contains one geodesic curve, then it contains infinitely many that are pairwise incommensurable.*

More specifically, we give a parametrization of the commensurability classes of geodesic curves that appear on a given Shimura surface. We note that there are indeed situations, for both $\mathbf{H}^2 \times \mathbf{H}^2$ and $\mathbf{H}^2_{\mathbb{C}}$, where there are no such curves. See Theorem 3.1 and Theorem 4.1 for the precise statements. We also note that this problem is equivalent to parametrizing the commensurability classes of what are often called $\mathbb{C}$-Fuchsian subgroups of $\pi_1(S)$.

The parametrization we give is via quaternion algebras over number fields. Briefly, this is the correct way to parametrize commensurability classes of arithmetic Fuchsian groups because the subalgebra of $\mathrm{M}_2(\mathbb{R})$ generated by a finite covolume Fuchsian group in $\mathrm{SL}_2(\mathbb{R})$ is a quaternion algebra (see [9, §3.2]). When the Fuchsian group is arithmetic, this quaternion algebra (along with the number field over which it is defined) is a complete commensurability invariant [9, §8.4].

These results originally appeared in a 2012 preprint [4], where we applied Theorem 1.1 to prove a result on generating fundamental groups of Shimura surfaces using supports of divisors made up of geodesic curves. That application can be strengthened using very recent work of Martin Möller and Domingo Toledo [13] or Vincent Koziarz and Julien Maubon [7]. Since the classification of geodesic curves is of independent interest, we decided to write the present note.

We now make some historical remarks. Such parametrizations are known for other classes of arithmetic lattices. The first result of this kind was the parametrization of Fuchsian subgroups of arithmetic Kleinian groups by Colin Maclachlan and A. W. Reid [10]. Jeffrey S. Meyer parameterized commensurability classes of geodesic hyperbolic submanifolds of certain arithmetic hyperbolic n-manifolds [12]. Very recently, Jouni Parkkonen and Frédéric Paulin gave a classification for Picard modular groups [15]. This is a special case of our work, but Parkkonen and Paulin state the classification in slightly different language; we describe the equivalence between the two, which follows from elementary quaternion algebra arithmetic, at the end of §4. Finally, recent work of the second author with Benjamin Linowitz studies the generalization of this problem to arbitrary products of hyperbolic 2- and 3-spaces [8], showing that the dichotomy "infinitely many commensurability classes or none at all" does not hold more generally.

The paper is organized as follows. In §2, we give some general facts on lattices in Lie groups and apply them to the cases of interest in this paper. In §3, we give the classification for $\mathbf{H}^2 \times \mathbf{H}^2$. In §4, we give the classification for $\mathbf{H}^2_{\mathbb{C}}$.

2. GENERALITIES ON LATTICES

Let G be a semisimple Lie group with finite center and associated symmetric space X of noncompact type. Then X is called *Hermitian symmetric* when it carries a G-invariant complex structure [6, Ch. VIII]. The quotient of X by any cocompact (nonuniform, respectively) lattice Γ in G is a projective (quasiprojective, respectively) algebraic variety [1]. Suppose H is a semisimple Lie subgroup of G with finite center and associated symmetric space Y. There is then a totally geodesic embedding $Y \to X$, and the totally geodesic subspaces of $\Gamma \backslash X$ arise from the subgroups H of this kind such that $\Gamma \cap H$ is a lattice in H; see [14, §1.4].

When Γ is an *arithmetic* subgroup of G (see [14, §3.6] for definitions), it follows that $\Gamma \cap H$ is an arithmetic subgroup of H. More precisely, suppose K_0 is a number field and that $\mathcal{G}$ is a K_0-algebraic group such that $\mathcal{G}(\mathbb{R} \otimes_{\mathbb{Q}} K_0)$ modulo compact factors is isomorphic to G. Let Γ be an arithmetic lattice commensurable with the lattice of K_0-integral points of $\mathcal{G}$. Then connected totally geodesic submanifolds of $\Gamma \backslash X$ are determined by semisimple F-algebraic subgroups of $\mathcal{G}$ for certain $F \subseteq K_0$.

The cases of interest here are where G is either $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SU}(2,1)$. The respective hermitian symmetric domains are the product $\mathbf{H}^2 \times \mathbf{H}^2$ of two hyperbolic planes or the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$. Therefore, the possible holomorphically embedded totally geodesic submanifolds are *Fuchsian curves*, that is, quotients of $\mathbf{H}^2$ by cocompact arithmetic subgroups of $\mathrm{SL}_2(\mathbb{R})$. These lattices are often called *arithmetic Fuchsian groups*. See [18] and [9] for an account of the basic theory of arithmetic Fuchsian groups.

We recall the construction of lattices in $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ in §3 and consider $\mathrm{SU}(2,1)$ in §4.

3. LATTICES ACTING ON $\mathbf{H}^2$ AND $\mathbf{H}^2 \times \mathbf{H}^2$

We recall from [3, §3] the construction of irreducible arithmetic lattices acting on products of hyperbolic planes. Let $\mathbf{H}^2$ be the hyperbolic plane and G (X, respectively) be the product of n copies of $\mathrm{SL}_2(\mathbb{R})$ ($\mathbf{H}^2$, respectively). Suppose that K is a totally real number field and A a quaternion algebra over K that is ramified at exactly n archimedean places of K. Then there is an algebra surjection $\pi : A \otimes_K \mathbb{R} \to \mathrm{M}_2(\mathbb{R})^n$ whose kernel is a product of $[K : \mathbb{Q}] - n$ copies of the Hamilton quaternion algebra over $\mathbb{R}$.

Let $\mathcal{O}$ be an order of A and let $\mathcal{O}^1$ be the multiplicative subgroup of elements with reduced norm one in $\mathcal{O}$. Then $\pi(\mathcal{O}^1)$ is an irreducible lattice in $\mathrm{SL}_2(\mathbb{R})^n$. An irreducible lattice $\Gamma < \mathrm{SL}_2(\mathbb{R})^n$ is arithmetic when it is commensurable with a group $\pi(\mathcal{O}^1)$ constructed in the above way. Two

arithmetic lattices Γ are commensurable if and only if they have the same associated totally real field k and quaternion algebra, modulo the action of $\mathrm{Aut}(k/\mathbb{Q})$ [9, Theorem 8.4.7]. An arithmetic lattice Γ is cocompact if and only if $A \not\cong \mathrm{M}_2(K)$.

When $n = 1$, such Γ are arithmetic Fuchsian groups. When $n = 2$, a lattice $\Gamma < G$ is irreducible if its projection onto any factor is dense in the analytic topology. A lattice fails to be irreducible in this case if and only if there is a finite index subgroup that is the direct product of two Fuchsian groups. All irreducible lattices in G are arithmetic by G. A. Margulis's Arithmeticity Theorem [11, p. 2].

Theorem 3.1. *Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ be an irreducible lattice defined via the quaternion algebra A over the totally real number field K. Then, there is a one-to-one correspondence between*

(1) *commensurability classes of arithmetic Fuchsian subgroups of Γ and*

(2) *degree two subfields $K_0 \subset K$ and $\mathrm{Aut}(K_0/\mathbb{Q})$-isomorphism classes of K_0-subalgebras $B \subset A$ such that $A = B \otimes_{K_0} K$.*

Thus, Γ contains infinitely many commensurability classes of arithmetic Fuchsian subgroups if and only if it contains one arithmetic Fuchsian subgroup.

Proof. We analyze the possible simple subalgebras of A. We first argue that a subalgebra as in (2) always produces an arithmetic Fuchsian subgroup of Γ. This is equivalent to showing that every such subalgebra splits at exactly one real place of K_0. This follows immediately from analyzing the possibly splitting behavior of quaternion algebras under tensor products. Indeed, at real place of K_0 at which B is isomorphic to Hamilton's quaternions ($\mathrm{M}_2(\mathbb{R})$, respectively) cannot extend to a real place of K at which A is isomorphic to $\mathrm{M}_2(\mathbb{R})$ (Hamilton's quaternions, respectively), since Hamilton's quaternions do not embed in $\mathrm{M}_2(\mathbb{R})$ and vice versa.

Conversely, let Σ be an arithmetic Fuchsian group with associated totally real field K_0 and K_0-quaternion algebra B, and suppose that Σ is a Fuchsian subgroup of Γ. The inclusion of Σ into Γ induces an embedding of algebras $B \to A$. Indeed, consider

$$\Sigma \subset \Gamma \subset A,$$

where we can assume that $\Gamma \subset A$ possibly after passing to a subgroup of finite index, which has no effect on the commensurability classes of arithmetic Fuchsian subgroups. If K_0 is the invariant trace field of Σ (see [9], [10]), then the K_0-subalgebra of A generated by Σ is K_0-isomorphic to B. It follows immediately that $A = B \otimes_{K_0} K$, so we must show that K_0 is a quadratic subfield of K. However, B splits at exactly one infinite place

of K_0 since it is the quaternion algebra associated with an arithmetic Fuchsian group. Then the number of real places of K at which $B \otimes_{K_0} K$ splits is $[K : K_0]$, which is 2 by assumption on Γ, so K_0 is a quadratic subfield of K. Therefore, Fuchsian subgroups of Γ arise from subalgebras as in (2).

The last statement of the theorem can be seen as follows. Let B' be another quaternion algebra over K_0 such that the places of K_0 that ramify in exactly one of B or B' do not split in K. Then $B \otimes_{K_0} K \cong B' \otimes_{K_0} K$, so B' defines another arithmetic Fuchsian subgroup of Γ. Taking B', not $\mathrm{Aut}(K_0/\mathbb{Q})$-conjugate to B implies that the associated arithmetic Fuchsian subgroups form a new commensurability class in Γ. The Chebotarev density theorem implies that there are infinitely many primes of K_0 that do not split in K; hence, there are infinitely many distinct choices of B' that produce distinct commensurability classes. $\qquad\square$

Example 3.2. To build arithmetic subgroups of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ containing no arithmetic Fuchsian subgroups, take K a totally real field with odd degree over $\mathbb{Q}$ and take a K-quaternion algebra unramified at exactly two real places. Then K contains no quadratic subfields, hence no arithmetic Fuchsian subgroups.

Example 3.3. We now describe how to parametrize the Fuchsian subgroups of Hilbert modular groups. Let d be a square-free positive integer, $K = \mathbb{Q}(\sqrt{d})$, and $\mathcal{O}_K$ be the ring of integers in K. Then the Hilbert modular groups are $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$, which are associated with the quaternion algebra $\mathrm{M}_2(K)$. The quaternion algebras associated with Fuchsian subgroups of Γ_K are then quaternion algebras $B/\mathbb{Q}$ such that

$$B \otimes_{\mathbb{Q}} K \cong \mathrm{M}_2(K)$$

and $B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R})$.

We claim that these are precisely the quaternion algebras B over $\mathbb{Q}$ that admit a Hilbert symbol

$$B = \left(\frac{D_K, n}{\mathbb{Q}} \right),$$

where D_K is the discriminant of K and $0 \neq n \in \mathbb{Z}$. Recall that this means B has a $\mathbb{Q}$-basis $\{1, I, J, IJ\}$ with $I^2 = D_K$, $J^2 = n$, and $IJ = -JI$.

Since $D_K > 0$, we certainly have that $B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R})$ (see [9, §2.5]), and so B is associated with a commensurability class of arithmetic Fuchsian groups. Furthermore, it is direct from the Hilbert symbol that $K \cong \mathbb{Q}(I)$ embeds in B; hence, $B \otimes_{\mathbb{Q}} K \cong \mathrm{M}_2(K)$ and B embeds in $\mathrm{M}_2(K)$. Conversely, suppose that K embeds in B. Then there is an element $I \in B$ such that $I^2 = D_K$. Since B is noncommutative, one shows that there exists an element $J \in B^*$ such that $IJ = -JI$. Scaling

J by an element of $\mathbb{Q}$, we can assume $J^2 = n \in \mathbb{Z}$, and thus we see that B has a Hilbert symbol as above.

4. Lattices Acting on $\mathbf{H}_{\mathbb{C}}^2$

First, we quickly recall the standard construction of $\mathbf{H}_{\mathbb{C}}^2$. See [5, Ch. 3] for further details. Let V be a rank 3 vector space over $\mathbb{C}$ and h a hermitian form on V of signature $(2,1)$. If V_- denotes the subspace of h-negative vectors, then $\mathbf{H}_{\mathbb{C}}^2$ is the space $\mathbb{P}(V_-)$ of h-negative lines. There is a natural biholomorphism from $\mathbf{H}_{\mathbb{C}}^2$ with the metric determined by h to the unit ball in $\mathbb{C}^2$ with the Bergman metric.

Totally geodesic holomorphic embeddings of $\mathbf{H}^2$ in $\mathbf{H}_{\mathbb{C}}^2$ arise from rank 2 subspaces of V on which the restriction of h to V has signature $(1,1)$. Taking the h-orthogonal complement, it follows that holomorphic totally geodesic embeddings of $\mathbf{H}^2$ into $\mathbf{H}_{\mathbb{C}}^2$ are in one-to-one correspondence with h-positive lines in V. See [5, §3.3.1] and [2, §12.2].

There are two constructions of arithmetic subgroups of $\mathrm{SU}(2,1)$. By [2, Ch. 8] or [13], the only arithmetic lattices in $\mathrm{SU}(2,1)$ that contain totally geodesic surfaces are those of so-called *simple type* (also sometimes called *first type*). These are defined as follows.

Recall that a *CM-pair* K/K_0 is a totally imaginary quadratic extension of a totally real number field. Let $z \mapsto \overline{z}$ be the nontrivial automorphism of K over K_0 and h be a hermitian form on K^3 such that h is indefinite at precisely one $\mathrm{Gal}(K/K_0)$-conjugate pair of embeddings of K into $\mathbb{C}$. If $\mathcal{O}$ is the integer ring of K, the K_0-algebraic group

$$\mathrm{SU}(h) = \{x \in \mathrm{SL}_3(K) \ : \ {}^t\overline{x}hx = h\},$$

contains the discrete subgroup $\Gamma_{\mathcal{O}}^1 = \mathrm{SU}(h) \cap \mathrm{SL}_r(\mathcal{O})$. Then $\Gamma_{\mathcal{O}}^1$ projects to a lattice in $\mathrm{SU}(2,1)$ since h is indefinite above exactly one real embedding of K_0. It is known that the commensurability class of $\Gamma_{\mathcal{O}}^1$ in $\mathrm{SU}(2,1)$ depends only on K, not the choice of hermitian form (see [16, §1.2]). The lattice $\Gamma_{\mathcal{O}}^1$ is cocompact if and only if $K_0 \neq \mathbb{Q}$.

Recall that the arithmetic quotients of $\mathbf{H}_{\mathbb{C}}^2$ that are not of simple type contain no totally geodesic curves. The following classifies and parametrizes the possible arithmetic Fuchsian subgroups of arithmetic lattices in $\mathrm{SU}(2,1)$ of simple type.

Theorem 4.1. *Let $\Gamma < \mathrm{SU}(2,1)$ be an arithmetic lattice of simple type with associated CM-pair K/K_0. Then there is a one-to-one correspondence between*

    (1) *commensurability classes of arithmetic Fuchsian subgroups of Γ corresponding to totally geodesic projective algebraic curves on $\Gamma \backslash \mathbf{H}_{\mathbb{C}}^2$ and*

(2) $\text{Aut}(K_0/\mathbb{Q})$-*isomorphism classes of K_0-quaternion algebras A that ramify at all but one infinite place of K_0 and at any finite set of nonarchimedean places that do not split in K/K_0.*

In particular, any such Γ contains infinitely many distinct commensurability classes of such arithmetic Fuchsian subgroups.

Proof. Assume that h is a hermitian form on $V = K^3$ that is indefinite above exactly one archimedean place of K_0. Let $W \cong K^2 \subset V$ be the h orthogonal complement of a line in V that is h-positive at the archimedean place of K_0 over which h is indefinite, and let h_W be the restriction of h to W. By our earlier discussion of holomorphic embeddings of $\mathbf{H}^2$ into $\mathbf{H}^2_{\mathbb{C}}$, the pair (W, h_W) determines a maximal arithmetic Fuchsian subgroup Σ_W of Γ associated with a holomorphic curve on $\Gamma \backslash \mathbf{H}^2_{\mathbb{C}}$, and all such curves arise in this way. We now proceed to determine the K_0-quaternion algebra associated to this arithmetic Fuchsian group Σ_W.

There is a natural homomorphism from Σ_W to $\text{SL}_2(K)$. Furthermore, the K_0-subalgebra B of $\text{M}_2(K)$ generated by Σ_W is four-dimensional and noncommutative, so it is a quaternion algebra. Since h_W is indefinite at exactly one infinite place of K_0 and B embeds into $\text{M}_2(K)$ (i.e., K splits B), B must satisfy condition (2) of the theorem.

Conversely, suppose B is a quaternion algebra over K_0 satisfying the conditions in (2). We must show that there is a commensurability class of arithmetic Fuchsian subgroups of Γ associated with B. Condition (2) implies that we have an embedding of K into B, so B is a two-dimensional left vector space over K. The nontrivial automorphism of K over K_0 extends to an anti-involution $z \to \overline{z}$ of B over K_0.

There is a left K-linear action of B on itself under which $z \in B$ acts on $x \in B$ by $x \mapsto x \cdot \overline{z}$. This embeds B into $\text{Mat}_2(K)$. Let $\text{Tr} : B \to K$ be the composition of this embedding with the trace. Identifying B with K^2, we obtain a hermitian form on K^2 by

$$h_B(x, y) = \text{Tr}(x\overline{y}),$$

for $x, y \in B$.

Now, consider the hermitian form H_B on K^3 defined as follows. Write $K^3 = W \oplus \ell$, where W is two-dimensional and ℓ is a line. Let the restriction of H_B to W be h_B, let ℓ be H_B-orthogonal to W, and let the H_B-norm of some generator of ℓ be the totally positive number $\det(h)/\det(h_B)$ in K_0. By [17, Ch. 10, Example 1.8(iii)], K^3 equipped with H_B is isomorphic as a hermitian vector space to K^3 equipped with h. This realizes the elements B^1 of reduced norm 1 inside $\text{SU}(h)$ as the stabilizer of the subspace W. Then $\Gamma \cap B$ is an arithmetic Fuchsian subgroup of Γ with associated quaternion algebra B. This completes the proof. $\qquad\square$

Example 4.2. In [15], Parkkonen and Paulin recently parameterized the $\mathbb{C}$-Fuchsian subgroups of the Picard modular groups. Recall that the Picard modular groups define the commensurability classes of nonuniform arithmetic lattices in $\mathrm{PU}(2,1)$ and are defined by hermitian forms over imaginary quadratic fields. In other words, the CM pair is $K/\mathbb{Q}$ for some imaginary quadratic field K.

Theorem 4.1 identifies Fuchsian subgroups of the Picard modular groups with $\mathbb{Q}$-quaternion algebras A that are unramified at the infinite place of $\mathbb{Q}$ and that ramify at a (possibly empty) set of rational primes that do not split in K. In particular, K embeds as a subfield of A. Arguing as in the above example of Hilbert modular groups, we see that A has a Hilbert symbol of the form

$$A = \left(\frac{D_K, n}{\mathbb{Q}} \right),$$

where D_K is the discriminant of K and we can assume that n is square-free. Then $A \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R})$ if and only if $n > 0$, and we obtain the classification given in [15].

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