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by

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STRONGLY SYMMETRIC COMPACTIFICATIONS

N. ADU, H. BOUSTIQUE, AND G. RICHARDSON

Abstract. Convergence approach spaces, defined by E. Lowen and R. Lowen [*A quasitopos containing CONV and MET as full subcategories*, Internat. J. Math. Math. Sci. **11** (1988)], possess both quantitative and topological properties. These spaces are equipped with a structure which provides information as to whether or not a sequence or filter approximately converges. Paul Brock and D. C. Kent [*Approach spaces, limit tower spaces, and probabilistic convergence spaces*, Appl. Categ. Structures **5** (1997)] show that the category of convergence approach spaces with contractions as morphisms is isomorphic to the category of limit tower spaces. Properties of the category of strongly symmetric limit tower spaces are studied here. In particular, a characterization of the limit tower spaces which possess a strongly symmetric compactification is given. Moreover, one-point strongly symmetric compactifications of limit tower spaces are studied.

1. Introduction and Preliminaries

The category AP of approach spaces was defined by R. Lowen in 1989 [10]. The category AP contains the categories TOP and MET as full subcategories and possesses both quantitative and topological-like properties. In particular, information as to whether a sequence or filter approximately converges is provided by the approach structure. E. Lowen and R. Lowen [9] embedded AP in the quasitopos CAP of convergence approach spaces. These and other results and references can be found in the monograph by R. Lowen [11].

The framework of the present paper is the category LTS of limit tower spaces. Paul Brock and D. C. Kent show in [3, Theorem 9] that CAP

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and LTS are isomorphic categories. The primary purpose of our work is to characterize the objects in LTS which possess a strongly symmetric compactification.

Let X be a set, $\mathbf{F}(X)$ the set of all filters on X, and 2^X the power set of X, and let $\dot{x}$ denote the filter on X whose base is $\{\{x\}\}$.

Definition 1.1. The pair (X, q) is called a *limit space* and q a *limit structure* on X provided $q : \mathbf{F}(X) \to 2^X$ satisfies

(L1) $x \in q(\dot{x})$ for each $x \in X$,
(L2) $\mathcal{F} \subseteq \mathcal{G}$ implies $q(\mathcal{F}) \subseteq q(\mathcal{G})$,
(L3) $q(\mathcal{F}) \cap q(\mathcal{G}) \subseteq q(\mathcal{F} \cap \mathcal{G})$.

The more intuitive notation $\mathcal{F} \xrightarrow{q} x$ ($\mathcal{F}$ q-converges to x) is used in place of $x \in q(\mathcal{F})$. A map $f : (X, q) \to (Y, p)$ between two limit spaces is said to be *continuous* if $f^{\to}\mathcal{F} \xrightarrow{p} f(x)$ whenever $\mathcal{F} \xrightarrow{q} x$, where $f^{\to}\mathcal{F}$ denotes the filter on Y whose base is $\{f(F) : F \in \mathcal{F}\}$. Let *LIM* denote the category consisting of all the limit spaces and continuous maps. Then LIM is a topological construct in the sense of Jiří Adámek et al. [1]. Define $LS(X)$ to be the set of all limit structures on X. If $p, q \in LS(X)$, $p \leq q$ means that $\mathcal{F} \xrightarrow{p} x$ whenever $\mathcal{F} \xrightarrow{q} x$. Then $(LS(X), \leq)$ is a poset with the largest (smallest) member the discrete (indiscrete) topology, respectively. Given $q_j \in LS(X)$, $j \in J$, $q = \bigvee_{j \in J} q_j$ exists, and is defined by $\mathcal{F} \xrightarrow{q} x$ if and only if for each $j \in J$, $\mathcal{F} \xrightarrow{q_j} x$. Hence, $(LS(X), \leq)$ is a complete lattice. A limit space (X, q) is said to be *Hausdorff* if each filter on X has at most one limit, and is called *regular* provided that $\mathrm{cl}_q \mathcal{F} \xrightarrow{q} x$ whenever $\mathcal{F} \xrightarrow{q} x$, where $\mathrm{cl}_q \mathcal{F}$ denotes the filter on X whose base is $\{\mathrm{cl}_q F : F \in \mathcal{F}\}$.

Definition 1.2. The pair $(X, \bar{q})$, where $\bar{q} = (q_\alpha)$, $0 \leq \alpha \leq \infty$, is a family of limit structures on X, is called a *limit tower space* and $\bar{q}$ a *limit tower* on X provided

(LT1) q_∞ is the indiscrete topology,
(LT2) $0 \leq \alpha \leq \beta \leq \infty$ implies that $q_\alpha \geq q_\beta$,
(LT3) $\bigvee_{\beta > \alpha} q_\beta = q_\alpha$, for each $0 \leq \alpha < \infty$ (right continuity).

A map $f : (X, \bar{q}) \to (Y, \bar{p})$ between two limit tower spaces is called a *contraction* whenever $f : (X, q_\alpha) \to (Y, p_\alpha)$ is continuous in LIM for each $0 \leq \alpha \leq \infty$. Let *LTS* denote the category of all limit tower spaces and contraction maps. Given limit towers $\bar{p}$ and $\bar{q}$ on X, define $\bar{q} \leq \bar{p}$ to mean that for each $0 \leq \alpha \leq \infty$, $q_\alpha \leq p_\alpha$ in $LS(X)$.

Definition 1.3. The pair $(X, \bar{q})$, where $\bar{q} = (q_\alpha)$, $0 \leq \alpha \leq \infty$, is a family of limit structures on X, is said to be a *generalized limit tower space* if it obeys (LT1) and (LT2).

A contraction between two generalized limit tower spaces is defined as in LTS. Let *GLTS* denote the category consisting of all the generalized limit tower spaces and contraction maps. Both GLTS and LTS are topological constructs in the sense of [1].

Definition 1.4. An object $(X, \bar{q}) \in |\text{GLTS}|$, where $\bar{q} = (q_\alpha)$, $0 \leq \alpha \leq \infty$, is called *Hausdorff* whenever (X, q_0) is a Hausdorff limit space and *strongly regular* provided that (X, q_α) is a regular limit space for each $0 \leq \alpha \leq \infty$.

The notions of regular and strongly regular limit tower spaces are defined and studied by Brock and Kent [4].

A generalized limit tower space $(X, \bar{q})$, where $\bar{q} = (q_\alpha)$, $0 \leq \alpha \leq \infty$, is said to be *compact* provided that each ultrafilter on X q_0-converges; equivalently, it follows from axiom LT2 that $(X, \bar{q})$ is compact if and only if each ultrafilter on X q_α-converges for each $0 \leq \alpha \leq \infty$. As usual, $((Y, \bar{p}), f)$ is called a *compactification* of $(X, \bar{q})$ in GLTS (LTS) whenever $(Y, \bar{p})$ is compact and $f : (X, \bar{q}) \to (Y, \bar{p})$ is a dense embedding in GLTS (LTS). An embedding $f : (X, \bar{q}) \to (Y, \bar{p})$ is said to be *dense* whenever $\text{cl}_{p_0} f(X) = Y$.

Definition 1.5. Assume that $(X, \bar{q}) \in |\text{GLTS}|$; define the *limit tower structure* $l\bar{q} = (l\bar{q})_\alpha$, $0 \leq \alpha \leq \infty$, of $\bar{q}$ as follows:

(i) $(l\bar{q})_\infty$ is the indiscrete topology on X;

(ii) for $0 \leq \alpha < \infty$, $\mathcal{F} \xrightarrow{(l\bar{q})_\alpha} x$ if and only if for each $\beta > \alpha$, $\mathcal{F} \xrightarrow{q_\beta} x$.

Observe that if $(X, \bar{q}) \in |\text{GLTS}|$, then $(X, l\bar{q}) \in |\text{LTS}|$. Given $(X, \bar{q}) \in |\text{GLTS}|$, let η denote the set of all ultrafilters on X which fail to q_0-converge. Define $\langle \mathcal{G} \rangle = \{\mathcal{G}\}$ for each $\mathcal{G} \in \eta$ and $X^* = X \cup \{\langle \mathcal{G} \rangle : \mathcal{G} \in \eta\}$, and let $j : X \to X^*$, $j(x) = x$ be the natural injection. For $A, B \subseteq X$, $A^* := A \cup \{\langle \mathcal{G} \rangle : A \in \mathcal{G}\}$, and note that $A^* \cap B^* = (A \cap B)^*$ and $(A \cup B)^* = A^* \cup B^*$. Let $\mathcal{F}^*$ denote the filter on X^* whose base is $\{F^* : F \in \mathcal{F}\}$ where $\mathcal{F} \in \mathbf{F}(X)$.

Definition 1.6. Given $(X, \bar{q}) \in |\text{GLTS}|$, define $\bar{p} = (p_\alpha)$, $0 \leq \alpha \leq \infty$, on X^* as follows:

(i) p_∞ is the indiscrete topology on X^*,

(ii) $\mathcal{H} \xrightarrow{p_\alpha} j(x)$ if and only if $\mathcal{H} \geq \mathcal{F}^*$ for some $\mathcal{F} \xrightarrow{q_\alpha} x$, $0 \leq \alpha < \infty$,

(iii) $\mathcal{H} \xrightarrow{p_\alpha} \langle \mathcal{G} \rangle$ if and only if $\mathcal{H} \geq \mathcal{G}^*$, $0 \leq \alpha < \infty$.

The following two theorems are proved in [2] and are used here.

Theorem 1.7. *Let $(X, \bar{q}) \in |GLTS|(|LTS|)$. Then $((X^*, \bar{p}), j)$ defined above is a compactification of $(X, \bar{q})$ in GLTS (LTS). Moreover, $(X^*, \bar{p})$ is Hausdorff whenever $(X, \bar{q})$ is Hausdorff.*

An object $(X, \bar{q}) \in |\text{GLTS}|$ is said to be q_0-*regular* if $\text{cl}_{q_0} \mathcal{F} \xrightarrow{q_\alpha} x$ whenever $\mathcal{F} \xrightarrow{q_\alpha} x$, where $\bar{q} = (q_\alpha)$, $0 \leq \alpha \leq \infty$. Note that this definition is weaker than strong regularity as given in Definition 1.4.

Theorem 1.8. *Assume that $(X, \bar{q}) \in |GLTS|$ and $((X^*, \bar{p}), j)$ is the compactification given in Theorem 1.7. Suppose that $f : (X, \bar{q}) \to (Y, \bar{r})$ is contraction, where $(Y, \bar{r}) \in |GLTS|$ is compact and r_0-regular. Then there exists a contraction $f^* : (X^*, \bar{p}) \to (Y, \bar{r})$ such that $f^* \circ j = f$.*

Definition 1.9 ([11])**.** The pair (X, λ) is called a *convergence approach space* provided $\lambda : \mathbf{F}(X) \to [0, \infty]^X$ satisfies the following:

(CAS1) $\lambda(\dot{x})(x) = 0$ for all $x \in X$,
(CAS2) $\mathcal{F} \subseteq \mathcal{G}$ implies $\lambda(\mathcal{G}) \leq \lambda(\mathcal{F})$,
(CAS3) $\lambda(\mathcal{F} \cap \mathcal{G}) = \lambda(\mathcal{F}) \vee \lambda(\mathcal{G})$.

A map $f : (X, \lambda) \to (Y, \sigma)$ between two convergence approach spaces is called a *contraction* if $\sigma(f^{\to}\mathcal{F})(f(x)) \leq \lambda(\mathcal{F})(x)$ for each $\mathcal{F} \in \mathbf{F}(X)$ and $x \in X$. Let *CAP* denote the category of all convergence approach spaces and contraction maps. Define $G : \text{LTS} \to \text{CAP}$ by $G(X, \bar{q}) = (X, \lambda)$ and $G(f) = f$, where $\lambda(\mathcal{F})(x) = \bigwedge\{\alpha \in L : \mathcal{F} \xrightarrow{q_\alpha} x\}$ whenever $\mathcal{F} \in \mathbf{F}(X)$ and $x \in X$. As mentioned earlier, it is shown in [3, Theorem 9] that the functor G is an isomorphism from LTS onto CAP. Moreover, given $(X, \bar{q}) \in |\text{LTS}|$, let $(X, \lambda) = G(X, \bar{q})$. G. Jäger [5] has constructed a compactification of any $(X, \lambda) \in |\text{CAP}|$. It can be shown that $G(X^*, \bar{p})$ coincides with Jäger's compactification of (X, λ) in CAP, where $((X^*, \bar{p}), j)$ is the compactification of $(X, \bar{q})$ in LTS given in Theorem 1.7 above.

2. Compactification

Symmetric limit spaces have additional features not possessed by regular limit spaces. This notion is extended to the category of limit tower spaces. The primary result of this section is the characterization of the limit tower spaces possessing a strongly symmetric compactification. First, some preliminary definitions are given.

Definition 2.1 ([7])**.** The object $(X, q) \in |\text{LIM}|$ is called *symmetric* provided that

(i) (X, q) is regular and
(ii) $\mathcal{F} \xrightarrow{q} z$ and $\dot{z} \xrightarrow{q} x$ imply that $\mathcal{F} \xrightarrow{q} x$.

Further, $(X, \bar{q}) \in |\text{GLTS}|$ is said to be *strongly symmetric* whenever each (X, q_α) is symmetric for each $0 \leq \alpha \leq \infty$.

Definition 2.2 ([8])**.** An object $(X, q) \in |\text{LIM}|$ is *reciprocal* if it is induced by a Cauchy space in the sense of [6]. Moreover, $(X, \bar{q}) \in |\text{GLTS}|$ is called *strongly reciprocal* provided that (X, q_α) is reciprocal for each $0 \leq \alpha \leq \infty$.

H. H. Keller [6] proved that a limit space is induced by a Cauchy space (complete Cauchy space) if and only if for each pair of distinct elements in X, their convergent filters either coincide or are disjoint.

Lemma 2.3. *Assume that* $(X, \bar{q}) \in |GLTS|$ *is strongly regular. Then* $(X, \bar{q})$ *is strongly symmetric if and only if it is strongly reciprocal.*

Proof. Suppose that $(X, \bar{q})$ is strongly symmetric and $\mathcal{G} \xrightarrow{q_\alpha} x, z$, for some $0 \leq \alpha \leq \infty$ and $x \neq z$. Then $\dot{z} \geq \mathrm{cl}_{q_\alpha} \mathcal{G}$, and thus $\dot{z} \xrightarrow{q_\alpha} x$. If $\mathcal{F} \xrightarrow{q_\alpha} z$, it follows that $\mathcal{F} \xrightarrow{q_\alpha} x$, and thus $\mathcal{F} \xrightarrow{q_\alpha} z$ if and only if $\mathcal{F} \xrightarrow{q_\alpha} x$. Hence, $(X, \bar{q})$ is strongly reciprocal. Conversely, assume that $(X, \bar{q})$ is strongly reciprocal and suppose that for some $0 \leq \alpha \leq \infty$, $\mathcal{F} \xrightarrow{q_\alpha} z$ and $\dot{z} \xrightarrow{q_\alpha} x$. It follows that $\mathcal{F} \xrightarrow{q_\alpha} z$ if and only if $\mathcal{F} \xrightarrow{q_\alpha} x$, and thus $(X, \bar{q})$ is strongly symmetric. $\square$

Given that $(X, q) \in |\mathrm{LIM}|$, let rq denote the finest regular limit structure on X which is coarser than q. Define σq to be the following convergence structure on X:

$$\mathcal{F} \xrightarrow{\sigma q} x \text{ iff } \exists\, z \in X \text{ such that } \mathcal{F} \xrightarrow{rq} z \text{ and } \dot{z} \xrightarrow{rq} x.$$

It is shown [7] that $(X, \sigma q)$ is the finest symmetric convergence space which is coarser than (X, q); however, σq may not be a limit structure. Let $L\sigma q$ denote the finest limit structure on X which is coarser that σq; that is,

$$\mathcal{H} \xrightarrow{L\sigma q} x \text{ iff } \mathcal{H} \geq \bigcap_{i=1}^{n} \mathcal{F}_i, \text{ for some } \mathcal{F}_i \xrightarrow{\sigma q} x, 1 \leq i \leq n.$$

Let *SGLTS* (*SLTS*) denote the full subcategory of GLTS (LTS) whose objects consist of all the strongly symmetric objects in GLTS (LTS).

Lemma 2.4. *The category SLTS is concretely reflective in GLTS, and for* $(X, \bar{q}) \in |GLTS|$*, the reflection morphism is given by* $\mathrm{id}_X : (X, \bar{q}) \to (X, lL\sigma\bar{q})$*. Also,* $(X, lL\sigma\bar{q}) \in |SLTS|$*.*

Proof. First, it is shown that for $0 \leq \alpha \leq \infty$ fixed, $(X, L\sigma q_\alpha)$ is symmetric. Observe that if $A \subseteq X$, then $\mathrm{cl}_{L\sigma q_\alpha} A = \mathrm{cl}_{\sigma q_\alpha} A$ and since $(X, \sigma q_\alpha)$ is regular, $(X, L\sigma q_\alpha)$ obeys Definition 2.1(i). Next, assume that $\mathcal{H} \xrightarrow{L\sigma q_\alpha} z$ and $\dot{z} \xrightarrow{L\sigma q_\alpha} x$. Then there exist $\mathcal{F}_i \xrightarrow{\sigma q_\alpha} z$ and $\mathcal{G}_j \xrightarrow{\sigma q_\alpha} x$ such that $\mathcal{H} \geq \bigcap_{i=1}^{n} \mathcal{F}_i$ and $\dot{z} \geq \bigcap_{j=1}^{m} \mathcal{G}_j$. It follows that $\dot{z} \geq \mathcal{G}_j$ for some $1 \leq j \leq m$, and thus $\dot{z} \xrightarrow{\sigma q_\alpha} x$. Since $(X, \sigma q_\alpha)$ is symmetric, $\mathcal{F}_i \xrightarrow{\sigma q_\alpha} x$ for each $1 \leq i \leq n$, and thus $\mathcal{H} \xrightarrow{L\sigma q_\alpha} x$. Hence, $(X, L\sigma q_\alpha)$ satisfies Definition

2.1(ii), and thus $(X, L\sigma q_\alpha)$ is symmetric for each $0 \leq \alpha \leq \infty$. It follows by construction that $(X, L\sigma q_\alpha)$ is the finest symmetric limit space which is coarser than (X, q_α), $0 \leq \alpha \leq \infty$, and hence $(X, L\sigma\bar{q}) \in |\mathrm{SGLTS}|$.

Since $(X, L\sigma\bar{q})$ is strongly regular, it follows from [2, Lemma 3.1] that $(X, lL\sigma\bar{q}) \in |\mathrm{LTS}|$ is also strongly regular. Moreover, suppose that $\mathcal{H} \xrightarrow{(lL\sigma\bar{q})_\alpha} z$ and $\dot{z} \xrightarrow{(lL\sigma\bar{q})_\alpha} x$, where $0 \leq \alpha \leq \infty$. Then for each $\beta > \alpha$, $\mathcal{H} \xrightarrow{L\sigma q_\beta} z$ and $\dot{z} \xrightarrow{L\sigma q_\beta} x$. Since $(X, L\sigma q_\beta)$ is a symmetric limit space, $\mathcal{H} \xrightarrow{L\sigma q_\beta} x$ for each $\beta > \alpha$, and thus $\mathcal{H} \xrightarrow{(lL\sigma\bar{q})_\alpha} x$. Hence, $(X, lL\sigma\bar{q}) \in |\mathrm{SLTS}|$ and $\mathrm{id}_X : (X, \bar{q}) \to (X, lL\sigma\bar{q})$ is a morphism in GLTS. Assume that $(Y, \bar{r}) \in |\mathrm{SLTS}|$ and $f : (X, \bar{q}) \to (Y, \bar{r})$ is a contraction in GLTS. Since contractions are preserved under the operations of σ, L, and l, it follows that $f : (X, lL\sigma\bar{q}) \to (Y, \bar{r})$ is a contraction in SLTS. Hence, SLTS is concretely reflective in GLTS. $\square$

Let $(X, q) \in |\mathrm{LIM}|$ and define δq to be the initial structure on X with respect to the source $f_j : X \to (\mathbb{R}, \tau)$, where τ is the usual topology on the reals and j indexes the set of all bounded, continuous real-valued functions defined on (X, q). Then δq is a completely regular topology. The subconstruct of all completely regular topological spaces is concretely reflective in LIM and, for $(X, q) \in |\mathrm{LIM}|$, the reflection morphism is $\mathrm{id}_X : (X, q) \to (X, \delta q)$. Further, if $(X, \bar{q}) \in |\mathrm{GLTS}|$, define $\delta\bar{q} = (\delta q_\alpha)$, $0 \leq \alpha \leq \infty$. Then $(X, \delta\bar{q}) \in |\mathrm{GLTS}|$, and $(X, \bar{q})$ is said to be *strongly completely regular* whenever $\bar{q} = \delta\bar{q}$.

Theorem 2.5 ([7])**.** *Let (X, q) be a compact symmetric limit space; then q and δq agree on ultrafilter convergence. In particular, $\mathrm{cl}_q A = \mathrm{cl}_{\delta q} A$ for each $A \subseteq X$.*

Theorem 2.6. *Assume that $(X, \bar{q}) \in |GLTS|(|LTS|)$. Then $(X, \bar{q})$ has a strongly symmetric compactification if and only if $(X, \bar{q})$ is strongly symmetric and $\bar{q}$ and $\delta\bar{q}$ agree on ultrafilter convergence. Moreover, each contraction $f : (X, \bar{q}) \to (Y, \bar{r})$ into a compact strongly symmetric object in GLTS (LTS) can be extended to a contraction on the corresponding compactification.*

Proof. Suppose that $((Y, \bar{p}), f)$ is a strongly symmetric compactification of $(X, \bar{q})$ in LTS. Since the property of being strongly symmetric is hereditary, $(X, \bar{q})$ is strongly symmetric. Moreover, assume that $\mathcal{F}$ is an ultrafilter for which $\mathcal{F} \xrightarrow{\delta q_\alpha} x$ for some $0 \leq \alpha \leq \infty$. Since $f : (X, \delta q_\alpha) \to (Y, \delta p_\alpha)$ is continuous, $f^{\to}\mathcal{F} \xrightarrow{\delta p_\alpha} f(x)$, and thus by Theorem 2.1, $f^{\to}\mathcal{F} \xrightarrow{p_\alpha} f(x)$. Hence, $\mathcal{F} \xrightarrow{q_\alpha} x$, and thus $\bar{q}$ and $\delta\bar{q}$ agree on ultrafilter convergence.

Conversely, assume that $(X, \bar{q})$ is strongly symmetric and $\bar{q}$ and $\delta\bar{q}$ agree on ultrafilter convergence. Let $((X^*, \bar{p}), j)$ denote the compactification of $(X, \bar{q})$ given in Theorem 1.7. According to Lemma 2.4, $(X^*, lL\sigma\bar{p}) \in |\mathrm{SLTS}|$ and is compact since $lL\sigma\bar{p} \leq \bar{p}$; hence, $j : (X, \bar{q}) \to (X^*, lL\sigma\bar{p})$ is a contraction. Conversely, suppose that $\mathcal{F} \in \mathbf{F}(X)$ such that $j^{\to}\mathcal{F} \xrightarrow{(lL\sigma\bar{p})_\alpha} j(x)$. Then for each $\beta > \alpha$, $j^{\to}\mathcal{F} \xrightarrow{(L\sigma\bar{p})_\beta} j(x)$. It remains to show that for each $\beta > \alpha$, $\mathcal{F} \xrightarrow{q_\beta} x$. Fix $\beta > \alpha$; then $j^{\to}\mathcal{F} \xrightarrow{(L\sigma\bar{p})_\beta} j(x)$ implies that $j^{\to}\mathcal{F} \geq \bigcap_{i=1}^{n} \mathcal{H}_i$ for some $\mathcal{H}_i \xrightarrow{\sigma p_\beta} j(x)$, $1 \leq i \leq n$. Since $j^{\to}\mathcal{F} \geq \bigcap_{i=1}^{n} \mathcal{H}_i$, one can assume that $j(X) \in \mathcal{H}_i$ for each $1 \leq i \leq n$.

Fix $1 \leq i \leq n$ and recall that $\mathcal{H}_i \xrightarrow{\sigma p_\beta} j(x)$ implies that there exists $y \in X^*$ such that $\mathcal{H}_i \xrightarrow{rp_\beta} y$ and $\dot{y} \xrightarrow{rp_\beta} j(x)$. First, suppose that $y = j(z)$ for some $z \in X$. As shown in [12], there exists $\mathcal{K} \xrightarrow{p_\beta} j(z)$ such that $\mathcal{H}_i \geq \mathrm{cl}^n_{rp_\beta} \mathcal{K}$. Employing the definition of p_β, there exists $\mathcal{L} \xrightarrow{q_\beta} z$ such that $\mathcal{K} \geq \mathcal{L}^*$. Then $\mathcal{H}_i \geq \mathrm{cl}^n_{rp_\beta} \mathcal{K} \geq \mathrm{cl}^n_{rp_\beta} \mathcal{L}^* \geq \mathrm{cl}^{n+1}_{rp_\beta} j^{\to}\mathcal{L} \geq \mathrm{cl}_{\delta p_\beta} j^{\to}\mathcal{L}$. Since $j(X) \in \mathcal{H}_i$, $j^{\leftarrow}\mathcal{H}_i \geq j^{\leftarrow}(\mathrm{cl}_{\delta p_\beta} j^{\to}\mathcal{L}) = \mathrm{cl}_{\delta q_\beta} \mathcal{L}$, and thus $\mathrm{cl}_{\delta q_\beta} \mathcal{L} = \mathrm{cl}_{q_\beta} \mathcal{L} \xrightarrow{q_\beta} z$. Hence, $j^{\leftarrow}\mathcal{H}_i \xrightarrow{q_\beta} z$, and a similar argument shows that $\dot{z} = j^{\leftarrow}(j^{\to}(\dot{z})) \xrightarrow{q_\beta} x$. Since $(X, \bar{q})$ is strongly symmetric, $j^{\leftarrow}\mathcal{H}_i \xrightarrow{q_\beta} x$.

Next, assume that $y = \langle \mathcal{G} \rangle$; then $\mathcal{H}_i \xrightarrow{rp_\beta} \langle \mathcal{G} \rangle$ and $\langle \dot{\mathcal{G}} \rangle \xrightarrow{rp_\beta} j(x)$. It is shown that $\mathcal{G} \xrightarrow{q_\beta} x$. Indeed, if $\mathcal{G} \nrightarrow q_\beta x$, then since q_β and δq_β agree on ultrafilter convergence, there exists a bounded, continuous $g : (X, \delta q_\beta) \to \mathbb{R}$ such that $g^{\to}\mathcal{G} \nrightarrow g(x)$ on $\mathbb{R}$. Since g has a continuous extension $h : (X^*, \delta p_\beta) \to \mathbb{R}$, $h^{\to}(j^{\to}\mathcal{G}) = g^{\to}\mathcal{G} \nrightarrow g(x)$ on $\mathbb{R}$. However, $\langle \dot{\mathcal{G}} \rangle \xrightarrow{rp_\beta} j(x)$ implies that $h^{\to}(\langle \dot{\mathcal{G}} \rangle) \to h(j(x)) = g(x)$ on $\mathbb{R}$. This is contrary to the fact that $h(\langle \mathcal{G} \rangle) = \lim h^{\to}(j^{\to}\mathcal{G}) = \lim g^{\to}\mathcal{G} \neq g(x)$ on $\mathbb{R}$. Hence, $\mathcal{G} \xrightarrow{q_\beta} x$, and thus $\mathcal{H}_i \geq \mathrm{cl}^n_{rp_\beta} j^{\to}\mathcal{G} \geq \mathrm{cl}_{\delta p_\beta} j^{\to}\mathcal{G}$ implies that $j^{\leftarrow}\mathcal{H}_i \geq \mathrm{cl}_{\delta q_\beta} \mathcal{G} = \mathrm{cl}_{q_\beta} \mathcal{G} \xrightarrow{q_\beta} x$. Therefore, $\mathcal{F} \geq \bigcap_{i=1}^{n} j^{\leftarrow}\mathcal{H}_i \xrightarrow{q_\beta} x$ for each $\beta > \alpha$, and thus $\mathcal{F} \xrightarrow{q_\alpha} x$. Hence, $j : (X, \bar{q}) \to (X^*, lL\sigma\bar{p}) \in |\mathrm{SLTS}|$ is a dense embedding.

According to Theorem 1.2, the contraction $f : (X, \bar{q}) \to (Y, \bar{r})$ can be extended to a contraction $f^* : (X^*, \bar{p}) \to (Y, \bar{r})$ with $f^* \circ j = f$. Due to invariance properties of contractions, $f^* : (X^*, lL\sigma\bar{p}) \to (Y, \bar{r})$ is a contraction in LTS. A similar argument holds whenever $(X, \bar{q}) \in |\mathrm{GLTS}|$. $\square$

Assuming that $(X, \bar{q})$ has a strongly symmetric compactification, it is shown below that the compactification constructed in Theorem 2.6 can be obtained by modifying the strongly regular compactification given in [2, Theorem 3.2] and stated below. The following terminology is needed: A limit space (X, q) is said to be *δ-regular* if $\mathrm{cl}_{\delta q}\,\mathcal{F} \xrightarrow{q} x$ whenever $\mathcal{F} \xrightarrow{q} x$; moreover, $(X, \bar{q}) \in |\mathrm{GLTS}|$ is called *strongly δ-regular* whenever each (X, q_α), $0 \le \alpha \le \infty$, is δ-regular.

Definition 2.7. Given $(X, \bar{q}) \in |\mathrm{GLTS}|$ and $(X^*, \bar{p})$ of Definition 1.6, define $\bar{s} = (s_\alpha)$, $0 \le \alpha \le \infty$, on X^* as follows:

(i) s_∞ is the indiscrete topology,

(ii) $\mathcal{H} \xrightarrow{s_\alpha} j(x)$ if and only if $\mathcal{H} \ge \mathrm{cl}_{\delta p_\alpha}\, j^{\rightarrow}\mathcal{F}$ for some $\mathcal{F} \xrightarrow{q_\alpha} x$, $0 \le \alpha \le \infty$,

(iii) $\mathcal{H} \xrightarrow{s_\alpha} \langle \mathcal{G} \rangle$ if and only if $\mathcal{H} \ge \mathrm{cl}_{\delta p_\alpha}\, j^{\rightarrow}\mathcal{G}$, where $\mathcal{G} \in \eta$.

Then $(X^*, \bar{s}) \in |\mathrm{GLTS}|$ and $(X^*, l\bar{s}) \in |\mathrm{LTS}|$.

Theorem 2.8 ([2])**.** *An object $(X, \bar{q}) \in |GLTS|(|LTS|)$ has a strongly regular compactification in GLTS (LTS) if and only if $(X, \bar{q})$ is strongly δ-regular. Moreover, if $(X, \bar{q}) \in |GLTS|(|LTS|)$ is strongly δ-regular, then $((X^*, \bar{s}), j)$ $(((X^*, l\bar{s}), j))$ is a strongly regular compactification in GLTS (LTS), respectively.*

Theorem 2.9. *Assume that $(X, \bar{q})$ has a strongly symmetric compactification in GLTS (LTS). Then $\sigma\bar{s} = \sigma\bar{p}$ and $((X^*, L\sigma\bar{s}), j)$, $(((X^*, lL\sigma\bar{s}), j))$ is a strongly symmetric compactification of $(X, \bar{q})$ in GLTS (LTS), respectively.*

Proof. It follows from the definition of $\bar{s}$ that $p_\alpha \ge s_\alpha \ge \delta p_\alpha$; hence, $\sigma p_\alpha \ge \sigma s_\alpha \ge \delta p_\alpha$ and $\delta p_\alpha = \delta s_\alpha$, $0 \le \alpha \le \infty$. Since $(X^*, \sigma p_\alpha)$ is a compact symmetric convergence space, it follows from Theorem 2.5 that $\mathrm{cl}_{\delta p_\alpha} = \mathrm{cl}_{\sigma p_\alpha}$. Therefore, if $\mathcal{H} \xrightarrow{s_\alpha} j(x)$, then $\mathcal{H} \ge \mathrm{cl}_{\delta p_\alpha}\, j^{\rightarrow}\mathcal{F}$ for some $\mathcal{F} \xrightarrow{q_\alpha} x$, and thus $\mathcal{H} \ge \mathrm{cl}_{\sigma p_\alpha}\, j^{\rightarrow}\mathcal{F} \xrightarrow{\sigma p_\alpha} j(x)$. Likewise, if $\mathcal{H} \xrightarrow{s_\alpha} \langle \mathcal{G} \rangle$, then $\mathcal{H} \ge \mathrm{cl}_{\sigma p_\alpha}\, j^{\rightarrow}\mathcal{G} \xrightarrow{\sigma p_\alpha} \langle \mathcal{G} \rangle$, and thus $s_\alpha \ge \sigma p_\alpha$. It follows that $\sigma p_\alpha = \sigma s_\alpha$, $0 \le \alpha \le \infty$, and hence $\sigma\bar{p} = \sigma\bar{s}$. In particular, $L\sigma\bar{s} = L\sigma\bar{p}$ $(lL\sigma\bar{s} = lL\sigma\bar{p})$. □

A compactification $((Y, \bar{r}), f)$ of $(X, \bar{q})$ in GLTS is called *strict* if $\mathcal{H} \xrightarrow{r_\alpha} y$ implies that there exists an $\mathcal{F} \in \mathbf{F}(X)$ such that $f^{\rightarrow}\mathcal{F} \xrightarrow{r_\alpha} y$ and $\mathcal{H} \ge \mathrm{cl}_{r_\alpha}\, f^{\rightarrow}\mathcal{F}$ for each $0 \le \alpha \le \infty$. It is shown below that every strongly symmetric compactification in GLTS can be modified to give a strict, strongly symmetric compactification in GLTS. The authors are unable to extend this result to the category LTS.

Theorem 2.10. *Suppose that $(X, \bar{q}) \in |GLTS|$ has a strongly symmetric compactification. Then it possesses a strict, strongly symmetric compactification in GLTS.*

Proof. Let $((Y,\bar{r}),f)$ denote a strongly symmetric compactification of $(X,\bar{q})$ in GLTS. Define $\bar{t}=(t_\alpha)$, $0\leq\alpha\leq\infty$ as

$$\mathcal{H}\xrightarrow{t_\alpha}y \text{ iff } \exists\,\mathcal{F}\in\mathbf{F}(X) \text{ such that } f^{\to}\mathcal{F}\xrightarrow{r_\alpha}y \text{ and } \mathcal{H}\geq \mathrm{cl}_{r_\alpha} f^{\to}\mathcal{F}.$$

Then $(Y,t_\alpha)\in|\mathrm{LIM}|$, $0\leq\alpha\leq\infty$, and t_∞ is the indiscrete structure. If $0\leq\alpha\leq\beta$, then $r_\alpha\geq r_\beta$ implies that $t_\alpha\geq t_\beta$, and thus $(Y,\bar{t})\in|\mathrm{GLTS}|$. Observe that $\bar{t}\geq\bar{r}$. Next, t_α and r_α agree on convergence of ultrafilters. Indeed, assume that $\mathcal{H}\xrightarrow{r_\alpha}y$ where $\mathcal{H}$ is an ultrafilter, and thus there exists an ultrafilter $\mathcal{F}$ on X such that $\mathcal{H}\geq\mathrm{cl}_{r_\alpha} f^{\to}\mathcal{F}$. Then $f^{\to}\mathcal{F}\xrightarrow{r_\alpha}z$ for some $z\in X$, and since (Y,r_α) is regular, $\mathrm{cl}_{r_\alpha} f^{\to}\mathcal{F}\xrightarrow{r_\alpha}z$. Hence, $\mathcal{H}\xrightarrow{r_\alpha}y,z$ and since (Y,r_α) is symmetric, y and z possess the same r_α-convergent filters. Then $\mathrm{cl}_{r_\alpha} f^{\to}\mathcal{F}\xrightarrow{r_\alpha}y$ and $\mathcal{H}\xrightarrow{t_\alpha}y$; therefore, t_α and r_α agree on convergence of ultrafilter.

It is shown that $(Y,\bar{t})$ is strongly regular. Indeed, suppose that $\mathcal{H}\xrightarrow{t_\alpha}y$; then there exists $\mathcal{F}\in\mathbf{F}(X)$ such that $f^{\to}\mathcal{F}\xrightarrow{r_\alpha}y$ and $\mathcal{H}\geq\mathrm{cl}_{r_\alpha} f^{\to}\mathcal{F}$. Since (Y,r_α) is compact and symmetric, $\mathrm{cl}_{r_\alpha}\mathcal{H}\geq\mathrm{cl}^2_{r_\alpha} f^{\to}\mathcal{F}=\mathrm{cl}_{r_\alpha} f^{\to}\mathcal{F}\xrightarrow{r_\alpha}y$. It follows that $\mathrm{cl}_{r_\alpha}\mathcal{H}\xrightarrow{t_\alpha}y$ and, since $\mathrm{cl}_{t_\alpha}\mathcal{H}=\mathrm{cl}_{r_\alpha}\mathcal{H}$, $\mathrm{cl}_{t_\alpha}\mathcal{H}\xrightarrow{t_\alpha}y$. Therefore, $(Y,\bar{t})$ is strongly regular. Observe that $((Y,t),f)$ is strict. Indeed, if $\mathcal{H}\xrightarrow{t_\alpha}y$, then there exists $\mathcal{F}\in\mathbf{F}(X)$ such that $f^{\to}\mathcal{F}\xrightarrow{r_\alpha}y$ and $\mathcal{H}\geq\mathrm{cl}_{r_\alpha} f^{\to}\mathcal{F}$. However, $f^{\to}\mathcal{F}\xrightarrow{t_\alpha}y$, and thus $\mathcal{H}\geq\mathrm{cl}_{r_\alpha} f^{\to}\mathcal{F}=\mathrm{cl}_{t_\alpha} f^{\to}\mathcal{F}$. Hence, $((Y,\bar{t}),f)$ is a strict, strongly regular compactification of $(X,\bar{q})$ in GLTS.

It remains to verify that $(Y,\bar{t})$ is strongly symmetric. Assume that $\mathcal{H}\xrightarrow{t_\alpha}z$ and $\dot{z}\xrightarrow{t_\alpha}y$; then there exists an $\mathcal{F}\in\mathbf{F}(X)$ such that $f^{\to}\mathcal{F}\xrightarrow{r_\alpha}z$ and $\mathcal{H}\geq\mathrm{cl}_{r_\alpha} f^{\to}\mathcal{F}$. Since $\dot{z}\xrightarrow{r_\alpha}y$ and (Y,r_α) is symmetric, $f^{\to}\mathcal{F}\xrightarrow{r_\alpha}y$. It follows that $\mathcal{H}\xrightarrow{t_\alpha}y$, and thus (Y,t_α) is symmetric. Hence, $((Y,\bar{t}),f)$ is a strict, strongly symmetric compactification of $(X,\bar{q})$ in GLTS. $\square$

3. One-Point Compactification

Necessary and sufficient conditions for a limit tower space to have a strongly symmetric compactification are provided in Theorem 2.6. One-point strongly symmetric compactifications are considered here. First, one-point symmetric compactifications in LIM are discussed. Let *SLIM* denote the full subcategory of LIM consisting of all the symmetric objects in LIM.

Definition 3.1. An object $(X,q)\in|\mathrm{LIM}|$ is called *locally compact* provided that each convergent filter contains a q-compact subset of X. Moreover, $(X,\bar{q})\in|\mathrm{GLTS}|$ is said to be *strongly locally compact* whenever each (X,q_α) is a locally compact limit space for each $0\leq\alpha\leq\infty$.

It is shown below that local compactness is a necessary condition for a limit space to have a one-point symmetric compactification. However, examples exist of symmetric limit spaces which fail to be locally compact but possess a regular compactification in LIM. Additional assumptions beyond strong local compactness are needed to guarantee the existence of a one-point strongly symmetric compactification in GLTS.

Theorem 3.2. *Assume that $(X, q) \in |SLIM|$ is not compact. Then (X, q) has a one-point symmetric compactification in LIM if and only if (X, q) is locally compact and q and δq agree on ultrafilter convergence.*

Proof. Suppose that $((Y, p), f)$ is a one-point symmetric compactification of (X, q) where $Y - f(X) = \{b\}$. Assume that $\mathcal{F} \xrightarrow{q} x$; then $\mathrm{cl}_q \mathcal{F} = \mathrm{cl}_{\delta q} \mathcal{F} \xrightarrow{q} x$. If $\mathcal{F}$ fails to have a q-compact member, then for each $F \in \mathcal{F}$ there exists an ultrafilter $\mathcal{H}_F$ such that $\mathrm{cl}_q F \in \mathcal{H}_F$ and $\mathcal{H}_F$ fails to q-converge. It follows that $f^{\rightarrow}\mathcal{H}_F \xrightarrow{p} b$, and thus $b \in \mathrm{cl}_p f(\mathrm{cl}_q F)$ for each $F \in \mathcal{F}$. Hence, $\dot{b} \geq \mathrm{cl}_p f^{\rightarrow}(\mathrm{cl}_q \mathcal{F}) \xrightarrow{p} f(x)$. Since (Y, p) is symmetric, b and $f(x)$ agree on ultrafilter convergence. Hence, for each $F \in \mathcal{F}$, $\mathcal{H}_F \xrightarrow{q} x$, which is a contradiction, and therefore (X, q) is locally compact.

Conversely, assume that (X, q) is locally compact, symmetric, and q and δq agree on ultrafilter convergence. Denote $\hat{X} = X \cup \{w\}$, $w \notin X$, and let $j : X \to \hat{X}$, $j(x) = x$, $x \in X$ be the natural injection. Define $\hat{q}$ as

(i) $\mathcal{H} \xrightarrow{\hat{q}} j(x)$ if and only if $j(X) \in \mathcal{H}$ and $j^{\leftarrow}\mathcal{H} \xrightarrow{q} x$;

(ii) $\mathcal{H} \xrightarrow{\hat{q}} w$ if and only if either $\mathcal{H} = \dot{w}$ or $\alpha_q(j^{\leftarrow}\mathcal{H}) = \emptyset$, where $\alpha_q(j^{\leftarrow}\mathcal{H})$ denotes the set of all adherence points of $j^{\leftarrow}\mathcal{H}$.

It easily follows that $(\hat{X}, \hat{q}) \in |\mathrm{LIM}|$. Observe that if $\mathcal{H} \xrightarrow{\hat{q}} j(x)$, then there exists an $\mathcal{F} \xrightarrow{q} x$ such that $\mathcal{H} \geq j^{\rightarrow}\mathcal{F}$. Since $\mathrm{cl}_q \mathcal{F} = \mathrm{cl}_{\delta q} \mathcal{F} \xrightarrow{q} x$ and (X, q) is locally compact, $\mathcal{F}$ contains a q-closed, q-compact set A, and it follows that $w \notin \mathrm{cl}_{\hat{q}} j(A)$. Hence, $\mathrm{cl}_{\hat{q}} \mathcal{H} \geq \mathrm{cl}_{\hat{q}} j^{\rightarrow}\mathcal{F} = j^{\rightarrow}(\mathrm{cl}_q \mathcal{F})$, and thus $\mathrm{cl}_{\hat{q}} \mathcal{H} \xrightarrow{\hat{q}} j(x)$. Next, suppose that $\mathcal{H} \xrightarrow{\hat{q}} w$. If $\mathcal{H} = \dot{w}$, then $\dot{w} \nrightarrow \hat{q}j(x)$ implies that $\mathrm{cl}_{\hat{q}} \mathcal{H} = \dot{w} \xrightarrow{\hat{q}} w$. Now consider the case that whenever $\mathcal{H} \xrightarrow{\hat{q}} w$ and $j^{\leftarrow}\mathcal{H} = \mathcal{G}$ exists, then $\alpha_q(\mathcal{G}) = \emptyset$. It remains to show that $\mathrm{cl}_{\hat{q}} \mathcal{H} \xrightarrow{\hat{q}} w$. Since $\dot{w} \nrightarrow \hat{q}j(x)$ and q and δq agree on ultrafilter convergence, it follows that $\alpha_q(j^{\leftarrow} \mathrm{cl}_{\hat{q}} \mathcal{H}) = \alpha_q(\mathrm{cl}_q \mathcal{G}) = \alpha_{\delta q}(\mathrm{cl}_{\delta q} \mathcal{G}) = \alpha_{\delta q}(\mathcal{G}) = \alpha_q(\mathcal{G}) = \emptyset$, and thus $\mathrm{cl}_{\hat{q}} \mathcal{H} \xrightarrow{\hat{q}} w$. Hence, $(\hat{X}, \hat{q})$ is regular.

It is shown that $(\hat{X}, \hat{q})$ is symmetric. Assume that $\mathcal{H} \xrightarrow{\hat{q}} t$ and $\dot{t} \xrightarrow{\hat{q}} j(x)$. If $t = j(z)$, then $\mathcal{H} \geq j^{\rightarrow}\mathcal{F}$ for some $\mathcal{F} \xrightarrow{q} z$, and $\dot{j(z)} \xrightarrow{\hat{q}} j(x)$ implies that $\dot{z} \xrightarrow{q} x$. Hence, $\mathcal{F} \xrightarrow{q} x$ and $\mathcal{H} \xrightarrow{\hat{q}} j(x)$. Since $\dot{w} \nrightarrow \hat{q}j(x)$, $t = w$ is not possible. Next, suppose that $\mathcal{H} \xrightarrow{\hat{q}} t$ and $\dot{t} \xrightarrow{\hat{q}} w$. Then $\dot{w} \xrightarrow{\hat{q}} t$, and thus

$t = w$; hence, $\mathcal{H} \xrightarrow{\hat{q}} w$. Therefore, $(\hat{X}, \hat{q})$ is symmetric, $j : (X, q) \to (\hat{X}, \hat{q})$ is a dense embedding, and $(\hat{X}, \hat{q})$ is compact; thus, $((\hat{X}, \hat{q}), j)$ is a one-point symmetric compactification of (X, q). $\square$

Let $(X, \bar{q}) \in |\mathrm{GLTS}|$ be symmetric and assume that (X, q_0) is not compact. Denote $\mathcal{A} = \{\mathcal{F} \in \mathbf{F}(X) : \alpha_{q_0}(\mathcal{F}) = \emptyset\}$ and, for each fixed $0 < \alpha < \infty$, define $B_\alpha = \{x \in X : \mathcal{G} \xrightarrow{q_\alpha} x$ for some $\mathcal{G} \in \mathcal{A}\}$. Consider the following axiom:

for every fixed $0 < \alpha < \infty$,

(A1) $\mathcal{F} \xrightarrow{q_\alpha} x$ for each $\mathcal{F} \in \mathcal{A}$ and $x \in B_\alpha$.

Observe that since (X, q_α) is symmetric, assumption (A1) implies that q_α-convergence coincides at all $x \in B_\alpha$. Let B_α^c denote the complement of B_α in X.

Lemma 3.3. *Assume that $(X, \bar{q}) \in |GLTS|$ obeys (A1) and possesses a strongly symmetric compactification and that (X, q_0) is not compact. Then for each fixed $0 < \alpha < \infty$,*

(i) $\dot{B}_\alpha \xrightarrow{q_\alpha} x$ *for each* $x \in B_\alpha$,

(ii) *no filter q_α-converges to both an $x \in B_\alpha$ and a $z \in B_\alpha^c$,*

(iii) *B_α is q_α-closed,*

(iv) $\mathcal{G} \xrightarrow{q_\alpha} z \in B_\alpha^c$ *implies that* $B_\alpha^c \in \mathrm{cl}_{q_\alpha} \mathcal{G}$.

Proof. (i) Since (X, q_α) is symmetric, it follows from (A1) that for $x \in B_\alpha$, $\dot{B}_\alpha \geq \mathrm{cl}_{q_\alpha} \dot{x} \xrightarrow{q_\alpha} x$. Hence, $\dot{B}_\alpha \xrightarrow{q_\alpha} x$ for each $x \in B_\alpha$.

(ii) Suppose that $\mathcal{K} \xrightarrow{q_\alpha} x, z$, where $x \in B_\alpha$ and $z \in B_\alpha^c$. According to the symmetry, x and z have the same q_α-convergent filters. This implies that $z \in B_\alpha$, and thus no such $\mathcal{K}$ exists.

(iii)–(iv) Verification follows from (i) and (ii). $\square$

Theorem 3.1 shows that if $(X, \bar{q})$ possesses a one-point strongly symmetric compactification in GLTS, then each (X, q_α), $0 \leq \alpha \leq \infty$, must necessarily be locally compact.

Theorem 3.4. *Assume that $(X, \bar{q}) \in |GLTS|$ is locally compact but not compact, satisfies (A1), and possesses a strongly symmetric compactification. Then $(X, \bar{q})$ has a one-point strongly symmetric compactification in GLTS.*

Proof. Denote $\hat{X} = X \cup \{w\}$ where $w \notin X$, and let $j : X \to \hat{X}$ be the natural injection. Define $\hat{q}_0$ on $\hat{X}$ as in Theorem 3.1:

$$\mathcal{H} \xrightarrow{\hat{q}_0} j(x) \text{ if and only if } j(X) \in \mathcal{H} \text{ and } j^{\leftarrow}\mathcal{H} \xrightarrow{q_0} x$$
$$\mathcal{H} \xrightarrow{\hat{q}_0} w \text{ if and only if either } \mathcal{H} = \dot{w} \text{ or } \alpha_{q_0}(j^{\leftarrow}\mathcal{H}) = \emptyset.$$

It is shown in Theorem 3.2 that $((\hat{X}, \hat{q}_0), j)$ is a one-point symmetric compactification of (X, q_0) in LIM. For each fixed $0 < \alpha < \infty$, define $\hat{q}_\alpha$ on $\hat{X}$ as follows:

(i) for $x \in X - B_\alpha$, $\mathcal{H} \xrightarrow{\hat{q}_\alpha} j(x)$ if and only if $j(X) \in \mathcal{H}$ and $j^{\leftarrow}\mathcal{H} \xrightarrow{q_\alpha} x$;

(ii) for $x \in B_\alpha$, $\mathcal{H} \xrightarrow{\hat{q}_\alpha} j(x), w$ if and only if either $\mathcal{H} = \dot{w}$ or $j^{\leftarrow}\mathcal{H} \xrightarrow{q_\alpha} x$;

(iii) for $B_\alpha = \emptyset$, $\mathcal{H} \xrightarrow{\hat{q}_\alpha} w$ if and only if either $\mathcal{H} = \dot{w}$ or $\alpha_{q_0}(j^{\leftarrow}\mathcal{H}) = \emptyset$.

Moreover, define $\hat{q}_\infty$ to be the indiscrete limit structure on $\hat{X}$, and denote $\bar{\hat{q}} = (\hat{q})_\alpha$, $0 \leq \alpha \leq \infty$. It easily follows that $(\hat{X}, \bar{\hat{q}}) \in |\mathrm{GLTS}|$.

Fix $0 < \alpha < \infty$; it is shown below that $(\hat{X}, \hat{q}_\alpha)$ is regular.

Case 1: $B_\alpha \neq \emptyset$ and $X - B_\alpha \neq \emptyset$.

(a): Assume that $\mathcal{H} \xrightarrow{\hat{q}_\alpha} j(x)$ where $x \in X - B_\alpha$; then $j(X) \in \mathcal{H}$ and $\mathcal{F} = j^{\leftarrow}\mathcal{H} \xrightarrow{q_\alpha} x$. According to Lemma 3.3(iv), $X - B_\alpha \in \mathrm{cl}_{q_\alpha} \mathcal{F}$. Choose $F \in \mathcal{F}$ such that $\mathrm{cl}_{q_\alpha} F \subseteq X - B_\alpha$. Suppose that $w \in \mathrm{cl}_{\hat{q}_\alpha} j(F)$; then there exists a $\mathcal{K} \in \mathbf{F}(X)$ such that $F \in \mathcal{K}$ and $j^{\rightarrow}\mathcal{K} \xrightarrow{\hat{q}_\alpha} w$. Moreover, $j^{\rightarrow}\mathcal{K} \xrightarrow{\hat{q}_\alpha} j(z)$ for each $z \in B_\alpha$, and thus $\mathcal{K} \xrightarrow{q_\alpha} z$ for each $z \in B_\alpha$. This contradicts the fact that $\mathrm{cl}_{q_\alpha} F \subseteq X - B_\alpha$. Hence, $w \notin \mathrm{cl}_{\hat{q}_\alpha} j(F)$ and $j(X) \in \mathcal{H}$ implies $\mathcal{H} = j^{\rightarrow}\mathcal{F}$; thus, $j(X) \in \mathrm{cl}_{\hat{q}_\alpha} \mathcal{H}$. Therefore, $j^{\leftarrow}(\mathrm{cl}_{\hat{q}_\alpha} \mathcal{H}) = \mathrm{cl}_{q_\alpha} \mathcal{F} \xrightarrow{q_\alpha} x$, and it follows that $\mathrm{cl}_{\hat{q}_\alpha} \mathcal{H} \xrightarrow{\hat{q}_\alpha} j(x)$.

(b): Suppose that $\mathcal{H} \xrightarrow{\hat{q}_\alpha} j(x)$ where $x \in B_\alpha$. If $\mathcal{H} = \dot{w}$, then $\mathrm{cl}_{\hat{q}_\alpha}\{w\} = j(B_\alpha) \cup \{w\}$, and thus by Lemma 3.3(i), $j^{\leftarrow}(\mathrm{cl}_{\hat{q}_\alpha} \dot{w}) = \dot{B}_\alpha \xrightarrow{q_\alpha} x$. Hence, $\mathrm{cl}_{\hat{q}_\alpha} \dot{w} \xrightarrow{\hat{q}_\alpha} j(x)$. Next, assume that $\mathcal{F} = j^{\leftarrow}\mathcal{H}$ exists; then $\mathcal{F} \xrightarrow{q_\alpha} x$ and $j^{\leftarrow}(\mathrm{cl}_{\hat{q}_\alpha} \mathcal{H}) = \mathrm{cl}_{q_\alpha} \mathcal{F} \xrightarrow{q_\alpha} x$. Therefore, $\mathrm{cl}_{\hat{q}_\alpha} \mathcal{H} \xrightarrow{\hat{q}_\alpha} j(x)$ whenever $x \in B_\alpha$. This also covers the case whenever $\mathcal{H} \xrightarrow{\hat{q}_\alpha} w$ and $B_\alpha \neq \emptyset$.

Case 2: $B_\alpha = \emptyset$.

(a): Assume that $\mathcal{H} \xrightarrow{\hat{q}_\alpha} j(x)$ where $x \in X$. Then $j(X) \in \mathcal{H}$ and $\mathcal{F} = j^{\leftarrow}\mathcal{H} \xrightarrow{q_\alpha} x$. Since (X, q_α) is locally compact, there exists a q_α-compact set $A \in \mathcal{F}$. Suppose that $\mathcal{U}$ is an ultrafilter on X such that $A \in \mathcal{U}$ and $j^{\rightarrow}\mathcal{U} \xrightarrow{\hat{q}_\alpha} w$. Then, by the definition of $\hat{q}_\alpha$, $\mathcal{U}$ fails to q_0-converge, and thus $\mathcal{U} \in \mathcal{A}$. However, since A is q_α-compact, $\mathcal{U} \xrightarrow{q_\alpha} z$ for some $z \in A$, and thus $z \in B_\alpha$. This contradiction implies that $w \notin \mathrm{cl}_{\hat{q}_\alpha} j(A)$, and thus $j(X) \in \mathrm{cl}_{\hat{q}_\alpha} \mathcal{H}$. Since $j^{\leftarrow}(\mathrm{cl}_{\hat{q}_\alpha} \mathcal{H}) = \mathrm{cl}_{q_\alpha} \mathcal{F} \xrightarrow{q_\alpha} x$, $\mathrm{cl}_{\hat{q}_\alpha} \mathcal{H} \xrightarrow{\hat{q}_\alpha} j(x)$.

(b): Suppose that $\mathcal{H} \xrightarrow{\hat{q}_\alpha} w$. If $\mathcal{H} = \dot{w}$, then $\dot{w} \nrightarrow \hat{q}_\alpha j(z)$ since $j(X) \notin \dot{w}$, and thus $\mathrm{cl}_{\hat{q}_\alpha} \dot{w} = \dot{w} \xrightarrow{\hat{q}_\alpha} w$. Next, assume that $\mathcal{F} = j^{\leftarrow}\mathcal{H}$ exists; then $\alpha_{q_0}(\mathcal{F}) = \emptyset$. Since q_0 and δq_0 agree on ultrafilter convergence, $\alpha_{q_0}(j^{\leftarrow} \mathrm{cl}_{\hat{q}_0} \mathcal{H}) = \alpha_{q_0}(\mathrm{cl}_{q_0} \mathcal{F}) = \alpha_{\delta q_0}(\mathrm{cl}_{\delta q_0} \mathcal{F}) = \alpha_{\delta q_0}(\mathcal{F}) = \alpha_{q_0}(\mathcal{F}) = \emptyset$. Hence, $\mathrm{cl}_{\hat{q}_\alpha} \mathcal{H} \xrightarrow{\hat{q}_\alpha} w$. It follows that $(\hat{X}, \bar{\hat{q}})$ is regular.

It remains to show that $(\hat{X}, \bar{\hat{q}})$ is symmetric. Suppose that $\mathcal{H} \xrightarrow{\hat{q}_\alpha} y_1$ and $\dot{y}_1 \xrightarrow{\hat{q}_\alpha} y_2$. Consider the following cases:

(i) $y_1 = j(z)$, $z \in B_\alpha$. It follows that either $y_2 = j(x)$ for some $x \in B_\alpha$ or $y_2 = w$, and thus $\mathcal{H} \xrightarrow{\hat{q}_\alpha} y_2$.

(ii) $y_1 = j(z)$, $z \in X - B_\alpha$. Note that $y_2 = j(x)$ for some $x \in X - B_\alpha$. Then $j(X) \in \mathcal{H}$ and $j^{\leftarrow}\mathcal{H} \xrightarrow{\hat{q}_\alpha} z$. Also, $\dot{z} \xrightarrow{q_\alpha} x$, and thus $j^{\leftarrow}\mathcal{H} \xrightarrow{q_\alpha} x$; hence, $\mathcal{H} \xrightarrow{q_\alpha} j(x)$.

(iii) $y_1 = w$ and $B_\alpha \neq \emptyset$. Since $\mathcal{H} \xrightarrow{\hat{q}_\alpha} w$ if and only if $\mathcal{H} \xrightarrow{\hat{q}_\alpha} j(z)$ for each $z \in B_\alpha$, $\mathcal{H} \xrightarrow{\hat{q}_\alpha} y_2$ by (i).

(iv) $y_1 = w$ and $B_\alpha = \emptyset$. Observe that $\dot{w} \xrightarrow{\hat{q}_\alpha} j(x)$ for some $x \in X$ is impossible since $B_\alpha = \emptyset$.

It follows that $(\hat{X}, \hat{q}_\alpha)$ is symmetric and, by construction, $j : (X, q_\alpha) \to (\hat{X}, \hat{q}_\alpha)$ is a dense embedding. Therefore, $((\hat{X}, \bar{\hat{q}}), j)$ is a one-point strongly symmetric compactification of $(X, \bar{q})$ in GLTS. □

Let us conclude with an example illustrating the significance of axiom (A1). Recall that axiom (A1) requires that if $\alpha > 0$, then each $\mathcal{F} \in \mathcal{A}$ must q_α-converge to x provided that some $\mathcal{G} \in \mathcal{A}$ q_α-converges to x. Property (A1) is needed to ensure that a one-point compactification is strongly symmetric.

Example 3.5. Suppose that $X = [0, 1)$ and let $\mathcal{G}_i$ denote the filter on X whose base is $\{(0, \epsilon) : \epsilon > 0\}$ ($\{(1 - \epsilon, 1) : \epsilon > 0\}$), $i = 0, 1$. Let τ denote the usual topology on $[0, 1)$ and define q_α as follows:

(i) if $0 \leq \alpha < 1$, $\mathcal{F} \xrightarrow{q_\alpha} x$ if and only if $\mathcal{F} \xrightarrow{\tau} x$, where $x > 0$, and $\mathcal{F} \xrightarrow{q_\alpha} 0$ if and only if $\mathcal{F} = \dot{0}$;

(ii) if $1 \leq \alpha < \infty$, $\mathcal{F} \xrightarrow{q_\alpha} x$ if and only if $\mathcal{F} \xrightarrow{\tau} x$;

(iii) if $\alpha = \infty$, $\mathcal{F} \xrightarrow{q_\alpha} x$ if and only if $\mathcal{F} \geq \dot{X}$.

Then $(X, \bar{q}) \in |\mathrm{LTS}|$ is locally compact and strongly symmetric; observe that $\mathcal{A} = \{\mathcal{G} \in \mathbf{F}(X) : \mathcal{G} \geq \mathcal{G}_0 \cap \mathcal{G}_1\}$. Note that $B_\alpha = \emptyset$ for $0 < \alpha < 1$ and, moreover, since $\mathcal{G}_0 \in \mathcal{A}$ and $\mathcal{G}_0 \xrightarrow{q_\alpha} 0$, $B_\alpha = \{0\}$ for $1 \leq \alpha < \infty$.

However, $\mathcal{G}_1 \in \mathcal{A}$, but $\mathcal{G}_1 \nrightarrow q_\alpha 0$, $1 \leq \alpha < \infty$, and hence $(X, \bar{q})$ fails to satisfy axiom (A1). It easily follows that $(X, \bar{q})$ fails to possess a one-point strongly symmetric compactification in GLTS. The modification $\mathcal{F} \xrightarrow{q_\alpha} 0$ if and only if $\mathcal{F} \geq \mathcal{G}_0 \cap \mathcal{G}_1 \cap \dot{0}$ for each $1 \leq \alpha < \infty$ is needed in order to obtain a one-point symmetric compactification.

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References

[1] Jiří Adámek, Horst Herrlich, and George E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. New York: John Wiley & Sons, Inc., 1990.

[2] H. Boustique and G. Richardson, *Compactification: Limit tower spaces*, Appl. Categor. Struct. **25** (2017), no. 3, 349–361.

[3] Paul Brock and D. C. Kent, *Approach spaces, limit tower spaces, and probabilistic convergence spaces*, Appl. Categ. Structures **5** (1997), no. 2, 99–110.

[4] Paul Brock and Darrell Kent, *On convergence approach spaces*, Appl. Categ. Structures **6** (1998), no. 1, 117–125.

[5] Gunther Jäger, *A Stone-Čech type compactification for convergence approach spaces*, Quaest. Math. **35** (2012), no. 2, 209–217.

[6] H. H. Keller, *Die limes-uniformisierbarkeit der limesräume*, Math. Ann. **176** (1968), 334–341.

[7] D. C. Kent and G. D. Richardson, *Completely regular and ω-regular spaces*, Proc. Amer. Math. Soc. **82** (1981), no. 4, 649–652.

[8] B. Losert and G. Richardson, *Convergence S-completions*, Topology Proc. **46** (2015), 15–27.

[9] E. Lowen and R. Lowen, *A quasitopos containing CONV and MET as full subcategories*, Internat. J. Math. Math. Sci. **11** (1988), no. 3, 417–438.

[10] R. Lowen, *Approach spaces: A common supercategory of TOP and MET*, Math. Nachr. **141** (1989), 183–226.

[11] R. Lowen, *Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad*. Oxford Mathematical Monographs. Oxford Science Publications. New York: The Clarendon Press, Oxford University Press, 1997.

[12] G. D. Richardson and D. C. Kent, *The regularity series of a convergence space*, Bull. Austral. Math. Soc. **13** (1975), no. 1, 21–44.

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