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PATCHWORK AND WEB SPACES

MARCEL ERNÉ

Abstract. Patch topologies are obtained by joining a given topology with a second topology having the dual specialization order. They provide a convenient passage from topological spaces to semi-qospaces (i.e., quasi-ordered sets equipped with a topology, making principal ideals and filters closed) that have better separation properties than the original spaces. Web spaces, originally defined by the existence of neighborhood bases consisting of webs (where a web around x contains with any y a lower bound of x and y), may be characterized by the condition that the interior operator preserves finite unions of saturated sets. More important in the patch game is that the web spaces are precisely those for which any patch space determines the original open sets as the upper sets generated by the patch open sets. Via suitable patch functors, the category of web spaces is concretely isomorphic to various categories of strongly convex web semi-qospaces. We apply the patch construction to semitopological semilattices (as specific web ordered spaces) and show that the T_0 web spaces are exactly the so-called $\downarrow$-consistent subspaces of semitopological semilattices with a compatible topology; similar representations are established for web ordered spaces. A look at regularity axioms for patch spaces concludes the study.

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A frame, patchwork, and a web space

1. Introduction

Before giving a survey of the intended program, let us fix some order-theoretical terminology and notation. A *quasi-ordered set* or *qoset* is a pair $Q = (X, \leq)$ with a reflexive and transitive relation $\leq$ on the set X. The dual relation is denoted by $\geq$ and the dual qoset $(X, \geq)$ by $\widetilde{Q}$. If $\leq$ is antisymmetric, we speak of a (*partial*) *order* and an *ordered set* or *poset*. A *lower set* of Q is a subset Y that coincides with its *down-closure* $\downarrow Y = \downarrow_Q Y = \{x \in X : \exists\, y \in Y\, (x \leq y)\}$. *Upper sets* and the *up-closure* $\uparrow Y$ are defined dually. The upper sets form the *upper Alexandroff topology* αQ, and the lower sets (the complements of the upper sets) form the *lower Alexandroff topology* $\alpha \widetilde{Q}$. Specifically, $\downarrow x = \downarrow\{x\}$ is the *principal ideal* and $\uparrow x = \uparrow\{x\}$ is the *principal filter* generated by the element x. A subset D of a qoset is (*up-*)*directed* or *filtered* (*down-directed*), respectively, if every finite subset of D has an upper or lower bound, respectively, in D; thus, directed sets and filtered sets are nonempty. A poset is *up-complete*, *directed complete*, or a *dcpo* if each directed subset has a join, that is, a least upper bound (supremum). We adopt the convention of calling arbitrary up-complete posets *domains* (in [13], the term *domain* is reserved for *continuous dcpos*, which play only a marginal role in the present paper). A *frame* or *locale* [15] is a complete lattice L enjoying the identity

$$\text{(d)} \quad x \wedge \bigvee Y = \bigvee \{x \wedge y : y \in Y\}$$

for all $x \in L$ and $Y \subseteq L$ (the symbol $\bigvee$ denotes joins, the symbol $\wedge$ binary meets); the dual identity characterizes *coframes*. An up-complete meet-semilattice satisfying (d) for all directed sets Y is called *meet-continuous*.

By a *space* we always mean a topological space. However, most of the definitions and results may be extended to the much more general class of *kernel spaces* or *closure spaces*, i.e., sets equipped with a collection of subsets closed under arbitrary unions or intersections, respectively. This idea and its history have been presented more extensively in [4] and [9].

Let $(X, \mathcal{S})$ be a space with $\mathcal{S}$ the frame of all open sets (that is, the topology) and $\mathcal{S}^c$ the coframe of all closed sets. The closure of a subset Y is denoted by $cl_{\mathcal{S}}Y$ and the interior by $int_{\mathcal{S}}Y$. A basic link between order and topology is provided by the *specialization order*, denoted by $\leq_{\mathcal{S}}$ or merely by $\leq$, and defined by

$$x \leq y \;\Leftrightarrow\; x \in cl_{\mathcal{S}}\{y\} \;\Leftrightarrow\; y \in \bigcap\{U \in \mathcal{S} : x \in U\}.$$

This is an order (i.e., antisymmetric) if and only if the space $(X, \mathcal{S})$ is T_0, but we shall speak of the *specialization order* also in the non-T_0 setting. The corresponding *specialization qoset* $(X, \leq_{\mathcal{S}})$ is denoted by $\Sigma^-(X, \mathcal{S})$. This gives rise to a concrete *specialization functor* Σ^- (denoted by Ω in [13]) from the category of spaces with continuous maps to the category of quasi-ordered sets with isotone (that is, order preserving) maps. Relative to the specialization order, the point closures are the principal ideals, while the *core* of a point x, the intersection of all neighborhoods of x, is the principal filter $\uparrow x$. The lower sets are the unions of point closures (or of arbitrary closed sets), and the upper sets are the unions of cores, but also the *saturated sets*, i.e., the intersections of open sets. A topology is closed under arbitrary intersections if and only if it is of the form αQ for a (unique) qoset Q (see [2]); spaces with that property are referred to as *Alexandroff (discrete) spaces* or *A-spaces*. Restricted to A-spaces, the specialization functor becomes a concrete categorical isomorphism.

A topology $\mathcal{S}$ on a set X is said to be *compatible* with a quasi-order $\leq$ on X if $Q = (X, \leq)$ is the specialization qoset of $(X, \mathcal{S})$ (in [13], compatibility has a different meaning). While the finest topology compatible with $\leq$ is the (upper) Alexandroff topology αQ, the coarsest one is the *weak upper topology* υQ, generated by the complements of principal ideals. In fact, compatibility of $\mathcal{S}$ with $\leq$ is equivalent to the inclusion $\upsilon Q \subseteq \mathcal{S} \subseteq \alpha Q$. The *weak lower topology* of a qoset Q is the weak upper topology $\upsilon \widetilde{Q}$ (elsewhere denoted by ωQ) of the order-dual $\widetilde{Q}$.

In §2, we record basic notions and facts concerning quasi-ordered spaces and introduce some relevant order-theoretical local convexity properties. Monotone nets and their convergence are basic tools in the theory of computation and also in topological aspects of order and domain theory. They provide abstract characterizations of the Scott and Lawson spaces of arbitrary domains (dcpos) by means of monotone convergence.

In §3, we associate in a systematic manner with any space a collection of *patch spaces*; these are quasi-ordered spaces whose order relation is the specialization order of the original space and whose topology is a patch topology, that is, the join of the original topology and another topology having the dual specialization order. The patch spaces are exactly the *strongly convex semi-qospaces*, that is, those quasi-ordered spaces whose principal ideals and filters are closed and whose open upper sets and open lower sets together generate the topology. A famous case is the patch space carrying the join of the original topology and the cocompact topology. Here, the patch construction gives rise to an equivalence between the category of stably compact spaces and the category of compact pospaces (see [13, VI-5] and [18]). A continuous map between compact pospaces has the property that preimages of compact sets are compact, so that the morphism part is unproblematic in that context. We shall find some interesting extensions to situations with no compactness assumptions or at most local compactness assumptions.

Of particular relevance are patch spaces that arise from *coselections* ζ, assigning to each topology $\mathcal{S}$ a subbase $\zeta\mathcal{S}$ of a topology $\tau_\zeta\mathcal{S}$ having the dual specialization order. The join $\mathcal{S}^{\zeta}$ of $\mathcal{S}$ and $\tau_\zeta\mathcal{S}$ is then a patch topology, and $\mathrm{P}_\zeta(X,\mathcal{S}) = (X, \leq_{\mathcal{S}}, \mathcal{S}^{\zeta})$ is a patch space. Of course, such patch constructions may be regarded as special instances of the passage between "complemented" bitopological spaces and the associated quasi-ordered spaces (see, for example, [11], [16], and [18]). In the opposite direction, we associate with any quasi-ordered space $(X, \leq, \mathcal{T})$ the *upper space* $(X, \mathcal{T}^{\leq})$, whose topology consists of all $\mathcal{T}$-open upper sets. We discuss order-theoretical convexity properties of patch spaces and the right choice of morphisms to make the above assignments functorial in both directions, and we determine the largest object classes for which these functors become mutually inverse isomorphisms.

In §4, we apply the patch construction to so-called *web spaces*, a particular class of path-connected spaces introduced in [8]. In such spaces, each point x has a neighborhood base of *webs*, containing x and with each point y a common lower bound of x and y. Surprisingly, web spaces may be characterized by an infinite distributive law: They are exactly those spaces for which not only the lattice of open sets but also that of closed sets is a frame [8]. A crucial role in the patch game is played by $\uparrow$-*stable* quasi-ordered spaces, in which the upper set generated by any open set is open, too (spaces with this and the dual property have been called *I-spaces* by Hans-Peter A. Künzi [16] and H. A. Priestley [20]). It turns out that the web spaces are exactly those spaces for which each patch space is $\uparrow$-stable and has the original space as upper space. We find diverse effective characterizations of *web quasi-ordered spaces*, i.e., $\uparrow$-stable quasi-ordered spaces with web neighborhood bases at each point. For every coselection ζ, the patch functor P_ζ induces an isomorphism between the category of web spaces and the category of ζ-*convex web semi-qospaces*;

these are $\uparrow$-stable web quasi-ordered semi-qospaces $(X, \leq, \mathcal{T})$ whose topology is generated by the upper topology $\mathcal{T}^{\leq}$ and its cotopology $\tau_{\zeta}(\mathcal{T}^{\leq})$.

In §5, we focus on semitopological semilattices as special instances of web ordered spaces, and we obtain abstract characterizations of the involved Scott and Lawson spaces. Then, in §6, we characterize T_0 web spaces and their patch spaces as certain "consistent" subspaces of strongly convex T_1-semitopological semilattices, using an ordered variant of the Vietoris topology. Finally, in §7, we touch upon situations where the considered spaces and their patch spaces satisfy higher separation axioms in form of regularity conditions.

These investigations are continued in a separate paper on core spaces (alias worldwide web spaces [8]; see also [4] and [6]), an infinitary analogue of web spaces, and their patch spaces.

For related material on quasi-uniform aspects, refer to Leopoldo Nachbin's pioneering monograph [19] and the work of Peter Fletcher and William F. Lindgren [11], Künzi [16], and J. D. Lawson [18].

If not otherwise stated, the results in the present paper are derived in a choice-free set-theoretical framework. That is, we shall not use the Axiom of Choice (AC) or weaker choice principles like the Ultrafilter Theorem (UT) not derivable in Zermelo-Fraenkel set theory (ZF), unless necessary.

Most of the functors considered in this paper are *concrete*; that is, they are functors between *constructs*, i.e., concrete categories over sets (with forgetful functors to the category of sets), and they keep fixed the underlying set functions of the morphisms. Such functors provide simultaneously different aspects of mathematical objects. For more background concerning these and related categorical topics, refer to [1].

The main reference for definitions and facts in the theory of order and topology is the monograph [13].

2. **Convexity and Monotone Convergence in Ordered Spaces**

By a *(quasi-)ordered space*, we mean here merely a (quasi-)ordered set equipped with a topology (no separation axioms required a priori). Some classical separation axioms extend to the ordered case as follows. Call a quasi-ordered space a *lower semi-qospace* if all principal ideals are closed, an *upper semi-qospace* if all principal filters are closed, and a *semi-qospace* if both conditions are fulfilled; that is, the quasi-order is *lower semiclosed*, *upper semiclosed*, or *semiclosed*, respectively, in the sense of [13, VI-1]. An ordered space with a semiclosed order is said to be a *semi-pospace* or *T_1-ordered* since it has to be T_1 (singletons are closed), and semiclosedness of the equality relation amounts to the T_1 axiom. The coarsest topology making a qoset Q a semi-qospace is the *interval topology* ιQ.

A space equipped with a closed quasi-order $\leq$ (regarded as a subset of the square of the space) is referred to as a *qospace* and as a *pospace* in case $\leq$ is a partial order [13]. Every qospace is a semi-qospace. In a T_2-*ordered space*, for $x \not\leq y$ there is an open upper set U and a disjoint open lower set V such that $x \in U$ and $y \in V$. An important class of T_2-ordered spaces is that of totally order-disconnected spaces [20] or totally order-separated spaces [15]; dropping the antisymmetry condition, we call a quasi-ordered space *totally order-separated* if, for $x \not\leq y$, there is a clopen upper set containing x but not y. A *Priestley space* is a compact totally order-separated ordered space. Notice that such spaces are zero-dimensional (i.e., they have a base of clopen sets), but in the absence of compactness, a totally order-separated or totally disconnected space need not be zero-dimensional (see [15, II–4]).

A quasi-ordered space is said to be *upper regular* if, for each open upper set O and each x in O, there is an open upper set U and a closed upper set B with $x \in U \subseteq B \subseteq O$. *Lower regular* spaces are defined dually. Note the non-invertible implications

compact pospace $\Rightarrow$ *upper/lower regular ordered* $\Rightarrow$ T_2*-ordered* $\Rightarrow$ *pospace.*

With a quasi-ordered space $(Q, \mathcal{T}) = (X, \leq, \mathcal{T})$, we associate the *upper space* $\mathrm{U}(Q, \mathcal{T}) = (X, \mathcal{T}^{\leq})$ and the *lower space* $\mathrm{L}(Q, \mathcal{T}) = (X, \mathcal{T}^{\geq})$, where

$\mathcal{T}^{\leq}$ is the topology of all open upper sets,
$\mathcal{T}^{\geq}$ is the topology of all open lower sets.

Note that for lower semi-qospaces $(X, \leq, \mathcal{T})$, the specialization order of $\mathcal{T}^{\leq}$ is $\leq$, while for upper semi-qospaces $(X, \leq, \mathcal{T})$, the specialization order of $\mathcal{T}^{\geq}$ is $\geq$. (In [13], [16], and [18], $\mathcal{T}^{\leq}$ is denoted by $\mathcal{T}^{\sharp}$ and $\mathcal{T}^{\geq}$ by $\mathcal{T}^{\flat}$.)

Some order-theoretical convexity properties will play a crucial role in our study. A subset of a qoset Q is (*order*) *convex* if it is the intersection of an upper and a lower set; equivalently, it contains with any two points y and z every x with $y \leq x \leq z$. In ordered vector spaces like $\mathbb{R}^n$ for $n > 1$, order convexity is incomparable to geometric convexity. Convex in the order-theoretical sense are all *intervals*, i.e., the whole space and all sets

$$[\,y) = \uparrow y,\ ]\,y) = \uparrow y \setminus \downarrow y = \{x : y < x\},\ (z\,] = \downarrow z,\ (z\,[= \downarrow z \setminus \uparrow z = \{x : x < z\},$$
$$[\,y, z\,] = [\,y) \cap (z\,],\ ]\,y, z\,] =]\,y) \cap (z\,],\ [\,y, z\,[= [\,y) \cap (z\,[,\ ]\,y, z\,[=]\,y) \cap (z\,[.$$

The converse holds in conditionally complete chains (e.g., in the real chain $\mathbb{R}$, but not in the rational chain $\mathbb{Q}$).

Now, let us call a quasi-ordered space $(Q, \mathcal{T}) = (X, \leq, \mathcal{T})$

- *locally (order) convex* if the convex open subsets form a base,
- *strongly (order) convex* if the topology $\mathcal{T}$ is generated by $\mathcal{T}^{\leq} \cup \mathcal{T}^{\geq}$,
- υ*-convex* if the topology $\mathcal{T}$ is generated by $\mathcal{T}^{\leq} \cup \upsilon\widetilde{Q}$.

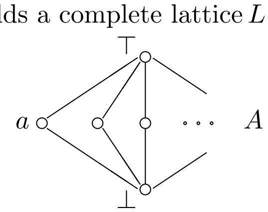
We define υ-convexity to mean that the sets $U \setminus \uparrow F$ with open upper sets U and finite sets F form an open base. Since the Latin prefix *hyper* is the Greek $\upsilon\pi\varepsilon\rho$ for *upper*, one might speak of *order hyperconvexity*, but clearly, this has not much to do with hyperconvexity of metric linear spaces. The following facts are readily checked (for (3), see [13, VI-1]).

Lemma 2.1. (1) *υ-convex quasi-ordered spaces are upper semi-qospaces.*
(2) *υ-convexity implies strong convexity, which implies local convexity.*
(3) *Compact qospaces are strongly convex.*

Example 2.2. On any qoset $Q = (X, \leq)$, an important compatible topology is the *Scott topology* σQ, which consists of all upper sets U that meet all directed subsets whose least upper bounds are contained in U (y is a *least upper bound* of D if $D \subseteq {\downarrow} z \Leftrightarrow y \leq z$); the *weak Scott topology* $\sigma_2 P$ (see [8] and [10]) consists of all upper sets U meeting each directed D with $U \cap \bigcap \{{\downarrow} y : D \subseteq {\downarrow} y\} \neq \emptyset$; it is always contained in σP, and on domains, both topologies coincide. The *Lawson topology* λQ is the join of (i.e., generated by) the Scott topology and the weak lower topology. As in [13], ΣQ denotes the *Scott space* $(X, \sigma Q)$, while ΛQ denotes the *quasi-ordered Lawson space* $(Q, \lambda Q)$, which is always an υ-convex semi-qospace, satisfying the equations $\sigma Q \vee \upsilon \widetilde{Q} = \lambda Q$ and $\lambda Q^{\leq} = \sigma Q$. Furthermore, we put $\Upsilon Q = (X, \upsilon Q)$ and call it the *weak upper space* of Q. For $\mathbb{R}^n$, coordinatewise ordered, $\Lambda(\mathbb{R}^n) = (\Lambda\mathbb{R})^n$ is the standard Euclidean space, while $\Upsilon(\mathbb{R}^n)$ is distinct from $(\Upsilon\mathbb{R})^n = (\Sigma\mathbb{R})^n = \Sigma(\mathbb{R}^n)$ if $n > 1$.

Example 2.3. A poset P with the discrete topology $\mathcal{T}$ is always strongly convex but rarely υ-convex. The topology $\mathcal{T}^{\leq}$ is the Alexandroff topology αP, whence the topology generated by αP and $\upsilon \widetilde{P}$ has a subbase consisting of the right half-open intervals $[\, y, z \,[= \uparrow y \setminus \uparrow z$. This topology is discrete on the integers $\mathbb{Z}$, but not on the rationals $\mathbb{Q}$, nor on the reals $\mathbb{R}$.

Example 2.4. Adding a bottom element $\perp$ and a top element $\top$ to an infinite antichain A yields a complete lattice L of height 2.



Enlarging the cofinite topology on L (which consists of the empty set and all complements of finite sets) by the singleton $\{a\}$ for some $a \in A$, one obtains a topology $\mathcal{T}$ making $(L, \mathcal{T})$ a semi-pospace (but not a pospace) that is locally convex (for $U \in \mathcal{T}$, either $U \setminus \{\perp\}$ or $U \setminus \{\top\}$ is convex and open); but it cannot be strongly convex, since any open upper set containing the point a meets any open lower set containing a in an infinite set and is certainly not contained in the least neighborhood $\{a\}$. This and Example 2.3 show the irreversibility of the implications in Lemma 2.1(2).

Scott spaces and Lawson spaces of domains may be characterized in terms of monotone nets, as defined in [13, O-1] (here, the word "monotone" is synonymous with "isotone"). We say an ordered space T is

(m1) *weakly mc-ordered* if any monotone net in T having a supremum converges to it,
(m2) *mc-ordered* if any monotone net in T has a supremum to which it converges,
(m3) *upper m-determined* if an upper set U in T is open whenever all monotone nets converging to points in U are eventually in U,

and *υmc-ordered* if it is an υ-convex semi-pospace satisfying (m2) and (m3). In the above definitions, monotone nets may be replaced by directed subsets of the space, regarded as nets. For T_0 spaces with the specialization order, (m1) defines the *weak monotone convergence spaces* or *weak mc-spaces* in [6], (m2) the *monotone convergence spaces* in [13] and the *mc-spaces* in [6], and (m3) the *m-determined* spaces (in [8], *monotone determined spaces*); without the monotonicity restriction, the condition in (m3) characterizes open sets in arbitrary spaces. By definition, the mc-(ordered) spaces are just the up-complete weak mc-(ordered) spaces. Furthermore, the mc-spaces coincide with the *d-spaces* or *temperate* spaces in the sense of Oswald Wyler [23], defined by the condition that the closure of any directed subset is the closure of a unique point.

Proposition 2.5. (1) *A strongly convex lower semi-pospace is (weakly) mc-ordered if and only if its upper space is a (weak) mc-space.*

(2) *A lower semi-pospace is upper m-determined if and only if its upper space is m-determined.*

(3) *A space $(X, \mathcal{S})$ with specialization poset P is a weak mc-space if and only if $\mathcal{S} \subseteq \sigma P$.*

(4) *$\sigma_2 P$ is the coarsest topology making P an m-determined space.*

(5) *Compact semi-pospaces are mc-ordered.*

Proof. (1) Let $(P, \mathcal{T}) = (X, \leq, \mathcal{T})$ be a (weakly) mc-ordered lower semi-pospace; recall that the specialization order of $\mathcal{T}^{\leq}$ is $\leq$. If D is a directed subset of P having a supremum x, then D converges to x in $(X, \mathcal{T})$, a fortiori in $(X, \mathcal{T}^{\leq})$ (since $\mathcal{T}^{\leq} \subseteq \mathcal{T}$). Thus, $(X, \mathcal{T}^{\leq})$ is a (weak) mc-space. Conversely, let $(X, \leq, \mathcal{T})$ be a strongly convex lower semi-pospace whose upper space is a (weak) mc-space, and let D be a directed subset with supremum x. Then D converges to x in $(X, \mathcal{T}^{\leq})$ but also in $(X, \mathcal{T}^{\geq})$, being a subset of any lower set containing x. By strong convexity, D also converges to x in $(X, \mathcal{T})$. Hence, $(X, \leq, \mathcal{T})$ is (weakly) mc-ordered.

(2) and (3) are immediate from the definitions.

(4) was shown in [8].

(5) is well known; see, e.g., [13, Proposition VI-1.3]. □

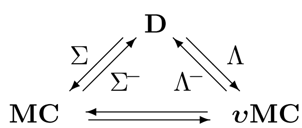
Corollary 2.6. *The Scott spaces of domains are exactly the m-determined mc-spaces.*

A map between quasi-ordered spaces is *lower semicontinuous* if and only if preimages of closed lower sets are closed (see §3). Putting all pieces together, we arrive at the following theorem.

Theorem 2.7. (1) *The Lawson spaces of domains are exactly the* υ*mc-ordered spaces, and their upper spaces are the Scott spaces, that is, the m-determined mc-spaces.*

(2) *The Scott functor* Σ *induces a concrete isomorphism between the category* **D** *of domains with maps preserving directed joins and the category* **MC** *of m-determined mc-spaces with continuous maps. The inverse functor is induced by the specialization functor* Σ^-.

(3) *The Lawson functor* Λ *induces a concrete isomorphism between* **D** *and the category* $\boldsymbol{\upsilon}$**MC** *of* υ*mc-ordered spaces with order preserving lower semicontinuous maps. The inverse functor* Λ^- *forgets the topology.*



In Theorem 3.10, we shall generalize the direct passage between Scott and Lawson spaces to much more general classes of (quasi-ordered) spaces.

3. Patch Spaces and Patch Functors

Spaces violating higher separation axioms may be "improved" by passing to so-called patch spaces that have better properties and still determine the original space. The resulting correspondence between spaces and their ordered counterparts will now be embedded in a general framework.

A *cotopology* (in [18] a *complementary topology*) for a topology $\mathcal{S}$ is a topology $\mathcal{S}'$ (on the same ground set) whose specialization order is dual to that of $\mathcal{S}$. The join topology $\mathcal{T} = \mathcal{S} \vee \mathcal{S}'$ generated by $\mathcal{S} \cup \mathcal{S}'$ is then referred to as a (*general*) *patch topology* and $(X, \leq_{\mathcal{S}}, \mathcal{T})$ as a *patch space* for $(X, \mathcal{S})$. The weak lower topology $\upsilon\widetilde{Q}$ is the weakest, i.e., coarsest cotopology for any topology with specialization qoset Q, and the lower Alexandroff topology $\alpha\widetilde{Q}$ is the strongest, i.e., finest cotopology.

By a *coselection*, we mean a function ζ assigning to each topology $\mathcal{S}$ a subbase $\zeta\mathcal{S}$ of a cotopology $\tau_\zeta\mathcal{S}$. This gives rise to the ζ-*patch topology*

$\mathcal{S}^{\zeta} = \mathcal{S} \vee \tau_{\zeta}\mathcal{S}$ and to the *ζ-patch space* $\mathrm{P}_{\zeta}(X, \mathcal{S}) = (X, \leq_{\mathcal{S}}, \mathcal{S}^{\zeta})$. In case each $\zeta\mathcal{S}$ is already a topology, we speak of a *topological coselection.*

Example 3.1. An important patch space $\mathrm{P}_{\pi}(X, \mathcal{S}) = (X, \leq_{\mathcal{S}}, \mathcal{S}^{\pi})$ results from joining $\mathcal{S}$ with the *cocompact topology* $\tau_{\pi}\mathcal{S}$, which has as a base the set $\pi\mathcal{S}$ of all complements of compact saturated subsets of $(X, \mathcal{S})$, and $\mathcal{S}^{\pi}$ is the *cocompact patch topology* (see, e.g., [13], [14], and [18] for prior applications to ring theory). The function π is a coselection (since cores are compact and saturated), but not a topological one.

Similarly, by an *upset selection* for quasi-ordered sets, we mean a function ζ assigning to each qoset Q a collection ζQ of upper sets such that $\uparrow y = \bigcap \{V \in \zeta Q : y \in V\}$ for all $y \in Q$. The topology generated by ζQ is then compatible, that is, Q is its specialization qoset. In case each ζQ is a topology, we call ζ a *topological upset selection* (for quasi-ordered sets). Any upset selection ζ gives rise to a coselection by putting $\zeta\mathcal{S} = \zeta\widetilde{Q}$ if Q is the specialization qoset of $\mathcal{S}$. In this way, we obtain patch spaces $\mathrm{P}_{\zeta}(X, \mathcal{S}) = (Q, \mathcal{S}^{\zeta})$, where $\mathcal{S}^{\zeta}$ is generated by $\mathcal{S} \cup \zeta\widetilde{Q}$.

The extremal topological upset selections are υ and α. An open base for the *weak patch topology* $\mathcal{S}^{\upsilon}$ is constituted by the sets $U \setminus \uparrow F$ with $U \in \mathcal{S}$ and finite subsets F of $\bigcup \mathcal{S}$, while the *strong patch topology* $\mathcal{S}^{\alpha}$ has a base consisting of the sets $U \setminus \uparrow Y$ with $U \in \mathcal{S}$ and arbitrary subsets Y. The strong patch topology coincides with the *Skula topology* generated by the open sets and the closed sets of the original topology (see [21]). Let us list a few intrinsic patch topologies on posets.

(1) The weak patch topology of the weak upper topology is the *interval topology.*

(2) The weak patch topology of the Scott topology is the *Lawson topology.*

(3) The weak patch topology of the upper Alexandroff topology is the *right half-open interval topology* with subbasic sets $[\, y, z\, [= \uparrow y \setminus \uparrow z$.

(4) The strong patch topology of the weak upper topology is the *left half-open interval topology* with subbasic sets $]\, y, z\,] = \downarrow z \setminus \downarrow y$.

(5) The strong patch topology of the Scott topology is the *finest topology* making the poset a strongly convex mc-ordered space.

(6) The strong patch topology of the upper Alexandroff topology is the *discrete topology.*

(7) The cocompact patch topology of the Scott topology on $\mathbb{R}^n$ is the Euclidean topology, but only for $n = 1$ does it coincide with the interval topology (see Example 2.2 and Example 3.6).

Any topological selection ζ gives rise to a concrete functorial isomorphism from the category of qosets with *ζ-continuous maps* $f : P \to Q$ (satisfying $f^{-1}[U] \in \zeta P$ for $U \in \zeta Q$) to the category of spaces with specialization qoset Q and topology ζQ. The inverse isomorphism is induced by the specialization functor Σ^-. An example is the Scott functor Σ, restricted to the category **D** of domains, with inverse Σ^-, restricted to the category **MC** of m-determined mc-spaces (Theorem 2.7 (2)).

Given a coselection ζ, a space $(X, \mathcal{S})$ is said to be *ζ-determined* if all ζ-patch open upper sets are open in the original space, that is, $\mathcal{S}^{\zeta\leq} \subseteq \mathcal{S}$. Since the reverse inclusion is always true, one has then the equality $\mathcal{S}^{\zeta\leq} = \mathcal{S}$, so that the original space is, in fact, determined by its ζ-patch space; in other words, the ζ-determined spaces are just the upper spaces of their ζ-patch spaces.

Example 3.2. All Alexandroff spaces and all spaces with a linear specialization order are ζ-determined for any coselection ζ.

Example 3.3. Scott spaces are υ-determined in view of the equations $\lambda P = \sigma P^{\upsilon}$ and $\sigma P = \lambda P^{\leq}$.

Example 3.4. Any T_1 space is υ-determined because its specialization order is the identity relation, and $\upsilon(X, =)$ is the cofinite topology, the coarsest T_1 topology on X.

Example 3.5. If the specialization qoset of a space $(X, \mathcal{S})$ is a complete lattice L with $\mathcal{S} \subseteq \lambda\widetilde{L}$, then the Ultrafilter Theorem (UT) ensures that $(X, \mathcal{S})$ is an υ- and σ-determined weak upper space (cf. [10] and [13, III]). For the proof, consider an ultrafilter $\mathcal{F}$ on X. The set $D = \{y \in X : \downarrow y \in \mathcal{F}\}$ is down-directed, whence $\mathcal{F}$ converges to $z = \bigwedge D$ in $\Sigma\widetilde{L}$. For $x \in X$,

$$\begin{aligned} \mathcal{F} \text{ converges to } x \text{ in } \Upsilon L \;&\Leftrightarrow\; x \in X \setminus \downarrow y \text{ implies } X \setminus \downarrow y \in \mathcal{F} \\ &\Leftrightarrow\; \downarrow y \in \mathcal{F} \text{ implies } x \leq y \;\Leftrightarrow\; x \leq \textstyle\bigwedge D = z. \end{aligned}$$

In particular, $\mathcal{F}$ converges to z in ΥL, hence also in $\Lambda\widetilde{L}$. Suppose now $\mathcal{F}$ converges to some x in ΥL, and a set $U \in \mathcal{S}^{\sigma\leq} \subseteq \lambda\widetilde{L}$ contains x and so z. Then U must be a member of $\mathcal{F}$, and therefore $\mathcal{F}$ converges to x in $(X, \mathcal{S}^{\sigma\leq})$. This establishes the inclusion $\mathcal{S}^{\sigma\leq} \subseteq \upsilon L$, which, together with the obvious reverse inclusions $\upsilon L \subseteq \mathcal{S} \subseteq \mathcal{S}^{\upsilon\leq} \subseteq \mathcal{S}^{\sigma\leq}$, yields the identity

$$\upsilon L = \mathcal{S} = \mathcal{S}^{\upsilon\leq} = \mathcal{S}^{\sigma\leq}.$$

The completeness assumption is essential, as we shall demonstrate in Example 3.12, presenting a lattice with the weak upper topology that is not υ-determined. But, of course, there are also non-complete lattices whose weak upper spaces are υ-determined.

Example 3.6. Let L be the conditionally complete lattice $\mathbb{R}^n$. For $n>1$, the interval topology $\iota L = \upsilon L^{\upsilon}$ is much coarser than the Euclidean topology $\lambda L = \sigma L^{\upsilon}$; in fact, ιL is irreducible (any two nonempty open sets intersect), and there are sequences converging to every point of that space. Nevertheless, like the Scott space ΣL, the weak upper space ΥL is υ-determined: any ι-open upper set is υ-open. This can be seen as follows. The sets $\Uparrow u \cup (L \setminus {\downarrow} z)$ with $\Uparrow u = \{x \in L : \forall i\,(u_i < x_i)\}$ $(u, z \in L)$ form a base for υL (that conclusion requires a few computations we omit here). Note that $\Uparrow u$ is $\wedge$-closed but not ι-open. Given an ι-open upper set O and a point $x \in O$, pick a basic set $W = \Uparrow u \cup (L \setminus {\downarrow} z) \in \upsilon L$ and a finite F with $x \in W \setminus {\uparrow} F \subseteq O$. For each $w \in L \setminus {\downarrow} z$, there is a $v \in {\downarrow} w \setminus {\uparrow} F \setminus {\downarrow} z \subseteq W \setminus {\uparrow} F$ (because $\mathbb{R}$ is a chain with no least element); thus, $L \setminus {\downarrow} z \subseteq {\uparrow}(W \setminus {\uparrow} F) \subseteq O$. If x lies in $\Uparrow u$, then $w \wedge x \in {\downarrow} w \cap \Uparrow u \setminus {\uparrow} F$ for all $w \in \Uparrow u$; hence, $x \in \Uparrow u \subseteq {\uparrow}(W \setminus {\uparrow} F) \subseteq O$, and so $x \in W \subseteq O$. Otherwise, $x \in L \setminus {\downarrow} z \subseteq O$. In any case, we conclude that O is υ-open.

Turning to the side of quasi-ordered spaces and considering a coselection ζ, we call a quasi-ordered space $(Q, \mathcal{T})$ *ζ-convex* if its topology is generated by $\mathcal{T}^{\leq} \cup \zeta(\mathcal{T}^{\leq})$, that is, $\mathcal{T}^{\leq\zeta} = \mathcal{T}$.

Generally, ζ-convexity implies strong convexity, and υ-convexity implies ζ-convexity. For $\zeta = \alpha$, we have the following lemma.

Lemma 3.7. *A quasi-ordered space is an α-convex lower semi-qospace if and only if it is strongly convex and principal ideals are clopen. Such a space is totally order-separated and zero-dimensional, and its lower space is an A-space.*

Proof. If $(Q, \mathcal{T})$ is an α-convex lower semi-qospace, then $\alpha\widetilde{Q}$ is contained in $\mathcal{T}$, hence equal to $\mathcal{T}^{\geq}$; thus, each principal ideal ${\downarrow} x$ is not only closed but also open. For any $x \in O \in \mathcal{T}$, there are $U \in \mathcal{T}^{\leq}$ and $V \in \alpha\widetilde{Q} = \mathcal{T}^{\geq}$ with $x \in U \cap {\downarrow} x \subseteq U \cap V \subseteq O$, proving strong convexity; furthermore, not only ${\downarrow} x$ but also U is clopen because U is the complement of a lower, hence open set. Thus, the space is zero-dimensional; and for $x \not\leq y$, the complement of ${\downarrow} y$ is a clopen upper set containing x but not y.

Conversely, if all principal ideals are clopen, then we have a lower semi-qospace whose lower sets are open, being unions of principal ideals, whence $\mathcal{T}^{\geq} = \alpha\widetilde{Q}$. And if $(Q, \mathcal{T})$ is strongly convex, then we obtain $\mathcal{T} = \mathcal{T}^{\leq} \vee \mathcal{T}^{\geq} = \mathcal{T}^{\leq\alpha}$. □

In general, every strongly convex semi-qospace is a patch space of its upper space, as $\mathcal{T}^{\geq}$ is a cotopology of $\mathcal{T}^{\leq}$. Conversely, every patch space is a strongly convex semi-qospace: The principal ideals are the point closures in the original topology, the principal filters are the point closures in the cotopology, and the patch topology $\mathcal{T} = \mathcal{S} \vee \mathcal{S}'$ is generated by $\mathcal{T}^{\leq} \cup \mathcal{T}^{\geq}$, since $\mathcal{S} \subseteq \mathcal{T}^{\leq}$ and $\mathcal{S}' \subseteq \mathcal{T}^{\geq}$. In particular, for any upset selection ζ, the patch topology $\mathcal{T} = \mathcal{S}^{\zeta}$ generated by $\mathcal{S} \cup \zeta \widetilde{Q}$ is also generated by $\mathcal{T}^{\leq} \cup \zeta \widetilde{Q}$. The following conclusions are now immediate (recall Proposition 2.5).

Proposition 3.8. *The patch spaces are the strongly convex semi-qospaces. If ζ is an upset selection and $(X, \mathcal{S})$ is any space, or if ζ is a coselection and $(X, \mathcal{S})$ is ζ-determined, then the patch space $\mathrm{P}_{\zeta}(X, \mathcal{S})$ is ζ-convex. Hence, weak patch spaces $\mathrm{P}_{\upsilon}(X, \mathcal{S})$ are υ-convex and strong patch spaces $\mathrm{P}_{\alpha}(X, \mathcal{S})$ are α-convex. The weak patch spaces of (weak) mc-spaces are exactly the υ-convex (weakly) mc-ordered semi-pospaces.*

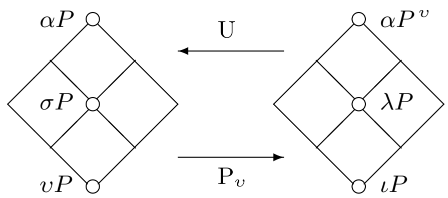
In the opposite direction, for any coselection ζ, the upper space $(X, \mathcal{T}^{\leq})$ of a ζ-convex lower semi-qospace $(X, \leq, \mathcal{T})$ is ζ-determined since $\mathcal{T}^{\leq \zeta} = \mathcal{T}$ entails $\mathcal{T}^{\leq \zeta \leq} = \mathcal{T}^{\leq}$. The argument is a bit subtle: The second $\leq$ in $\mathcal{T}^{\leq \zeta \leq}$ refers to the specialization order of $\mathcal{T}^{\leq}$, and it coincides with the first $\leq$ if and only if $(X, \leq, \mathcal{T})$ is a lower semi-qospace.

Now, we are going to declare the appropriate morphisms in the respective categories to make P_{ζ} functorial. The most obvious choice is to take the continuous maps between spaces; these maps are isotone, but not always continuous as maps between the associated patch spaces.

Example 3.9. For the real line $\mathbb{R}$ with the usual linear order $\leq$, the step function $s : \mathbb{R} \to \mathbb{R}$ with $s(x) = 0$ for $x \leq 0$ and $s(x) = 1$ for $x > 0$ is isotone and continuous as a map on the Scott space $\Sigma\,\mathbb{R}$, but not as a map on the associated weak patch space $\Lambda\,\mathbb{R}$, whose topology is here the Euclidean topology.

Notice that a map f between quasi-ordered spaces $(Q, \mathcal{T})$ and $(Q', \mathcal{T}')$ is lower semicontinuous if and only if it is continuous as a map from $(Q, \mathcal{T})$ to the upper space $\mathrm{U}(Q', \mathcal{T}')$ (see §2; for the more specific situation of maps into the reals, cf. [13, O-2]). Hence, an isotone map between quasi-ordered spaces is lower semicontinuous if and only if it is continuous as a map between the associated upper spaces (since a map is isotone if and only if it is α-continuous). Thus, we have a concrete *upper space functor* U from the category of quasi-ordered spaces with isotone lower semicontinuous maps as morphisms to the category of topological spaces – and, indeed, this is even a categorical equivalence, because U is full, faithful, and onto on objects since arbitrary spaces $(X, \mathcal{S})$ coincide with $\mathrm{U}(Q, \mathcal{S})$ for the specialization qoset $Q = (X, \leq_{\mathcal{S}})$. But observe that isomorphic quasi-ordered spaces in the above category need not be homeomorphic! Now, one sees that a map between ζ-determined spaces is continuous if and only if it is isotone (preserves the specialization order) and lower semicontinuous as a map between the associated ζ-patch spaces. Let us summarize these thoughts.

Theorem 3.10. *Any coselection ζ gives rise to a concrete patch functor P_ζ from the category of ζ-determined spaces with continuous maps to the category of quasi-ordered spaces with isotone lower semicontinuous maps, and P_ζ induces an isomorphism onto the category of ζ-convex semi-qospaces. The inverse isomorphism is induced by the upper space functor* U*, sending a semi-qospace $(X, \leq, \mathcal{T})$ to $(X, \mathcal{T}^{\leq})$. Specifically, the weak patch functor P_υ induces a concrete isomorphism between the category of υ-determined spaces and the category of υ-convex semi-qospaces.*



A more limited morphism class is formed by so-called proper maps, whose definition varies in the literature, depending on the context and the desired results. A (*strongly*) *proper map* would be a continuous map for which preimages of compact sets are compact (automatic for continuous maps from compact spaces to Hausdorff spaces). A wider convention is to call a continuous map *proper* if all preimages of cores are compact. However, that kind of map is still not general enough to provide the desired categorical correspondence if one wishes to receive all isotone continuous maps between the patch spaces.

Example 3.11. Every isotone and continuous real function is continuous for the Scott topology $\sigma\mathbb{R} = \upsilon\mathbb{R}$, but if the range has a lower bound b, then the preimage of the principal filter generated by b (the core of b in $\Sigma\mathbb{R}$) is the whole line $\mathbb{R}$, which is not compact (neither in $\sigma\mathbb{R}$ nor in $\lambda\mathbb{R}$).

A similar task would be to require that preimages of cores be closed in the weak lower topology (which amounts to continuity in that topology). This, together with continuity in the original topology, entails continuity relative to the weak patch topologies. But the reverse implication fails.

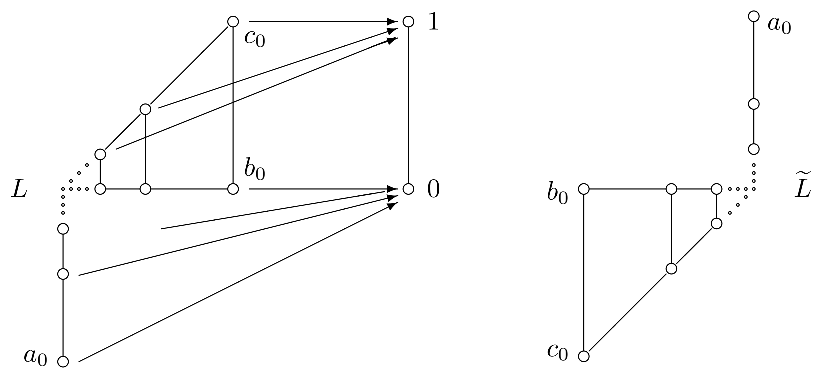
Example 3.12. Consider the following sublattice of the plane $\mathbb{R}^2$:

$$L = \{a_n,\, b_n,\, c_n : n \in \omega\} \text{ with}$$

$$a_n = (0, -2^{-n}),\ b_n = (2^{-n}, 0),\ c_n = (2^{-n}, 2^{-n}).$$

Being a sublattice of the distributive lattice $\mathbb{R}^2$, L is distributive, too. Clearly, L is not complete. Indeed, a directed subset can have a join only if it has a greatest element, and dually. Therefore, the Scott topology σL coincides with the upper Alexandroff topology αL. The patch topology

$\lambda L = \sigma L^{\upsilon}$ is discrete and coincides with the interval topology $\iota L = \upsilon L^{\upsilon}$, although σL greatly differs from υL. Define a map f from L to the chain $C = \{0, 1\}$ by $f(a_n) = f(b_n) = 0,\ f(c_n) = 1$.



This map is isotone and trivially continuous for the (discrete) weak patch topologies and also for the Scott (= Alexandroff) topologies. But the preimage of the complement $\{0\}$ of the core $\uparrow 1 = \{1\}$ is the principal ideal $\downarrow b_0 = \{a_n, b_n : n \in \omega\}$, which is not open in the weak lower topology, since it does not contain any neighborhood $L \setminus \uparrow F$ of b_0 (F finite).

Being an A-space, the Scott space ΣL is ζ-determined for any coselection ζ. Now, look at the dual lattice $\widetilde{L}$ and the cotopology $\sigma\widetilde{L}$ of $\sigma L = \alpha L$. The Scott space $\Sigma\widetilde{L}$ is an A-space, hence ζ-determined, too, while the weak lower space $\Upsilon\widetilde{L}$ of L is not ζ-determined for any coselection ζ; as $\upsilon\widetilde{L}^{\zeta} = \iota L$ is discrete, $(\upsilon\widetilde{L}^{\zeta})^{\leq} = \alpha\widetilde{L}$ is distinct from $\upsilon\widetilde{L}$. Compare this situation with the observation about complete lattices in Example 3.5.

It turns out that the appropriate morphisms for the intended functorial isomorphism are those continuous maps for which preimages of open sets in the ζ-cotopology are ζ-patch open. We call them ζ-*proper* maps.

Corollary 3.13. *Let ζ be an arbitrary coselection. Then a map f between ζ-determined spaces is ζ-proper if and only if it is isotone and continuous as a map between the associated ζ-patch spaces. The patch functor* P_{ζ} *induces an isomorphism between the category of ζ-determined spaces with ζ-proper maps and the category of ζ-convex semi-qospaces with isotone continuous maps.*

Proof. If f is ζ-proper, then f preserves the specialization order (by continuity) and is ζ-patch continuous, as preimages of subbasic patch open sets are patch open. Conversely, if $f : (X, \leq, \mathcal{S}^{\zeta}) \to (X, \leq', \mathcal{S}'^{\zeta})$ is isotone and continuous, then $U' \in \mathcal{S}'$ implies $U' \in \mathcal{S}'^{\zeta\leq'}$; hence, $f^{-1}[U'] \in \mathcal{S}^{\zeta\leq} = \mathcal{S}$, provided $(X, \mathcal{S})$ is ζ-determined. Thus, the map $f\colon (X, \mathcal{S}) \to (X', \mathcal{S}')$ is continuous, ζ-patch continuous, and therefore ζ-proper. $\square$

4. **Upwards-Stable and Web Quasi-Ordered Spaces**

We call a quasi-ordered space $(Q, \mathcal{T})$ *upwards-stable* (*$\uparrow$-stable*) if $O \in \mathcal{T}$ implies $\uparrow O \in \mathcal{T}$. Here are some alternative characterizations of $\uparrow$-stability.

Lemma 4.1. *For a quasi-ordered space $(Q, \mathcal{T})$, the following conditions are equivalent:*

(u1) *$(Q, \mathcal{T})$ is $\uparrow$-stable.*
(u2) *$\mathcal{T}^{\leq} = \{\uparrow O : O \in \mathcal{T}\}$.*
(u3) *The interior of each upper set is an upper set.*
(u4) *The closure of each lower set is a lower set.*

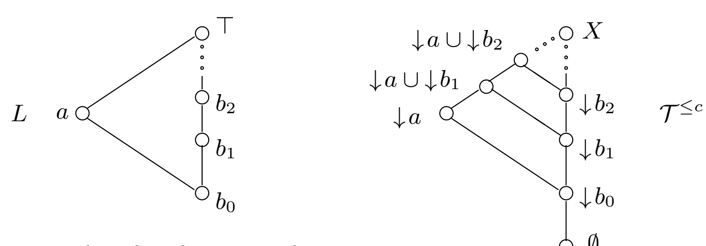
Upwards-stability is frequently fulfilled in concrete situations (for instance, in Euclidean spaces $\Lambda\mathbb{R}^n$). However, there are also rather simple examples of compact pospaces that are not $\uparrow$-stable, although in compact pospaces, $\uparrow A$ and $\downarrow A$ are closed for any closed subset A (see [13, VI-1]).

Example 4.2. Let $P = (X, \leq)$ be a poset with top element $\top$ such that all principal ideals except $\downarrow\top = X$ are finite. The topology

$$\mathcal{T} = \{U \subseteq X : \top \notin U\} \cup \{X \setminus F : F \text{ finite}, \top \notin F\}$$

makes $(P, \mathcal{T})$ a compact pospace, in fact, a Priestley space: Since all points except $\top$ are isolated, for $x \not\leq y$ the principal ideal $\downarrow y$ is a clopen lower set containing y but not x, and the space is compact because every neighborhood of $\top$ misses only a finite number of points. If there is at least one $a \in X \setminus \{\top\}$ for which $X \setminus \uparrow a$ is infinite, then $(P, \mathcal{T})$ cannot be $\uparrow$-stable because $\{a\}$ is open, while $\uparrow a$ is not.

A typical instance is the complete lattice $L = (X, \leq)$ sketched below. Neither L nor the coframe $\mathcal{T}^{\leq c}$ of closed lower sets is meet-continuous.



$X = \{a, \top\} \cup \{b_n : n \in \omega\}$,
$x \leq y \iff x = y$ or $x = b_0$ or $y = \top$ or $x = b_i, y = b_j, i < j$.

In this example, the above-defined topology $\mathcal{T}$ coincides with the Lawson topology. Hence, $(L, \mathcal{T})$ is an υ-convex but not an $\uparrow$-stable Priestley space.

Such situations cannot occur in *semitopological meet-semilattices* (whose unary meet operations $\wedge_x : y \mapsto x \wedge y$ are continuous): They are always $\uparrow$-stable, since $\uparrow U = \bigcup \{{\wedge_x}^{-1}[U] : x \in U\}$. A much more general result holds for so-called web quasi-ordered spaces, which we now are going to define. A *web* around a point x in a qoset $Q = (X, \leq)$ is a set W containing x and, for each $y \in W$, a common lower bound of x and y. If $\leq$ is the specialization order of a space, the latter condition means that the closures of x and y have a common point in W. Any union of down-directed sets that contain a given point x is a web around it, and conversely. Every web is connected in the order-theoretical sense: any two elements of a web are joined by a path of length at most 4. A straightforward verification shows that for each subset U of a qoset Q and each point $x \in U$, the set

$$U_x = U \cap \uparrow (U \cap \downarrow x)$$

is the greatest web around x in the subqoset U, called the *web component* of U containing x. In spite of the similarity to (path) components in qosets and spaces, distinct web components need not be disjoint. However, the web components of any order convex subset are order convex, too. By a *web (quasi-)ordered space*, we mean an $\uparrow$-stable (quasi-)ordered space in which every point has a neighborhood base of webs around it. In the case of a space equipped with its specialization order, $\uparrow$-stability is automatic, and the previous definition amounts to that of a *web space* as given in [8], where it was shown that web spaces are locally path-connected and generalize meet-continuous semilattices and dcpos; in [13, III-2], *meet-continuous dcpos* are defined by the condition that for directed subsets D, $x \leq \bigvee D$ implies $x \in cl_\sigma(\downarrow x \cap \downarrow D)$, which means that their Scott spaces are web spaces. The next theorem provides a list of characteristic properties of web quasi-ordered spaces and, in particular, of web spaces equipped with the specialization order (see [8]).

Theorem 4.3. *For any quasi-ordered space* $(Q, \mathcal{T}) = (X, \leq, \mathcal{T})$*, the following seven conditions are equivalent:*

(1) $(Q, \mathcal{T})$ *is web quasi-ordered.*
(2) $U \in \mathcal{T}$ *implies* $\uparrow (U \cap V) \in \mathcal{T}$ *for all lower sets* V.
(3) $x \in U \in \mathcal{T}$ *implies* $\uparrow (U \cap \downarrow x) \in \mathcal{T}$.
(4) $x \in \downarrow cl_{\mathcal{T}} Y$ *implies* $x \in cl_{\mathcal{T}}(\downarrow x \cap \downarrow Y)$ *for all* $Y \subseteq X$.
(5) $\downarrow x \cap \downarrow cl_{\mathcal{T}} Y \subseteq cl_{\mathcal{T}}(\downarrow x \cap \downarrow Y)$ *for all* $x \in X$ *and* $Y \subseteq X$.
(6) *The interior operator induces a homomorphism from* αQ *to* $\mathcal{T}^{\leq}$.
(7) *The closure operator induces a homomorphism from* $\alpha \widetilde{Q}$ *to* $\mathcal{T}^{\leq c}$.

These conditions are equivalent to $\uparrow$*-stability plus one of the following:*

(1′) *The upper space* $\mathrm{U}(Q, \mathcal{T}) = (X, \mathcal{T}^{\leq})$ *is a web space.*
(2′) *Every point has a neighborhood base of open webs around it.*

(3′) *The web components of any open subset are open.*
(4′) *The lattice $\mathcal{T}^{\leq}$ of open upper sets is a coframe.*
(5′) *The lattice $\mathcal{T}^{\leq c}$ of closed lower sets is meet-continuous (a frame).*
(6′) *The interior operator preserves finite unions of upper sets.*
(7′) *The closure operator preserves finite intersections of lower sets.*

Proof. (1) ⇒ (2) For $U \in \mathcal{T}$, $V = \downarrow V$, and $y \in \uparrow(U \cap V)$, there is an $x \in U \cap V \cap \downarrow y$. The web component U_x satisfies $x \in int_{\mathcal{T}}\, U_x$ and $W = \uparrow int_{\mathcal{T}}\, U_x \subseteq \uparrow U_x \subseteq \uparrow(U \cap \downarrow x) \subseteq \uparrow(U \cap V)$. Then $y \in \uparrow x \subseteq W$ and, by ↑-stability, $W \in \mathcal{T}$. Hence, $\uparrow(U \cap V) \in \mathcal{T}$.

(2) ⇒ (3) is trivial.

(3) ⇒ (4) by contraposition: If $x \in U = X \setminus cl_{\mathcal{T}}(\downarrow x \cap \downarrow Y)$, then we get $W = \uparrow(U \cap \downarrow x) \in \mathcal{T}$ and $x \in W$. Now, the equation $U \cap \downarrow x \cap \downarrow Y = \emptyset$ entails $W \cap Y = \emptyset$; hence, $W \cap cl_{\mathcal{T}} Y = \emptyset$ and $x \notin \downarrow cl_{\mathcal{T}} Y$ (as $x \in W = \uparrow W$).

(4) ⇒ (3) For $x \in U \in \mathcal{T}$ and $Y = X \setminus \uparrow(U \cap \downarrow x)$, we get $U \cap \downarrow x \cap \downarrow Y = \emptyset$ and $x \notin cl_{\mathcal{T}}(\downarrow x \cap \downarrow Y)$ (as $x \in U$), and then $x \notin \downarrow cl_{\mathcal{T}} Y$ and $cl_{\mathcal{T}} Y \subseteq X \setminus \uparrow x \subseteq Y$ (as $\uparrow(U \cap \downarrow x) \subseteq \uparrow x$); hence, $\uparrow(U \cap \downarrow x) = X \setminus Y \in \mathcal{T}$.

(4) ⇒ (5) $z \in \downarrow x \cap \downarrow cl_{\mathcal{T}} Y$ implies $z \in cl_{\mathcal{T}}(\downarrow z \cap \downarrow Y) \subseteq cl_{\mathcal{T}}(\downarrow x \cap \downarrow Y)$.

(5) ⇒ (7) First, let us show that (5) entails ↑-stability. If Y is a lower set, then $x \in \downarrow cl_{\mathcal{T}} Y$ implies $x \in \downarrow x \cap \downarrow cl_{\mathcal{T}} Y \subseteq cl_{\mathcal{T}}(\downarrow x \cap Y) \subseteq cl_{\mathcal{T}} Y$; hence, $cl_{\mathcal{T}} Y = \downarrow cl_{\mathcal{T}} Y$, and Lemma 4.1 applies. It follows that the closure operator induces a join-preserving map from $\alpha\widetilde{Q}$ onto $\mathcal{T}^{\leq c}$, and if we can prove the inclusion $cl_{\mathcal{T}} Y \cap cl_{\mathcal{T}} Z \subseteq cl_{\mathcal{T}}(Y \cap Z)$ for all lower sets Y and Z, then the restricted and corestricted closure operator from $\alpha\widetilde{Q}$ onto $\mathcal{T}^{\leq c}$ is a lattice homomorphism (and even a frame homomorphism).

By (5), we have $\downarrow x \cap cl_{\mathcal{T}} Y \subseteq cl_{\mathcal{T}}(\downarrow x \cap Y)$ for each $x \in Z = \downarrow Z$; hence,

$$(*) \quad cl_{\mathcal{T}} Y \cap Z \subseteq cl_{\mathcal{T}}(Y \cap Z).$$

Now, observing that $cl_{\mathcal{T}} Z$ is a lower set by ↑-stability (see Lemma 4.1 again) and using $(*)$ twice (once for $cl_{\mathcal{T}} Z$ in place of Z and once with Y and Z exchanged), we get the inclusion

$$cl_{\mathcal{T}} Y \cap cl_{\mathcal{T}} Z \subseteq cl_{\mathcal{T}}(Y \cap cl_{\mathcal{T}} Z) \subseteq cl_{\mathcal{T}} cl_{\mathcal{T}}(Y \cap Z) = cl_{\mathcal{T}}(Y \cap Z).$$

(6) ⇔ (7) The lower sets are the complements of the upper sets, and the closure of a lower set V is complementary to the interior of $X \setminus V$.

(7) ⇒ (1) will follow from the implications established below; the required hypothesis of ↑-stability is a consequence of (7) by Lemma 4.1.

(3) ⇒ (3′) ⇒ (2′), (6) ⇒ (6′), and (7) ⇒ (7′) are clear.

(7′) ⇒ (5′) Since the restricted and corestricted closure operator from the frame $\alpha\widetilde{Q}$ to $\mathcal{T}^{\leq c}$ preserves finite meets and arbitrary joins, its range is a frame, too. As $\mathcal{T}^{\leq c}$ is always distributive, it suffices to postulate meet-continuity in order to ensure the frame property.

$(1') \Leftrightarrow (4') \Leftrightarrow (5')$ has been shown in [8].

$(6') \Leftrightarrow (7')$ is obtained by passing to complements.

Now, assume that $(Q, \mathcal{T})$ is $\uparrow$-stable. Then $(2')$ clearly implies (1).

$(1') \Rightarrow (4)$ If $x \in \downarrow cl_{\mathcal{T}} Y \subseteq \downarrow cl_{\mathcal{T}}(\downarrow Y)$, then $x \in cl_{\mathcal{T}^{\leq}}(\downarrow Y)$ by Lemma 4.1(u4), and the proven implication $(1) \Rightarrow (4)$ for $\mathcal{T}^{\leq}$ instead of $\mathcal{T}$ yields $x \in cl_{\mathcal{T}^{\leq}}(\downarrow x \cap \downarrow Y) = cl_{\mathcal{T}}(\downarrow x \cap \downarrow Y)$, again by Lemma 4.1(u4).

In all, we have established the following implication circuits:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (1),$$

and under the hypothesis that $(Q, \mathcal{T})$ is $\uparrow$-stable,

$$(1) \Rightarrow (7) \Rightarrow (7') \Rightarrow (6') \Rightarrow (5') \Rightarrow (4') \Rightarrow (1') \Rightarrow (4) \Rightarrow (3) \Rightarrow (3') \Rightarrow (2') \Rightarrow (1).$$

□

As shown in [8], in case Q is a meet-semilattice, $(1')$ is equivalent to

$(1'')$ The unary meet operations $\wedge_x : y \mapsto x \wedge y$ are $\mathcal{T}^{\leq}$-continuous.

That $\uparrow$-stability is indispensable for the equivalence of conditions $(1')$ to $(7')$ can be checked easily by a review of Example 4.2.

Example 4.4. Consider once again the compact but not $\uparrow$-stable Lawson pospace $(L, \mathcal{T})$ in Example 4.2; recall that neither L nor the coframe $\mathcal{T}^{\leq c}$ is meet-continuous. Nevertheless, the closure operator preserves intersections of arbitrary collections $\mathcal{Y}$ of lower sets: The only non-closed lower sets are $A = X \setminus \{a, \top\}$ and $B = X \setminus \{\top\}$, and one easily verifies

$$cl_{\mathcal{T}}(\bigcap \mathcal{Y}) = \bigcap \mathcal{Y} \cup \{\top\} = \bigcap \{cl_{\mathcal{T}} Y : Y \in \mathcal{Y}\} \text{ in case } \bigcap \mathcal{Y} \in \{A, B\},$$

and $cl_{\mathcal{T}}(\bigcap \mathcal{Y}) = \bigcap \mathcal{Y} = \bigcap \{cl_{\mathcal{T}} Y : Y \in \mathcal{Y}\}$ in all other cases. Hence, conditions $(6')$ and $(7')$ are fulfilled, whereas $(4')$ and $(5')$ and therefore all seven conditions $(1) - (7)$ are violated. Furthermore, $(1')$ and $(1'')$ cannot hold either, because the unary meet operation $\wedge_a$ is not continuous with respect to the topology $\mathcal{T}^{\leq} = \alpha L \setminus \{\uparrow a, \uparrow \top\}$; whereas $U = X \setminus \downarrow b_0$ belongs to $\mathcal{T}^{\leq}$, the preimage $\wedge_a^{-1}[U] = \uparrow a$ does not. While $(2')$ holds, $(3')$ fails because the web component U_a of the open set U is the non-open set $\uparrow a$.

Example 4.5. Every linearly ordered lower semi-pospace is a web ordered space, as all nonempty subsets are filtered; for $\uparrow$-stability, observe that

$$\uparrow U = U \cup \{X \setminus \downarrow x : x \in U\}.$$

But there are linearly ordered spaces whose topology of open upper sets is a coframe, while $\uparrow$-stability fails. An example is the real line $\mathbb{R}$ with the topology consisting of $\mathbb{R}$ and all Euclidean open subsets of $\mathbb{R} \setminus \mathbb{Z}$. Here, the upper topology is the 2-element coframe $\{\emptyset, \mathbb{R}\}$.

Example 4.6. The conditionally complete lattice $\mathbb{R}^n$ with the interval topology (see Example 3.6) is $\uparrow$-stable, but for $n>1$, it badly fails to be a web ordered space: no point has any web neighborhood except $\mathbb{R}^n$.

We are ready for diverse further characterizations of web spaces, showing that they may always be recovered from their patch spaces.

Proposition 4.7. *For any space $S = (X, \mathcal{S})$, the following are equivalent:*

(1) *S is a web space.*
(2) *$\uparrow(U \cap V) \in \mathcal{S}$ for each $U \in \mathcal{S}$ and each lower set V.*
(3) *The topology $\mathcal{S}$ is a coframe.*
(4) *The interior operator induces a homomorphism from $\alpha(X, \leq_{\mathcal{S}})$ to $\mathcal{S}$.*
(5) *Any patch space of S is a web quasi-ordered space with upper space S.*
(6) *S is ζ-determined with $\uparrow$-stable ζ-patch space for any coselection ζ.*
(7) *S is α-determined, and its strong patch space $\mathrm{P}_\alpha S$ is $\uparrow$-stable.*
(8) *$\mathcal{S} = \{\uparrow O : O \in \mathcal{T}\}$ for any patch topology $\mathcal{T}$ of $\mathcal{S}$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) Apply Theorem 4.3 to the quasi-ordered space $(Q, \mathcal{T}) = (X, \leq_{\mathcal{S}}, \mathcal{S})$ (see [8]).

(2) $\Rightarrow$ (5) As the up-closure operator $\uparrow$ preserves arbitrary unions, for $\uparrow$-stability of a patch space $(X, \leq, \mathcal{T})$ of $(X, \mathcal{S})$, it suffices to assure that $O \in \mathcal{T}$ implies $\uparrow O \in \mathcal{T}$ for basic sets of the form $O = U \cap V$ with $U \in \mathcal{S}$ and lower sets V. By the implication (2) $\Rightarrow$ (1), for $x \in O$, there is a web neighborhood W with $x \in W \subseteq U$, and then $W \cap V$ is a web neighborhood of x in the patch space $(X, \leq, \mathcal{T})$ with $W \cap V \subseteq O$.

In order to see that the upper space of $(X, \leq, \mathcal{T})$ is $(X, \mathcal{S})$, consider any $O \in \mathcal{T}^{\leq}$ and $x \in O$. There are $U \in \mathcal{S}$ and $V = \downarrow V$ with $x \in U \cap V \subseteq O$. By (2), it follows that $W = \uparrow(U \cap V) \in \mathcal{S}$, and we get $x \in W \subseteq \uparrow O = O \in \mathcal{S}$.

(5) $\Rightarrow$ (6) $\Rightarrow$ (7) and (5) $\Rightarrow$ (8) $\Rightarrow$ (2) are straightforward.

(7) $\Rightarrow$ (2) $U \in \mathcal{S}$ and $V = \downarrow V$ imply $U \cap V \in \mathcal{S}^{\alpha}$; now $\uparrow$-stability yields $\uparrow(U \cap V) \in \mathcal{S}^{\alpha\leq} = \mathcal{S}$. □

By a *web semi-qospace*, we mean a web quasi-ordered semi-qospace. Now, Proposition 3.8, Theorem 3.10, and Theorem 4.3 lead to the following.

Corollary 4.8. *The strongly convex (ζ-convex, respectively) web semi-qospaces are exactly the patch (ζ-patch, respectively) spaces of web spaces.*

Finally, let us note that a nonempty product of (quasi-ordered) spaces is a web (quasi-ordered) space if all factors are web (quasi-ordered) spaces and all but a finite number are filtered, i.e., down-directed (see [3]).

5. **Categories of Web Ordered Spaces and Semilattices**

We now are going to apply the previous results to (meet-)semilattices. Let us record a consequence of Proposition 4.7 and the remark thereafter.

Lemma 5.1. *Any semitopological meet-semilattice is a web ordered space, and its upper space is a web space, hence a semitopological semilattice.*

Boolean lattices provide interesting examples showing that patch spaces of T_0 web spaces (in fact, of meet-continuous lattices with the Scott topology) need not be Hausdorff, while T_1 web spaces must already be discrete.

Example 5.2. Every complete Boolean lattice is a frame, hence meet-continuous, but the Axiom of Choice guarantees that it is continuous in the sense of Scott [13] if and only if it is atomic, i.e., isomorphic to a power set (see [10] and [13, Theorem I-4.20]). Thus, every Scott space of a complete Boolean lattice is a web space, while the only complete Boolean lattices whose Lawson space is Hausdorff are the atomic ones (see [13, Theorem III-2.11]).

It is important to distinguish between continuity of the unary meet operations $\wedge_x$ and continuity of the binary meet. While both properties coincide for compact Hausdorff semilattices (see [17]), we have the following.

Example 5.3. The complete lattice of all regular open subsets of the Euclidean real line is Boolean, hence meet-continuous, but the binary meet operation is not continuous relative to the Lawson topology [10], and this topology is not Hausdorff (the discovery of such "pathological" lattice topologies is due to E. E. Floyd [12]).

Recall from §2 that an *υmc-ordered space* is an υ-convex, mc-ordered, and upper m-determined semi-pospace. Corollary 2.6, Theorem 2.7, Proposition 4.7, and Lemma 5.1 together yield the following facts.

Proposition 5.4. (1) *The Scott spaces of meet-continuous domains are exactly those m-determined mc-spaces which are web spaces.*

(2) *The Lawson spaces of meet-continuous domains are exactly those υmc-ordered spaces which are web ordered.*

Corollary 5.5. (1) *The Scott spaces of meet-continuous semilattices are exactly those m-determined mc-spaces which are semitopological meet-semilattices relative to the specialization order.*

(2) *The Lawson spaces of meet-continuous semilattices are exactly the υmc-ordered semitopological meet-semilattices.*

From Theorem 3.10, Theorem 4.3, and Proposition 4.7, we deduce for $\zeta = \alpha$ that any α-convex $\uparrow$-stable semi-qospace is web quasi-ordered, being the strong patch space of its upper space, which is a web space. Let us reformulate some of the previous results in categorical terms.

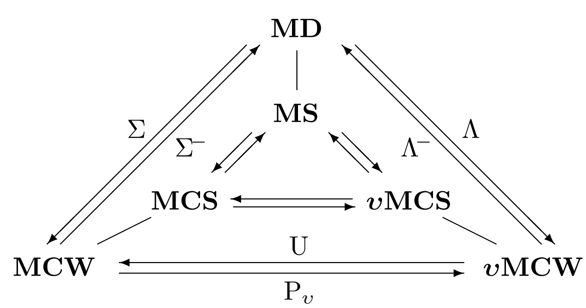
Theorem 5.6. (1) *For any coselection ζ, the patch functor P_ζ induces a concrete isomorphism between the category of web spaces with ζ-proper (continuous, respectively) maps and the category of ζ-convex web semi-qospaces with isotone continuous (lower semicontinuous, respectively) maps.*

(2) *Via P_α, the category of web spaces is isomorphic to the category of α-convex $\uparrow$-stable semi-qospaces; and the category of semitopological meet-semilattices with compatible topologies is isomorphic to the category of α-convex T_1-ordered semitopological meet-semilattices.*

(3) *Via P_υ, the category of web spaces is isomorphic to the category of υ-convex web semi-qospaces; and the category of semitopological meet-semilattices with compatible topologies is isomorphic to the category of υ-convex T_1-ordered semitopological meet-semilattices.*

(4) *Via Σ, the category* **MD** *of meet-continuous domains is isomorphic to the category* **MCW** *of m-determined mc-spaces that are web spaces, and via Λ to the category* $\boldsymbol{\upsilon}$**MCW** *of υmc- and web ordered spaces.*

(5) *Via Σ, the category* **MS** *of meet-continuous semilattices is isomorphic to the category* **MCS** *of m-determined mc-spaces that are semitopological semilattices, and via Λ, it is isomorphic to the category* $\boldsymbol{\upsilon}$**MCS** *of υmc-ordered semitopological semilattices.*

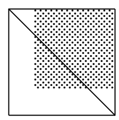


The Fundamental Theorem of Compact Semilattices [13, VI-3] states that (in ZF+AC) the compact T_2-topological semilattices with small semilattices are exactly the Lawson spaces of continuous complete semilattices. Our considerations show that here the existence of enough small semilattices may be replaced by the property of being upper m-determined.

6. **Subspaces of Web (Ordered) Spaces**

By a *subspace* of a (quasi-)ordered space, we mean one that carries not only the induced topology but also the induced (quasi-)order. While strong convexity is inherited by subspaces, the web property is not.

Example 6.1. The real frame $T = [0,1]^2$ with the Scott topology is a topological lattice, hence a web ordered space. As a subspace, the antichain $S = \{(x, 1-x) : x \in [0,1]\}$ carries the Euclidean topology, whereas a T_1 web space must already be discrete. Thus, S cannot be web ordered.



However, we shall characterize the T_0 web spaces as certain "consistent" subspaces of semitopological semilattices and prove a similar theorem for strongly convex web semi-pospaces, that is, for patch spaces of T_0 web spaces. Let $(Q, \mathcal{V})$ be a quasi-ordered space with down-closure operator $\downarrow$. A subspace $T = (X, \leq, \mathcal{T})$ of $(Q, \mathcal{V})$ is called $\downarrow$-*consistent* if

$$X \cap cl_{\mathcal{V}}(\downarrow Y \cap \downarrow Z) \subseteq cl_{\mathcal{T}}(Y \cap Z) \text{ for all lower sets } Y \text{ and } Z \text{ in } T.$$

The reverse inclusion is always fulfilled. Hence, $\downarrow$-consistency entails

$$X \cap cl_{\mathcal{V}}(\downarrow Y) = cl_{\mathcal{T}} Y \text{ for all lower sets } Y \text{ in } T,$$

which implies that $\uparrow$-stability is inherited by T from $(Q, \mathcal{V})$ (Lemma 4.1), and that $X \cap cl_{\mathcal{V}}(\downarrow Y) \cap cl_{\mathcal{V}}(\downarrow Z) = cl_{\mathcal{T}} Y \cap cl_{\mathcal{T}} Z$ for lower sets Y and Z in T. Hence, Theorem 4.3 yields the following lemma.

Lemma 6.2. *Every $\downarrow$-consistent subspace of a web (quasi-ordered) space is a web (quasi-ordered) space, too.*

Corollary 6.3. *Every $\downarrow$-consistent subspace of a semitopological meet-semilattice is a web ordered space.*

Example 6.4. Any subspace of a linearly mc-ordered lower semi-pospace is $\downarrow$-consistent.

The *cospectrum* of a complete lattice L is the set P of all *coprimes* p (satisfying $p \in \downarrow F$ for all finite $F \subseteq L$ with $p \leq \bigvee F$), equipped with the topology $\{P \setminus \downarrow y : y \in L\}$. A subset X of L is *join-dense* in L if $y = \bigvee (X \cap \downarrow y)$ for all y in L.

Proposition 6.5. *For a space S, the following conditions are equivalent:*

(1) *S is a subspace of the cospectrum of a frame L and join-dense in L.*
(2) *S is a $\downarrow$-consistent subspace of a semitopological meet-semilattice with a compatible topology (in fact, with the weak upper topology).*

(3) *S is a $\downarrow$-consistent subspace of a T_0 web space.*
(4) *S is a T_0 web space.*

The frames L in (1) *are isomorphic to the lattice of closed subsets of S.*

Proof. (1) $\Rightarrow$ (2) For any frame L, the ordered space $(L, \upsilon L)$ is a semitopological semilattice: The unary meet operations $\wedge_x$ are residuated (i.e., preimages of principal ideals are principal ideals), hence υ-continuous. Let $S = (X, \mathcal{S})$ be a subspace of the cospectrum of L and join-dense in L. Then, given any $Y \subseteq X$, one obtains $Y \subseteq C$ for $C = X \cap \downarrow \bigvee Y \in \mathcal{S}^c$, and if $Y \subseteq A = X \cap \downarrow a \in \mathcal{S}^c$, then $C = X \cap \downarrow \bigvee Y \subseteq X \cap \downarrow a = A$. This proves $cl_{\mathcal{S}} Y = X \cap \downarrow \bigvee Y$. Now, for arbitrary lower sets $Y = X \cap \downarrow Y$, $Z = X \cap \downarrow Z$ in S, one computes, using the join-density of X in L,

$$X \cap cl_{\upsilon L}(\downarrow Y \cap \downarrow Z) \subseteq X \cap \downarrow \bigvee (\downarrow Y \cap \downarrow Z) = X \cap \downarrow \bigvee (Y \cap Z) = cl_{\mathcal{S}}(Y \cap Z),$$

which shows that S is a $\downarrow$-consistent subspace of $(L, \upsilon L)$.

(2) $\Rightarrow$ (3) Lemma 5.1.

(3) $\Rightarrow$ (4) Lemma 6.2.

(4) $\Rightarrow$ (1) Put $S = (X, \mathcal{S})$, $Y' = \{\downarrow y : y \in Y\}$ $(Y \subseteq X)$, $\mathcal{S}' = \{U' : U \in \mathcal{S}\}$, and $S' = (X', \mathcal{S}')$. Then $\eta_S : S \to S'$, $x \mapsto \downarrow x = cl_{\mathcal{S}}\{x\}$ is a homeomorphism, and S' is a subspace of the cospectrum of $\mathcal{S}^c$, which is a frame. Any $A \in \mathcal{S}^c$ is a union of point closures, hence the join of the set A'. Thus, X' is join-dense in $\mathcal{S}^c$, and an isomorphic copy L of $\mathcal{S}^c$ gives (1).

On the other hand, if L is a frame as in (1), then the map $i : L \to \mathcal{S}^c$ with $i(y) = X \cap \downarrow y$ is an isomorphism since $A \in \mathcal{S}^c$ implies $A = X \cap \downarrow y$ for a $y \in L$; the embedding property is assured by join-density of X in L. □

We now are going to establish a similar result for web ordered spaces. It remains open whether any web ordered space is a subspace of a semitopological semilattice, but we are able to give a positive answer for $\uparrow$-*costable* ordered spaces, in which $\uparrow C$ is closed for every closed subset C. (In [16] and [20], pospaces with this and the dual property are referred to as *C-spaces*, whereas in [4], [6], and elsewhere, C-spaces have a different meaning, namely, that each point has a neighborhood base of cores.) All compact pospaces are $\uparrow$-costable (see, e.g., [13, VI-1]). For the intended subspace theorem, we need an ordered variant of the classical Vietoris topology ([22]; see also [13, Example VI-3.8] and [15]). Let L be the coframe $\mathcal{T}^{\leq c}$ of all closed lower sets in a lower semi-pospace $T = (X, \leq, \mathcal{T})$, and let $\mathcal{V}_T$ be the topology on L generated by

the upper sets $U^{\leq} = \{A \in L : A \cap U \neq \emptyset\}$ $(U \in \mathcal{T}^{\leq})$

and the lower sets $V^{\geq} = \{A \in L : A \subseteq V\}$ $(V \in \mathcal{T}^{\geq})$.

We call the ordered space $\mathrm{V}T = (L, \mathcal{V}_T)$ the (*ordered*) *Vietoris space* of T. The principal ideal map (which here is not the point closure map!)

$$e_T : X \to L,\ x \mapsto \downarrow x$$

is a well-defined order embedding (as $x \le y \Leftrightarrow \downarrow x \subseteq \downarrow y$), and it is a topological embedding of T in $\mathrm{V}T$ in case T is strongly convex, since

$$e_T^{-1}[U^{\le}] = \{x \in X : \downarrow x \cap U \ne \emptyset\} = U \text{ for } U \in \mathcal{T}^{\le},$$
$$e_T^{-1}[V^{\ge}] = \{x \in X : \downarrow x \subseteq V\} \qquad = V \text{ for } V \in \mathcal{T}^{\ge}.$$

By slight abuse of language, we speak of a *semitopological frame* if we mean a frame L with a topology $\mathcal{V}$ making the operations $\wedge_x$ continuous. And we say a subspace $(X, \le, \mathcal{T})$ of $(L, \mathcal{V})$ is *join-generating* if

(g1) X is join-dense in L,
(g2) $cl_{\mathcal{T}} Y = X \cap \downarrow_L \bigvee Y$ for all lower sets Y of $(X, \le)$, and
(g3) $\{L \setminus \downarrow_L y : y \in L\} \cup \{L \setminus \uparrow_L C : C \in \mathcal{T}^{\ge c}\}$ is a subbase for $\mathcal{V}$.

After these preparations, we are ready for the main result in this section.

Theorem 6.6. *Any strongly convex semi-pospace T, i.e., any patch space of a T_0 space, enjoys the implications* $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$, *where*

(1) *T is $\uparrow$-stable and $\mathrm{V}T$ is a semitopological frame,*
(2) *T is a join-generating subspace of a semitopological frame $(L, \mathcal{V})$,*
(3) *T is a $\downarrow$-consistent subspace of a semitopological meet-semilattice,*
(4) *T is a $\downarrow$-consistent subspace of a web ordered space,*
(5) *T is a web ordered space.*

If T is $\uparrow$-costable, all five conditions are equivalent.

The semitopological frames in (2) *are isomorphic and homeomorphic to $\mathrm{V}T$. The ordered spaces in* (3) *and* (4) *may be chosen strongly convex.*

Proof. $(1) \Rightarrow (2)$ Identifying T with $T' = (X', \subseteq, \mathcal{T}')$ as in the proof of Proposition 6.5, we may say T is a join-dense subspace of $\mathrm{V}T = (L, \mathcal{V}_T)$. For any $Y \in \alpha(X, \ge)$, one concludes that $cl_{\mathcal{T}} Y$ coincides with $X \cap \downarrow \bigvee Y$, using that T is $\uparrow$-stable: By Lemma 4.1, $A = cl_{\mathcal{T}} Y$ lies in $L = \mathcal{T}^{\le c}$, whence $A \subseteq X \cap \downarrow \bigvee Y \subseteq X \cap \downarrow \bigvee A = A$, by a similar clue as in Proposition 6.5.

For $U \in \mathcal{T}^{\le}$, we have $X \setminus U \in \mathcal{T}^{\le c} = L$ and

$$U^{\le} = \{A \in L : A \not\subseteq X \setminus U\} = L \setminus \downarrow_L \{X \setminus U\}.$$

For $V \in \mathcal{T}^{\ge}$ and $C = X \setminus V$, we obtain a closed upper set C' in T' with

$$V^{\ge} = \{A \in L : A \cap C = \emptyset\} = \{A \in L : \forall\, x \in C\, (\downarrow x \not\subseteq A)\} = L \setminus \uparrow_L C'.$$

This shows that $\mathcal{V}_T$ has the claimed specified subbase in (g3).

(2) ⇒ (3) The subbase in (g3) ensures that $(L, \mathcal{V})$ is strongly convex. The meet of lower sets Y and Z in T is a lower set, too. Now from (g1), (g2), and (g3), one derives the inclusion

$$X \cap cl_{\mathcal{V}}(\downarrow Y \cap \downarrow Z) \subseteq X \cap \downarrow \bigvee(\downarrow Y \cap \downarrow Z) = X \cap \downarrow \bigvee(Y \cap Z) = cl_{\mathcal{T}}(Y \cap Z),$$

which shows that T is a $\downarrow$-consistent subspace of $(L, \mathcal{V})$.

(3) ⇒ (4) Lemma 5.1.

(4) ⇔ (5) Lemma 6.2.

(5) ⇒ (1) By Proposition 4.7, $L = \mathcal{T}^{\leq c}$ is a frame. If T is $\uparrow$-costable, then for $B \in L = \mathcal{T}^{\leq c}$, $U \in \mathcal{T}^{\leq}$, and $V \in \mathcal{T}^{\geq}$, the operation $\wedge_B$ satisfies

$$\wedge_B^{-1}[U^{\leq}] = \{A \in L \colon A \cap B \cap U \neq \emptyset\} = U_B{}^{\leq} \text{ for } U_B = X \setminus C : B \in \mathcal{T}^{\leq},$$

where $C = X \setminus U \in L$ and $C : B = \max\{A \in L : A \cap B \subseteq C\}$,

$$\wedge_B^{-1}[V^{\geq}] = \{A \in L \colon A \cap B \subseteq V\} = V_B{}^{\geq} \text{ for } V_B = X \setminus \uparrow(B \setminus V) \in \mathcal{T}^{\geq}.$$

Finally, if $(L, \mathcal{V})$ is as in (2), then the map $e : L \to \mathcal{T}^{\leq c}$, $y \mapsto X \cap \downarrow y$, is an isomorphism and a homeomorphism between $(L, \mathcal{V})$ and $\mathrm{V}T$, since

$$e^{-1}[U^{\leq}] = \{y \in L : \downarrow y \cap U \neq \emptyset\} = L \setminus \downarrow z \text{ for } U \in \mathcal{T}^{\leq} \text{ and } z = \bigvee(X \setminus U),$$
$$e^{-1}[V^{\geq}] = \{y \in L : X \cap \downarrow y \subseteq V\} = L \setminus \uparrow_L C \text{ for } V \in \mathcal{T}^{\geq} \text{ and } C = X \setminus V. \quad \square$$

7. Regularity Axioms

Concerning regularity (see §2), we note without proof the following facts.

Lemma 7.1. (1) *Compact qospaces are upper and lower regular.*

(2) *Strongly convex upper and lower regular quasi-ordered spaces, as well as υ-convex upper regular semi-qospaces, are regular.*

(3) *$\uparrow$-stable and $\uparrow$-costable regular quasi-ordered spaces are upper regular.*

(4) *If T is upper regular, then the Vietoris space $\mathrm{V}T$ is T_2-ordered.*

Given any coselection ζ, we say a space $(X, \mathcal{S})$ is *ζ-regular* if for all $x \in O \in \mathcal{S}$, there are $U \in \mathcal{S}$ and $V \in \zeta\mathcal{S}$ such that $x \in U \subseteq X \setminus V \subseteq O$. Every space is α-regular (take $U = X \setminus V = O$). The π-regular spaces are the locally compact ones, and for $\eta\mathcal{S} = \{X \setminus \uparrow F \colon F \text{ finite}\}$, the η-regular spaces are the locally hypercompact ones [8]; any such space is υ-regular.

On the other hand, call a qospace $(X, \leq, \mathcal{T})$ a *ζ-qospace* if it is ζ-convex and upper regular, and *ζ-reflexive* if $\zeta(\mathcal{T}^{\leq}) = \mathcal{T}^{\geq}$. Similarly, call a space $(X, \mathcal{S})$ *ζ-reflexive* if $\mathcal{S} = \mathcal{S}^{\zeta\leq}$ and $\zeta S = \mathcal{S}^{\zeta\geq}$. Then we have the following consequences of Theorem 3.10.

Proposition 7.2. *ζ-patch spaces of ζ-determined ζ-regular spaces are ζ-qospaces. The functor P_ζ induces a concrete isomorphism between the category of ζ-reflexive ζ-regular spaces and that of ζ-reflexive ζ-qospaces.*

The first part, together with Lemma 3.7 and Corollary 4.8, yields the following.

Corollary 7.3. *The strong patch space of a web space is an α-qospace. The patch space $\mathrm{P}_\pi S$ of a locally compact web space S is a π-qospace. The weak patch space of a locally hypercompact web space is an υ-qospace.*

If $(X, \mathcal{S})$ and $(X, \zeta\mathcal{S})$ are ζ-determined ζ-regular spaces with $\zeta\zeta\mathcal{S} = \mathcal{S}$, we call $(X, \mathcal{S})$ a *ζ-symmetric space*. Similarly, a strongly convex, upper and lower regular qospace $(X, \leq, \mathcal{T})$ with $\mathcal{T}^{\geq} = \zeta(\mathcal{T}^{\leq})$ and $\mathcal{T}^{\leq} = \zeta(\mathcal{T}^{\geq})$ is referred to as a *ζ-symmetric qospace*. Then an easy check confirms the following theorem.

Theorem 7.4. *The patch functor P_ζ induces a concrete isomorphism between the category of ζ-symmetric spaces and that of ζ-symmetric qospaces.*

This theorem has several interesting instances. First, for $\zeta = \alpha$, it gives the categorical equivalence between (T_0) A-spaces and qosets (posets). Second, using results in [7] and [13], one deduces from Alexander's Subbase Lemma (which is equivalent to UT, see [5]) that the π-symmetric T_0 spaces are the stably compact spaces and the π-symmetric pospaces are the compact pospaces. Thus, Theorem 7.4 includes for $\zeta = \pi$ the equivalence between stably compact spaces and compact pospaces. And third, for $\zeta = \upsilon$, UT assures that the interval space $\mathrm{I}L = (L, \upsilon L^{\upsilon})$ of a complete lattice is compact [5], and Theorem 7.4 yields an equivalence between locally hypercompact weak upper spaces of complete lattices and lattices with compact T_2 interval topologies.

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