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H. Bruin and S. Štimac

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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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ON ISOTOPY OF SELF-HOMEOMORPHISMS OF QUADRATIC INVERSE LIMIT SPACES

H. BRUIN AND S. ŠTIMAC

Abstract. We prove that every self-homeomorphism on the inverse limit space of a quadratic map is isotopic to some power of the shift map.

1. Introduction

The two most prominent families of unimodal maps are the family of quadratic maps Q_a, $a \in [1,4]$, and the family of tent maps T_s, $s \in [1,2]$. The inverse limit spaces of quadratic and tent maps share a lot of common properties. For example, if f is a map from one of these families, then 0 is a fixed point of f; the point $\bar{0} := (\dots, 0, 0, 0)$ is contained in $\varprojlim([0,1], f)$ and is an end-point. The arc-component C of $\varprojlim([0,1], f)$ which contains $\bar{0}$ is a ray converging to, but (provided $a < 4$ and $s < 2$) disjoint from the inverse limit of the core $\varprojlim([c_2, c_1], f)$, and $\varprojlim([0,1], f) = C \cup \varprojlim([c_2, c_1], f)$, where the critical or turning point is denoted as c and $c_k := f^k(c)$. If c is periodic with (prime) period N, then $\varprojlim([c_2, c_1], f)$ contains N end-points.

The relationships between quadratic and tent maps and between their inverse limits are mostly well understood. Each quadratic map Q_a with positive topological entropy is semi-conjugate to a tent map T_s with $\log s = h_{top}(Q_a)$, and this semi-conjugacy collapses (pre)periodic intervals to points [6]. If a quadratic map is not renormalizable and does not

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have an attracting periodic point, then this semi-conjugacy is a conjugacy indeed.

It is clear that if two interval maps are topologically conjugate, then their inverse limit spaces are homeomorphic. The effect of renormalization on the structure of the inverse limit is also well understood [2]: It produces proper subcontinua that are periodic under the shift homeomorphism and homeomorphic with the inverse limit space of the renormalized map.

A very interesting question is to characterize groups of homeomorphisms which act on inverse limits of unimodal maps. In [5] we proved that for every homeomorphism $h : \varprojlim([0,1], T_s) \to \varprojlim([0,1], T_s)$ there exists $R \in \mathbb{Z}$ such that h is isotopic to σ^R, where σ is the standard shift map (see also [3] and [4]). Thus, it is natural to ask the same question for the "fuller" quadratic family, which includes (infinitely) renormalizable maps. The answer does not follow in a straightforward way from [5]. Some work should be done, and the following result is what we have proved in this paper.

Theorem 1.1. *Let $H : \varprojlim([0,1], Q) \to \varprojlim([0,1], Q)$ be a homeomorphism. Then H is isotopic to σ^R for some $R \in \mathbb{Z}$.*

The paper is organized as follows. Section 2 gives basic definitions. Section 3 gives the major step for the isotopy result from tent map inverse limits to quadratic map inverse limits. In section 4 we show how homeomorphisms act on p-points and prove our main theorem. These last proofs depend largely on the results obtained in [5] and [1].

2. Preliminaries

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We consider two families of unimodal maps, the family of quadratic maps $Q_a : [0,1] \to [0,1]$, with $a \in [1,4]$, defined as $Q_a(x) = ax(1-x)$, and the family of tent maps $T_s : [0,1] \to [0,1]$ with slope $\pm s$, $s \in [1,2]$, defined as $T_s(x) = \min\{sx, s(1-x)\}$. Let f be a map from any of these two families. The *critical* or *turning* point is $c := 1/2$. Write $c_k := f^k(c)$. The closed f-invariant interval $[c_2, c_1]$ is called the *core*.

The inverse limit space $\varprojlim([0,1], f)$ is the collection of all backward orbits

$$\{x = (\dots, x_{-2}, x_{-1}, x_0) : f(x_{-i-1}) = x_{-i} \in [0, c_1] \text{ for all } i \in \mathbb{N}_0\},$$

equipped with metric $d(x,y) = \sum_{n \le 0} 2^n |x_n - y_n|$ and *induced*, or *shift homeomorphism*

$$\sigma(x) := \sigma_f(\dots, x_{-2}, x_{-1}, x_0) = (\dots, x_{-2}, x_{-1}, x_0, f(x_0)).$$

Let $\pi_k : \varprojlim([0,1], f) \to [0, c_1]$, $\pi_k(x) = x_{-k}$ be the k^{th} projection map. For any point $x \in \varprojlim([0,1], f)$, the *composant* of x in $\varprojlim([0,1], f)$ is the union of all proper subcontinua of $\varprojlim([0,1], f)$ containing x, and the *arc-component* of x in $\varprojlim([0,1], f)$ is the union of all arcs in $\varprojlim([0,1], f)$ containing x.

We review some of the main tools introduced in [1] and which are necessary here as well. We define *p-points* as those points $x = (\dots, x_{-2}, x_{-1}, x_0) \in \varprojlim([0,1], f)$ such that $x_{-p-k} = c$ for some $k \in \mathbb{N}_0$. The supremum k of the set of integers with this property is called the *p-level* of x, $L_p(x) := k$. Note that k can be ∞, and this will happen if, for example, the turning point is periodic, say of period N. In this case the corresponding inverse limit will have $N+1$ end-points: one is $\bar{0}$ and the others are p-points with p-level ∞ for every p.

Among the p-points of C there are special ones, called salient, which are center points of symmetries in C. Homeomorphisms preserve these symmetries to such an extent that it is possible to prove that salient points map close to salient points.

We call a p-point $y \in C$ *salient* if $0 \le L_p(x) < L_p(y)$ for every p-point $x \in (\bar{0}, y)$. Let $(s_p^i)_{i\in\mathbb{N}}$ be the sequence of all salient p-points of C, ordered such that $s_p^i \in (\bar{0}, s_p^{i+1})$ for all $i \ge 1$. Since by definition $L_p(s_p^i) > 0$, for all $i \ge 1$, we have $L_p(s_p^1) = 1$. Also, since $s_p^i = \sigma^{i-1}(s_p^1)$, we have $L_p(s_p^i) = i$ for every $i \in \mathbb{N}$. Therefore, for every p-point x of $\varprojlim([0,1], f)$ with $L_p(x) \neq 0$, there exists a unique salient p-point s_p^k such that $L_p(x) = L_p(s_p^k) = k$. Note that the salient p-points depend on p: If $p \ge q$, then the salient p-point s_p^i equals the salient q-point s_q^{i+p-q}.

A continuum is *chainable* if for every $\varepsilon > 0$, there is a cover $\{\ell^1, \dots, \ell^n\}$ of open sets (called *links*) of diameter less than ε such that $\ell^i \cap \ell^j \neq \emptyset$ if and only if $|i - j| \le 1$. Such a cover is called a *chain*. Clearly, the interval $[0, c_1]$ is chainable. Throughout, we will use sequences of chains $\mathfrak{C}_p$ of $\varprojlim([0,1], f)$ satisfying the following properties:

(1) there is a chain $\{I_p^1, \dots, I_p^n\}$ of $[0, c_1]$ such that $\ell_p^j := \pi_p^{-1}(I_p^j)$ are the links of $\mathfrak{C}_p$;
(2) each point $x \in \bigcup_{i=0}^p f^{-i}(c)$ is a boundary point of some link I_p^j;
(3) for each i there is j such that $f(I_{p+1}^i) \subset I_p^j$.

If $\max_j |I_p^j| < \varepsilon s^{-p}/2$, then $\mathrm{mesh}(\mathfrak{C}_p) := \max\{\mathrm{diam}(\ell_p) : \ell_p \in \mathfrak{C}_p\} < \varepsilon$, which shows that $\varprojlim([0,1], f)$ is indeed chainable. Property (3) ensures that $\mathfrak{C}_{p+1}$ refines $\mathfrak{C}_p$ (written $\mathfrak{C}_{p+1} \preceq \mathfrak{C}_p$).

Note that all p-points of p-level k belong to the same link of $\mathfrak{C}_p$. (This follows by property (1) of $\mathfrak{C}_p$ because $L_p(x) = L_p(y)$ implies $\pi_p(x) =$

$\pi_p(y)$.) Therefore, every link of $\mathfrak{C}_p$ which contains a p-point of p-level k contains also the salient p-point s_p^k.

Let $\ell^0, \ell^1, \dots, \ell^k$ be those links in $\mathfrak{C}_p$ that are successively visited by an arc $A \subset C$ (hence, $\ell^i \neq \ell^{i+1}$, $\ell^i \cap \ell^{i+1} \neq \emptyset$, and $\ell^i = \ell^{i+2}$ is possible if A turns in ℓ^{i+1}). We call the arc A *p-link-symmetric* if $\ell^i = \ell^{k-i}$ for $i = 0, \dots, k$ and *maximal p-link-symmetric* if it is p-link-symmetric and there is no p-link-symmetric arc $B \supset A$ which passes through more links than A. In any of these cases, k is even, and the link $\ell^{k/2}$ is called the *central link* of A.

As we have already mentioned in the introduction, in [5] we proved that for a tent map T and every homeomorphism $h : \varprojlim([0,1], T) \to \varprojlim([0,1], T)$, there exists $R \in \mathbb{Z}$ such that h is isotopic to σ^R. The only places in the proofs of [5] that rely on properties of tent maps are those where Theorem 4.1, Proposition 4.2, or Theorem 1.3 from [1] are cited. All other results, although stated for the tent maps, work for the quadratic maps as well. Therefore, the goal of this paper is to prove the analogs of these three results for the quadratic family, since, in that case, the proof of our main theorem will follow from [5]. For the reader's convenience, we state the analogs of these three results.

Let $H : \varprojlim([0,1], Q) \to \varprojlim([0,1], Q)$ be a homeomorphism. Let p and q be such that $H(\mathfrak{C}_p) \prec \mathfrak{C}_q$, where $\mathfrak{C}_k$ denotes a natural chain of $\varprojlim([0,1], Q)$. Let x be a p-point of C and let y be a q-point of C. We will write that $H(x) \approx y$ if $H(x)$ and y belong not only to the same link of $\mathfrak{C}_q$, but also to the same arc-component of that link.

Theorem 2.1 (The analog of [1, Theorem 4.1])**.** *There exists $R \in \mathbb{Z}$ such that $H(s_i) \approx \sigma^R(s_i)$ for all sufficiently large integers $i \in \mathbb{N}$.*

Proposition 2.2 (The analog of [1, Proposition 4.2])**.** *Let R be as in Theorem* 2.1*. For every p-point $x \in C$ of p-level i there exists a q-point $y \in C$ of p-level $i+R$ such that $H(x) \approx y$ for all sufficiently large integers $i \in \mathbb{N}$.*

Theorem 2.3 (The analog of [1, Theorem 1.3])**.** *Let R be as in Theorem* 2.1*. Then H and σ^R, restricted to the core $\varprojlim([c_2, c_1], Q)$, are pseudo-isotopic; i.e., they permute the composants of the core of the inverse limit in the same way.*

At the end of these preliminaries let us recall the definition of itineraries which we will need later on: For $x \in X$, define the *itinerary* $i(x) = (i_n)_{n \in Z}$

as the sequence where

$$i_n(x) = \begin{cases} 0, & x_n \le c, \\ 1, & x_n \ge c, \end{cases}$$

and if $x_n = c$, we write $i_n(x) = {}^0_1$.

3. Pseudo-isotopy

Let Q be a quadratic map of entropy $h_{top}(Q) > \frac{1}{2}\log 2$, which is renormalizable, but (due to the entropy restraint) of period $N > 2$. Let $\{J_j\}_{j=0}^{N-1}$ be the periodic cycle of intervals numbered so that $c \in J_0$ and $Q(J_i) = J_{i+1 \pmod N}$. Denote $J = \bigcup_{j=0}^{N-1} J_j$. Also assume that $J_i = [p_i, \hat{p}_i]$, where p_i is N-periodic and $Q^N(\hat{p}_i) = p_i$. Let $T := T_s$ be the semiconjugate tent map with $s = \exp(h_{top}(Q))$. Then T has an N-periodic critical point.

Let $X := \varprojlim([c_2, c_1], Q)$ and $\tilde{X} := \varprojlim([\tilde{c}_2, \tilde{c}_1], T)$. Let $G_j := \{x \in X : \pi_k(x) \in J_{j-k \pmod N}, k \in \mathbb{N}_0\}$, $j \in \{0, \dots, N-1\}$. These are the (maximal) proper subcontinua of X that are not arcs or points. On the other hand, all proper subcontinua of $\tilde{X}$ are arcs and points and $\tilde{X}$ has N endpoints e_j, where $\tilde{\pi}_0(e_j) = \tilde{c}_j$. Here $\pi_i : X \to [c_2, c_1]$ and $\tilde{\pi}_i : \tilde{X} \to [\tilde{c}_2, \tilde{c}_1]$ are the coordinate projections.

There is a unique arc-component Z_j of $X \cap \pi_0^{-1}(J_j)$ that compactifies exactly on G_j. This is a ray, and we can extend it on one side with an arc Z_j^* such that $\pi_0(Z_j^*) = [r_{j,0}, p_j]$ (where the point $r_{j,0}$ close to p_j is chosen below).

Theorem 3.1. *There exists a continuous onto map $\phi : \varprojlim([c_2, c_1], Q) \to \varprojlim([\tilde{c}_2, \tilde{c}_1], T)$ such that $\phi(G_j) = e_j$ and ϕ is one-to-one on $\varprojlim([c_2, c_1], Q) \setminus \bigcup_{j=0}^{N-1} G_j$.*

Proof. For the quadratic map, let $(r_{j,k})_{k \in \mathbb{N}_0}$ be a monotone sequence of points such that $r_{j,k}$ belong to a single component of $[c_2, c_1] \setminus J$, and $r_{j,k} \to p_j$ (see Figure 1).

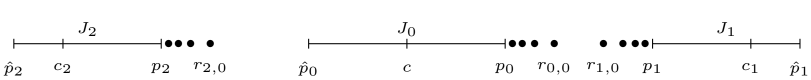

Figure 1. The intervals J_j with sequences $r_{j,k} \to p_j$.

Without loss of generality, we can set $Q(r_{j,k}) = r_{j+1,k}$ for $0 \le j < N-1$ and $Q(r_{N-1,k+1}) = r_{0,k}$ for $k \ge 0$. This means that the sequence

$(r_{j,k} : j = 0, \dots N-1, k \in \mathbb{N}_0)$, starting from $r_{0,0}$, forms a single backward orbit.

Similarly, for the tent map T, for fixed $0 \le j < N$, let $(\tilde{r}_{j,k})_{k\in\mathbb{N}_0}$ be a monotone sequence of points such that $\tilde{r}_{j,k}$ belong to a single component of $[\tilde{c}_2, \tilde{c}_1] \setminus \mathrm{orb}_T(\tilde{c})$, and $\tilde{r}_{j,k} \to \tilde{c}_j$. Again we set $T(\tilde{r}_{j,k}) = \tilde{r}_{j+1,k}$ for $0 \le j < N-1$ and $T(\tilde{r}_{N-1,k+1}) = \tilde{r}_{0,k}$ for $k \ge 0$.

Note that if $x \in X$ is such that $\pi_n(x) \notin J$ for all $n \in \mathbb{Z}$, then there is a unique point $\tilde{x} \in \tilde{X}$ such that x and $\tilde{x}$ have the same itinerary. Without loss of generality, we can indeed assume that $\mathrm{orb}(r_{0,0}) \cap J = \emptyset$, and then indeed choose $\tilde{r}_{0,0}$ with the same itinerary as $r_{0,0}$. Then $r_{j,k}$ and $\tilde{r}_{j,k}$ have the same itinerary for every j and k.

Now take $x \in X$. Depending on whether $\pi_0(x) \in \bigcup_j (r_{j,0}, \hat{p}_j)$ or not, and on whether G_j are the Knaster continua or not, we have different algorithms to define $\phi(x)$.

Case 1. $\pi_0(x) \notin \bigcup_j (r_{j,0}, \hat{p}_j)$. Suppose that x belongs to a component W of $X \setminus \pi_0^{-1}(\bigcup_j (r_{j,0}, \hat{p}_j))$. There is a unique component $\tilde{W}$ of $\tilde{X} \setminus \tilde{\pi}_0^{-1}(\bigcup_j (\tilde{r}_{j,0}, \tilde{c}_j))$ which has the same backward itinerary as W. Define $\phi : W \to \tilde{W}$ such that $\tilde{\pi}_0 \circ \phi \circ \pi^{-1}$ is an affine map from $\pi_0(W)$ to $\tilde{\pi}_0(\tilde{W})$.

Case 2. $\pi_0(x) \in \bigcup_j (r_{j,0}, \hat{p}_j)$, non-Knaster construction. Assume that $\pi_0(x) \in [r_{j,0}, \hat{p}_j]$ for some j, and let W be the component of $\pi_0^{-1}([r_{j,0}, \hat{p}_j])$ containing x and $V \subset W$ be the corresponding component of $\pi_0^{-1}([p_j, \hat{p}_j])$. Let

$$\Lambda = \sup\{L(y) : y \in V \text{ is a } p\text{-point}\},$$

where $L := L_p$ is the p-level of a p-point. The definition of $\phi|_W$ will depend on the value of Λ (see Figure 2).

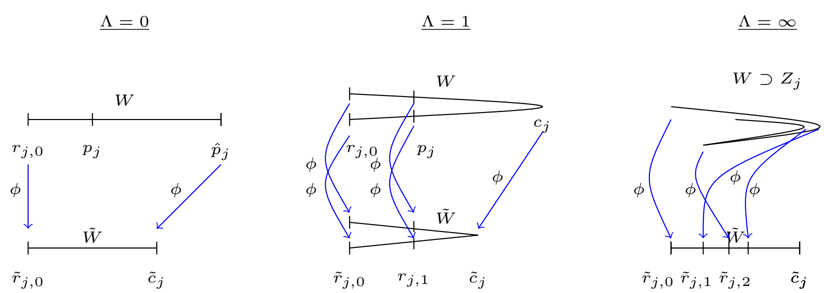

FIGURE 2. The non-Knaster case: The arcs W and their images under ϕ. The labels refer to the π_0-images of the points.

(2.1). If $\Lambda = 0$, then $\pi_0 : W \to [r_{j,0}, \hat{p}_j]$ is injective. Let $\tilde{W} \subset \tilde{X}$ be the unique component of $\tilde{\pi}_0^{-1}([\tilde{r}_{j,0}, \tilde{c}_j])$ that has the same backward itinerary as W. Define $\phi : W \to \tilde{W}$ to be the homeomorphism such that $\tilde{\pi}_0 \circ \phi \circ \pi^{-1}$ is the affine map from $[r_{j,0}, \hat{p}_j] = \pi_0(W)$ onto $[\tilde{r}_{j,0}, \tilde{c}_j] = \tilde{\pi}_0(\tilde{W})$ so that $\tilde{\pi}_0 \circ \phi \circ \pi^{-1}(r_{j,0}) = \tilde{r}_{j,0}$.

(2.2). If $0 < \Lambda < \infty$, then let $m \in W$ be the p-point of level $L(m) = \Lambda$. In fact, m is the unique point with this property: it is the midpoint of W. Furthermore, there are two finite sequences of points $\{v_k\}_{k=1}^{\lambda}$ and $\{\hat{v}_k\}_{k=1}^{\lambda}$, $\lambda \le \Lambda$, inside V such that $\partial V = \{v_1, \hat{v}_1\}$, $v_\lambda = \hat{v}_\lambda = m$, v_{k+1} is the p-point in $[v_k, m]$ such that $L(v_{k+1}) > L(v_k)$ and no point in (v_k, v_{k+1}) has level larger than the level of v_k, and $\hat{v}_{k+1}$ is the p-point in $[\hat{v}_k, m]$ such that $L(\hat{v}_{k+1}) > L(\hat{v}_k)$ and no point in $(\hat{v}_k, \hat{v}_{k+1})$ has level larger than the level of $\hat{v}_k$. (Note that if V and V' are two different components of $\pi_0^{-1}([p_j, \hat{p}_j])$ with levels of their midpoints satisfying $\Lambda < \Lambda'$, then $\lambda < \lambda'$, and, in fact, the levels of the points $v'_k \in V'$ are a superset of the levels of the points $v_k \in V$.)

Let $\tilde{W} \subset \tilde{X}$ be the component of $\tilde{\pi}_0^{-1}([\tilde{r}_{j,0}, \tilde{c}_j]$ so that the two endpoints of $\tilde{W}$ have the same itineraries as the corresponding endpoints of W. Note that the midpoint $\tilde{m}$ of $\tilde{W}$ has the level $L(\tilde{m}) = \Lambda$.

Define $\phi : W \to \tilde{W}$ to be the homeomorphism such that

(a) $\tilde{\pi}_0 \circ \phi \circ \pi^{-1}$ maps $[r_{j,0}, p_j]$ affinely onto $[\tilde{r}_{j,0}, \tilde{r}_{j,1}]$;
(b) $[v_k, v_{k+1}]$ and $[\hat{v}_k, \hat{v}_{k+1}]$ are mapped to the two components of $\tilde{\pi}_0^{-1}([\tilde{r}_{j,k}, \tilde{r}_{j,k+1}]) \cap \tilde{W}$, for $0 \le k < \lambda - 1$;
(c) $[v_{\lambda-1}, \hat{v}_{\lambda-1}]$ is mapped onto $\tilde{\pi}_0^{-1}([\tilde{r}_{j,\lambda-1}, \tilde{c}_j]) \cap \tilde{W}$ in such a way that $\tilde{\pi}_0 \circ \phi(m) = \tilde{c}_j$.

(2.3). If $\Lambda = \infty$ and $x \in Z_j$ (but $Z_j \cap G_j = \emptyset$ since we are in the non-Knaster case), then $V = Z_j \subset W$ is a ray, and we can define an infinite sequence $\{v_k\}_{k=1}^{\infty}$ so that $\pi_0(v_1) = p_j$ and v_{k+1} is the p-point on $Z_j \setminus [v_1, v_k]$ such that $L(v_{k+1}) > L(v_k)$ and no p-point on (v_k, v_{k+1}) has the level larger than the level of v_k.

Let $\tilde{W}$ be the component of $\tilde{\pi}_0^{-1}([\tilde{r}_{j,0}, \tilde{c}_j))$ having e_j as boundary point. Define $\phi : W \to \tilde{W}$ to be the homeomorphism such that

(a) $\tilde{\pi}_0 \circ \phi \circ \pi^{-1}$ maps $[r_{j,0}, p_j]$ affinely onto $[\tilde{r}_{j,0}, \tilde{r}_{j,1}]$;
(b) $[v_k, v_{k+1}]$ is mapped to $\tilde{\pi}_0^{-1}([\tilde{r}_{j,k}, \tilde{r}_{j,k+1}]) \cap \tilde{W}$, for $k \ge 1$.

(2.4). $\phi(x) = e_j$ for every $x \in G_j$.

Case 3. $\pi_0(x) \in \bigcup_j (r_{j,0}, \hat{p}_j)$, Knaster construction. Now we adapt the construction for the case that the renormalization $Q^N|_{J_0}$ is a full unimodal map (i.e., G_j is the Knaster continuum). In this case $Z_j \subset G_j$ and the construction of $\phi : W \to \tilde{W}$ for (2.1) remains the same. Item (2.2) is changed into (3.2) below, and (2.3) and (2.4) are combined into (3.4).

(3.2). With W, $\tilde{W}$, and sequence $\{v_k\}_{k=1}^{\lambda}$ and $\{\hat{v}_k\}_{k=1}^{\lambda}$ with $v_\lambda = \hat{v}_\lambda = m$ as before, define $\phi : W \to \tilde{W}$ to be the homeomorphism such that

(a) $\tilde{\pi}_0 \circ \phi \circ \pi^{-1}$ maps $[r_{j,0}, p_j]$ affinely onto $[\tilde{r}_{j,0}, \tilde{r}_{j,\lambda+1}]$;
(b) $[v_k, v_{k+1}]$ and $[\hat{v}_k, \hat{v}_{k+1}]$ are mapped to the two components of $\tilde{\pi}_0^{-1}([\tilde{r}_{j,\lambda+k}, \tilde{r}_{j,\lambda+k+1}]) \cap \tilde{W}$, for $1 \leq k < \lambda - 1$.
(c) $[v_{\lambda-1}, \hat{v}_{\lambda-1}]$ is mapped onto $\tilde{\pi}_0^{-1}([\tilde{r}_{j,2\lambda-1}, \tilde{c}_j]) \cap \tilde{W}$ in such a way that $\tilde{\pi}_0 \circ \phi(m) = \tilde{c}_j$ (see Figure 3).

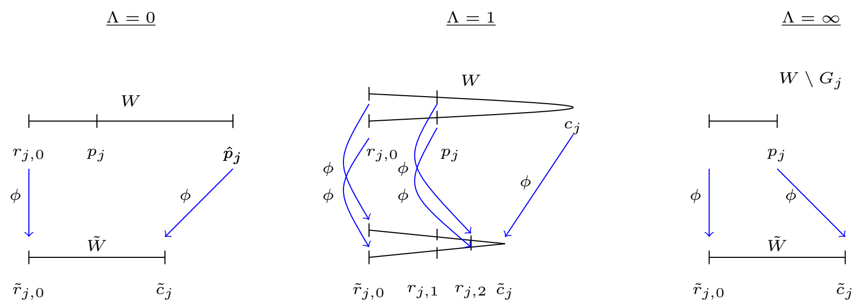

FIGURE 3. The Knaster case: the arcs W and their images under ϕ. The labels refer to the π_0-images of the points.

(3.4). If $\Lambda = \infty$ and $x \in Z_j^* \cup G_j$, then let $\tilde{W}$ be the component of $\tilde{X}$ with boundary point e_j and $\tilde{\pi}_0(\tilde{W}) = [\tilde{r}_{j,0}, \tilde{c}_j)$. Let $\phi : Z_j^* \to \tilde{W}$ be such that $\tilde{\pi}_0 \circ \phi \circ \pi_0^{-1} : [r_{j,0}, p_j) \to [\tilde{r}_{j,0}, \tilde{c}_j)$ is affine and $\phi(x) = e_j$ if $x \in G_j$.

By construction, $\phi : W \to \tilde{W}$ is always a homeomorphism for all the components W in this construction, and their union, together with $\bigcup_{j=0}^{N-1} G_j$, is the entire X, just as the union of all $\tilde{W}$, together with $\bigcup_{j=0}^{N-1} \{e_j\}$, is the entire $\tilde{X}$. Therefore, ϕ is onto and one-to-one on $\tilde{X} \setminus \bigcup_{j=0}^{N-1} G_j$.

The construction of $\phi : W \to \tilde{W}$ essentially depends only on the value of Λ, in the sense that without loss of generality,

$$\tilde{\pi}_0 \circ \phi \circ \pi^{-1} : \pi_0(W) \to \tilde{\pi}_0(\tilde{W}) \tag{3.1}$$

can be chosen to depend only on $\Lambda = \Lambda(W)$. This makes ϕ continuous on $X \setminus \bigcup_{j=0}^{N-1} (G_j \cup Z_j)$. Finally, in the non-Knaster case, $W \to Z_j \cup G_j$ in the Hausdorff metric if and only if $\Lambda(W) \to \infty$ and $\Lambda(W) \pmod N = j$. Recall that we specified $\phi|_{Z_j}$ to have the limit dynamics of $\phi|_W$ for $W \to Z_j \cup G_j$, thus achieving continuity of ϕ on each $Z_j \cup G_j$.

In the Knaster case, $\mathrm{diam}(\phi(W)) \to 0$ as $W \to G_j$, so here ϕ is also continuous on each G_j. □

Corollary 3.2. *Given a homeomorphism $h : X \to X$, the map $\tilde{h} := \phi \circ h \circ \phi^{-1}$ is a well-defined homeomorphism on $\varprojlim([\tilde{c}_2, \tilde{c}_1], T)$.*

Proof. The endpoints e_j are the only points where ϕ^{-1} is not single-valued. Therefore, $\tilde{h}$ is well defined on $\varprojlim([\tilde{c}_2, \tilde{c}_1], T) \setminus \{e_0, \dots, e_{N-1}\}$. However, since h is a homeomorphism, it has to permute the subcontinua G_j; therefore, $h \circ \phi^{-1}(e_j) = G_i$ for some i and $\phi \circ h \circ \phi^{-1}(e_j) = e_i$. Therefore, $\tilde{h}$ is defined (in a single-valued way) also at the endpoints, and it permutes them by the same permutation as h permutes the subcontinua G_j.

Since ϕ is continuous and injective outside G_j, $\tilde{h}$ is continuous on $\tilde{X} \setminus \{e_0, \dots, e_{N-1}\}$. To conclude continuity of $\tilde{h}$ at the endpoints e_j, observe that for every $\Lambda_0 \in \mathbb{N}$ there is a neighborhood U of G_j such that $\pi_0^{-1}(J) \cap W \subset U$ only if $\Lambda(W) \geq \Lambda_0$ (where the components W and their maximal levels $\Lambda(W)$ are as in the proof of Theorem 3.1). On the other hand, for every neighborhood $\tilde{U}$ of $e_i = \tilde{h}(e_j)$, $\phi(W) \cap \tilde{\pi}_0^{-1}(\tilde{c}_i) \cap \tilde{U} \neq \emptyset$ only if $\Lambda(W)$ is sufficiently large. Therefore, $\tilde{h}$ maps small neighborhoods of e_j into small neighborhoods of e_i, and the continuity of $\tilde{h}$ at $\{e_0, \dots, e_{N-1}\}$ follows. □

Recall that $\varprojlim([0,1], Q) = C \cup X$, where $X = \varprojlim([c_2, c_1], Q)$ and C is the ray containing $\bar{0}$ that compactifies on X. Analogously, $\varprojlim([0,1], T) = \tilde{C} \cup \tilde{X}$.

Remark 3.3. Let $H : \varprojlim([0,1], Q) \to \varprojlim([0,1], Q)$ be a homeomorphism. In a straightforward way it is possible to expand our construction of $\phi : X \to \tilde{X}$ to get the continuous map $\Phi : \varprojlim([0,1], Q) \to \varprojlim([0,1], T)$ such that $\Phi|_X = \phi$ and the map $\tilde{H} := \Phi \circ H \circ \Phi^{-1}$ is a well-defined homeomorphism on $\varprojlim([0,1], T)$. Obviously, $\tilde{H}|_{\tilde{X}} = \tilde{h}$.

In [5] we proved that every homeomorphism on $\varprojlim([0,1], T)$ is isotopic to some power of the shift map. Therefore, $\tilde{H}$ is isotopic to σ^R for some $R \in \mathbb{Z}$. Since Φ is injective on $\varprojlim([0,1], Q) \setminus (\bigcup_{i=0}^{N-1} G_i)$ and $\Phi \circ H = \tilde{H} \circ \Phi$, H restricted to $\varprojlim([0,1], Q) \setminus (\bigcup_{i=0}^{N-1} G_i)$ is pseudo-isotopic to σ^R, and H permutes the G_is in the same way as σ^R.

4. Isotopy

Let $\mathfrak{C}_k$ denote a natural chain of $\varprojlim([0,1],Q)$. Then, by construction of Φ, $\Phi(\mathfrak{C}_k)$ is a natural chain of $\varprojlim([0,1],T)$. Also, for every $k,l \in \mathbb{N}$, Φ maps the salient k-point of C of k-level l to the salient k-point of $\tilde{C}$ of the same k-level l.

Let p and q be such that $H(\mathfrak{C}_p) \prec \mathfrak{C}_q$. Then, by construction of Φ, $\tilde{H}(\Phi(\mathfrak{C}_p)) \prec \Phi(\mathfrak{C}_q)$. Let $\{t_i : i \in \mathbb{N}\}$ denote all salient p-points of C and $\{s_i : i \in \mathbb{N}\}$ all salient p-points of $\tilde{C}$. Let $\{t'_i : i \in \mathbb{N}\}$ denote all salient q-points of C and $\{s'_i : i \in \mathbb{N}\}$ all salient q-points of $\tilde{C}$.

Let A_i be the maximal p-link-symmetric arc centered at t_i. Since A_i is p-link-symmetric, and $H(\mathfrak{C}_p) \preceq \mathfrak{C}_q$, the image $D_i := H(A_i)$ is q-link-symmetric and therefore has a well-defined central link ℓ_q, and a well-defined center, we denote it as m'_i. In fact, $H(t_i)$ and m'_i belong to the central link ℓ_q and m'_i is the q-point with the highest q-level of all q-points of the arc component of ℓ_q which contains $H(t_i)$. Recall that we write $H(d) \approx b$ if $H(d)$ and b belong not only to the same link, but also to the same arc-component of that link. Thus, $H(t_i) \approx m'_i$.

As we explained in the preliminaries, to prove our main theorem we should prove Theorem 2.1, Proposition 2.2, and Theorem 2.3.

Proof of Theorem 2.1. We will prove that there exists $R \in \mathbb{Z}$ such that $m'_i = t'_{i+R}$ for all sufficiently large integers $i \in \mathbb{N}$.

Let Φ and H be as in Remark 3.3. Let R be such that $\tilde{H}$ is isotopic to σ^R. Recall that Φ maps the salient p-point of C of p-level i to the salient p-point of $\tilde{C}$ of the same p-level i; that is, $\Phi(t_i) = s_i$, for all $i \in \mathbb{N}$, and analogously for salient q-points, $\Phi(t'_i) = s'_i$, for all $i \in \mathbb{N}$. Since by [7, Theorem 4.12] and [1, Theorem 4.1], $\tilde{H}(s_i) \approx s'_{i+R}$ for all i sufficiently large, and $\tilde{H} \circ \Phi = \Phi \circ H$, it follows that $H(t_i) \approx t'_{i+R}$ for all sufficiently large $i \in \mathbb{N}$. $\square$

Now, in the same way as in the proof of [1, Proposition 4.2], but using Theorem 2.1 instead of [1, Theorem 4.1], it follows that for every p-point $x \in C$ of p-level i there exists a q-point $x' \in C$ of p-level $i+R$ such that $H(x) \approx x'$ for all sufficiently large integers $i \in \mathbb{N}$, which is the statement of Proposition 2.2. Further, in the same way as in the proof of [1, Theorem 1.3], but using Theorem 2.1 instead of [1, Theorem 4.1] and Proposition 2.2 instead of [1, Proposotion 4.2], it follows that the homeomorphism $H : \varprojlim([0,1],Q) \to \varprojlim([0,1],Q)$ is pseudo-isotopic to σ^R.

Having now the analogous results of Theorem 4.1, Proposition 4.2, and Theorem 1.3 from [1] proved for the quadratic family, the proof of our main theorem follows from [5].

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(Bruin) Faculty of Mathematics; University of Vienna; Oskar Morgensternplatz 1; 1090 Wien, Austria
E-mail address: `henk.bruin@univie.ac.at`

(Štimac) Department of Mathematics; Faculty of Science; University of Zagreb; Bijenička 30; 10 000 Zagreb, Croatia
E-mail address: `sonja@math.hr`