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by

Nobuyuki Kemoto

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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THE STRUCTURE OF THE LINEARLY ORDERED COMPACTIFICATIONS OF GO-SPACES

NOBUYUKI KEMOTO

Abstract. A linearly ordered extension of a GO-space X is a LOTS L such that the LOTS L contains the GO-space X as a subspace and the order $<_L$ on L extends the order $<_X$ on X; moreover, if X is dense in L, then L is called a linearly ordered d-extension. A linearly ordered compactification of a GO-space X is a compact linearly ordered d-extension of X. We will visualize all linearly ordered compactifications of a given GO-space in a certain way. For a given linearly ordered set $\langle X, <_X\rangle$, $\mathbb{L}_X$ denotes the class of all linearly ordered compactifications of GO-spaces whose underlying linearly ordered set is $\langle X, <_X\rangle$. We will also see the partial order structure $\langle \mathbb{L}_X, \leq\rangle$, where $L_0 \leq L_1$ if there is a continuous map $f : L_1 \to L_0$ such that $f(x) = x$ for every $x \in X$, is order isomorphic to the product $\langle \mathcal{P}(A), \subseteq\rangle \times \langle \mathcal{P}(B), \subseteq\rangle \times \langle \mathcal{P}(C), \subseteq\rangle$ for some sets A, B, and C, where $\langle \mathcal{P}(A)), \subseteq\rangle$ denotes the partial ordered set of the set of all subsets of A with the usual inclusion. The sets A, B, and C will be described exactly. Moreover, we will see that the partial order structure on the class of all linearly ordered compactifications of a fixed GO-space depends only on its underlying linearly ordered set, not on its topology.

1. Introduction

We assume that all topological spaces have cardinality at least 2. At first, we give precise definitions for later arguments.

A linearly ordered set $\langle L, <_L\rangle$ (see [1]) has a natural T_2-topology $\lambda(<_L)$ called the *interval topology* which is the topology generated by $\{(\leftarrow, u)_L :$

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$u \in L\} \cup \{(u, \rightarrow)_L : u \in L\}$ as a subbase, where $(\leftarrow, u)_L = \{w \in L : w <_L u\}$ and $(u, \rightarrow)_L = \{w \in L : u <_L w\}$. Also, we denote $\{w \in L : u <_L w \leq_L v\}$ by $(u, v]_L$, and $[u, v]_L$, $(u, v]_L$ ..., etc., are similarly defined, where $w \leq_L v$ means $w <_L v$ or $w = v$. If the contexts are clear, we write $<$ and $(u, v]$ instead of $<_L$ and $(u, v]_L$, respectively. Note that this subbase induces a base by convex subsets (e.g., the collection of all intersections of at most two members of this subbase), where a subset B of L is *convex* if for every $u, v \in B$ with $u <_L v$, $[u, v]_L \subseteq B$. The triple $\langle L, <_L, \lambda(<_L)\rangle$ is called a *linearly ordered topological space* (*LOTS*) and simply denoted by LOTS L. Observe that if $u \in U \in \lambda(<_L)$ and $(\leftarrow, u)_L \neq \emptyset$, then there is $v \in L$ such that $v <_L u$ and $(v, u]_L \subseteq U$. Also observe its analogous result. Unless otherwise stated, the real line $\mathbb{R}$ is considered as a linearly ordered set (hence LOTS) with the usual order; similarly so are the set $\mathbb{Q}$ of rationals, the set $\mathbb{P}$ of irrationals, and an ordinal α.

A triple $\langle L, <_L, \tau\rangle$, where $<_L$ is a linear order on L and τ is a T_2 topology on L, is called a *generalized ordered space* (*GO-space*) if τ has a base consisting of convex sets, simply denoted by GO-space L; see [4]. The pair $\langle L, <_L\rangle$ (the triple $\langle L, <_L, \lambda(<_L)\rangle$) is said to be the *underlying linearly ordered set* (the *underlying LOTS*, respectively) of the GO-space L and such a topology τ is called a *GO-space topology* on L. It is easy to verify that τ as described above is stronger than the topology $\lambda(<_L)$ of the underlying linearly ordered set; that is, $\tau \supset \lambda(<_L)$. Obviously, every LOTS is a GO-space but not conversely; the Sorgenfrey line $\mathbb{S}$ is such an example.

Let $L = \langle L, <_L, \lambda(<_L)\rangle$ be a LOTS and $X = \langle X, <_X, \tau\rangle$ a GO-space with $X \subseteq L$. If $<_L$ extends $<_X$ and the space $\langle X, \tau\rangle$ is a subspace of $\langle L, \lambda(<_L)\rangle$, that is, $\tau = \lambda(<_L) \upharpoonright X = \{U \cap X : U \in \lambda(<_L)\}$, then the LOTS L is called a *linearly ordered extension* of X. Moreover, if X is dense in L, then the LOTS L is called a *linearly ordered d-extension* of X; see [5]. A compact linearly ordered d-extension is called a *linearly ordered compactification*; see [2], [3], and [6].

A pair $\langle A, B\rangle$ of subsets of a linearly ordered set $\langle L, <_L\rangle$ is called a *cut* if $A \cup B = L$, and if $u \in A$ and $v \in B$ then $u <_L v$. A cut is called a *jump* if A has a maximal element (denoted by $\max A$) and B has a minimal element (denoted by $\min B$). A cut $\langle A, B\rangle$ is called a *gap* if A has no maximal element (we write, A has no max) and B has no minimal element (B has no min). In particular, if $A = \emptyset$ or $B = \emptyset$, then $\langle A, B\rangle$ is called an *end gap*; other gaps are called *middle gaps*. Usually if $\langle \emptyset, X\rangle$ is a gap, then it is written as $-\infty$. Similarly, if $\langle X, \emptyset\rangle$ is a gap, then it is written as ∞. It is easy to verify:

- A compact GO-space is a LOTS.
- A LOTS L is compact iff the linearly ordered set L has no gaps.

Now let $X = \langle X, <_X, \tau\rangle$ be a GO-space and $\lambda = \lambda(<_X)$. Note that for every $x \in X$, $(\leftarrow, x]_X \notin \lambda$ iff $(x, \rightarrow)_X$ is non-empty and has no min, also analogously $[x, \rightarrow)_X \notin \lambda$ iff $(\leftarrow, x)_X$ is non-empty and has no max. Let

$$X_R = \{x \in X : (\leftarrow, x]_X \notin \lambda\},$$
$$X_L = \{x \in X : [x, \rightarrow)_X \notin \lambda\}.$$

Note that the definitions of X_R and X_L only depend on the underlying LOTS. Also let

$$X^+_\tau = \{x \in X : (\leftarrow, x]_X \in \tau \setminus \lambda\},$$
$$X^-_\tau = \{x \in X : [x, \rightarrow)_X \in \tau \setminus \lambda\}.$$

Obviously, $X^+_\tau \subseteq X_R$ and $X^-_\tau \subseteq X_L$. Note that $X^+_\tau \cap X^-_\tau$ might be non-empty. If there is no confusion, we usually simply write X^+ and X^- instead of X^+_τ and X^-_τ. The following two lemmas are straightforward.

Lemma 1.1. *In the situation above, the topology τ coincides with the topology generated by $\{(\leftarrow, x)_X : x \in X\} \cup \{(x, \rightarrow)_X : x \in X\} \cup \{(\leftarrow, x]_X : x \in X^+_\tau\} \cup \{[x, \rightarrow)_X : x \in X^-_\tau\}$ as a subbase.*

Lemma 1.2. *Let $\langle X, <_X\rangle$ be a linearly ordered set with $A \subseteq X_R$ and $B \subseteq X_L$. Moreover, let $\tau(A, B)$ be the topology generated by $\{(\leftarrow, x)_X : x \in X\} \cup \{(x, \rightarrow)_X : x \in X\} \cup \{(\leftarrow, x]_X : x \in A\} \cup \{[x, \rightarrow)_X : x \in B\}$ as a subbase. Then $\tau(A, B)$ is a GO-space topology and $A = X^+_{\tau(A,B)}$ and $B = X^-_{\tau(A,B)}$.*

In the case $X = \mathbb{R}$, note $X_R = X_L = \mathbb{R}$. The Sorgenfrey line $\mathbb{S}$ is the GO-space $\langle \mathbb{R}, <_\mathbb{R}, \tau(\emptyset, \mathbb{R})\rangle$ and the Michael line $\mathbb{M}$ is the GO-space $\langle \mathbb{R}, <_\mathbb{R}, \tau(\mathbb{P}, \mathbb{P})\rangle$. Given a linearly ordered set $\langle X, <_X\rangle$, let GT_X be the set of all GO-space topologies on $\langle X, <_X\rangle$, i.e.,

$$GT_X = \{\tau : \langle X, <_X, \tau\rangle \text{ is a GO-space}\}.$$

We consider GT_X as a partially ordered set $\langle GT_X, \subseteq\rangle$ with the usual inclusion, where $\langle \mathbb{P}, \leq\rangle$ is a partially ordered set if $\leq$ is reflexive ($p \leq p$), transitive ($p \leq q$, $q \leq r \rightarrow p \leq r$), and antisymmetric ($p \leq q$, $q \leq p \rightarrow p = q$). For two partially ordered sets $\langle \mathbb{P}, \leq_\mathbb{P}\rangle$ and $\langle \mathbb{Q}, \leq_\mathbb{Q}\rangle$, one can define the partial order $\leq_{\mathbb{P}\times\mathbb{Q}}$ on the product $\mathbb{P} \times \mathbb{Q}$; that is, $\langle p, q\rangle \leq_{\mathbb{P}\times\mathbb{Q}} \langle p', q'\rangle$ if and only if $p \leq_\mathbb{P} p'$ and $q \leq_\mathbb{Q} q'$. This partially ordered set is denoted by $\langle \mathbb{P}, \leq_\mathbb{P}\rangle \times \langle \mathbb{Q}, \leq_\mathbb{Q}\rangle$. Similarly, we can define the product of three (and so on) partially ordered sets. Now, the two lemmas above show the following.

Proposition 1.3. *Let $\langle X, <_X\rangle$ be a linearly ordered set. Then the partially ordered set $\langle GT_X, \subseteq\rangle$ is order isomorphic to the partially ordered set $\langle \mathcal{P}(X_R), \subseteq\rangle \times \langle \mathcal{P}(X_L), \subseteq\rangle$.*

Here two partially ordered sets $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle$ are said to be *order isomorphic* if there is a 1-1 onto map $f : \mathbb{P} \to \mathbb{Q}$ such that $p \leq_{\mathbb{P}} p'$ if and only if $f(p) \leq_{\mathbb{P}} f(p')$. In the case $X = \mathbb{R}$, the structure $\langle GT_{\mathbb{R}}, \subseteq \rangle$ is order isomorphic to $\langle \mathcal{P}(\mathbb{R}), \subseteq \rangle \times \langle \mathcal{P}(\mathbb{R}), \subseteq \rangle$.

Given two linearly ordered sets L_0 and L_1, one can define an order $<_L$ on $L = L_0 \times L_1$, called the *lexicographic order*, by

$$\langle u, v \rangle <_L \langle u', v' \rangle \text{ iff } u <_{L_0} u', \text{ or } (u = u' \text{ and } v <_{L_1} v').$$

In the case $Z \subseteq L_0 \times L_1$, the restricted order $<_{L_0 \times L_1} \restriction Z$ of the lexicographic order $<_{L_0 \times L_1}$ to Z is also called the *lexicographic order* on Z and denoted by $<_Z$.

Now for a given GO-space, $X = \langle X, <_X, \tau \rangle$, let

$$X^* = \big(X^- \times \{-1\}\big) \cup \big(X \times \{0\}\big) \cup \big(X^+ \times \{1\}\big)$$

and consider the lexicographic order $<_{X^*}$ on X^* induced by the lexicographic order on $X \times \{-1, 0, 1\}$; here, of course, $-1 < 0 < 1$. We usually identify X as $X = X \times \{0\}$ in the obvious way (i.e., $x = \langle x, 0 \rangle$); thus, we may consider $X^* = \big(X^- \times \{-1\}\big) \cup X \cup \big(X^+ \times \{1\}\big)$. It is easy to verify that X^* is a linearly ordered d-extension of X. Moreover, under the trivial identification, we may consider that X^* is the smallest linearly ordered d-extension of X; that is, if L is a linearly ordered d-extension of X, then $X^* \subseteq L$ (see [5, Theorem 2.1]). Note that $(\leftarrow, x]_X = (\leftarrow, \langle x, 1 \rangle)_{X^*} \cap X \in \lambda(<_{X^*}) \restriction X$ whenever $x \in X^+$, and also its analogue. Using this fact and easy arguments, one can show the following lemma.

Lemma 1.4. *Let $X = \langle X, <_X, \tau \rangle$ be a GO-space and consider the LOTS $X^* = \langle X^*, <_{X^*}, \lambda(<_{X^*}) \rangle$ defined above. Let L be a linearly ordered compactification of X. Regarding $X^* \subseteq L$, the following hold:*

(1) *if $x \in X^+$, then $(x, \langle x, 1 \rangle)_L = \emptyset$;*
(2) *if $x \in X^-$, then $(\langle x, -1 \rangle, x)_L = \emptyset$;*
(3) *if $u \in L$, $v \in X^- \times \{-1\}$, and $u <_L v$, then $(u, v)_L \cap X \neq \emptyset$;*
(4) *if $u \in L$, $v \in X^+ \times \{1\}$, and $v <_L u$, then $(v, u)_L \cap X \neq \emptyset$;*
(5) *if $u, v \in X^* \setminus X$ and $u <_{X^*} v$, then $(u, v)_{X^*} \cap X \neq \emptyset$.*

Let $X = [0, 1) \cup (2, 3]$ and $L = [0, 1] \cup [2, 3]$ be the subspaces of $\mathbb{R}$. We may consider that X is a GO-space and L is a linearly ordered compactification of X. In (5) above, X^* cannot be replaced by L witnessed by the case $u = 1$ and $v = 2$.

2. Compact LOTS

In this section, we will present a machine from a compact LOTS making another compact LOTS.

First, let L be a LOTS. For a subset $W \subseteq L$, $L[W]$ denotes the LOTS $L \times \{0\} \cup W \times \{1\}$ with the lexicographic order $<_{L[W]}$. Also, as above, we identify $L \times \{0\}$ with L, so we may consider as $L[W] = L \cup W \times \{1\}$. Obviously, the interval topology $\lambda(<_L)$ is weaker than the subspace topology $\lambda(<_{L[W]}) \restriction L$ and, in general, not equal. Remark that L is not a subspace of $L[W]$ whenever $u \in \mathrm{Cl}_L(u, \rightarrow)_L$ for some $u \in W$ because $u \notin \mathrm{Cl}_{L[W]}(u, \rightarrow)_L$, where Cl_L denotes the closure with respect to L. Later, we use the following easy lemma.

Lemma 2.1. *Let $f : L_1 \to L_0$ be an order preserving (i.e., $u <_{L_1} v \to f(u) \leq_{L_0} f(v)$) onto map between LOTS's L_1 and L_0. Then the following hold.*

1. *If for each $y \in L_0$, $f^{-1}[\{y\}]$ has a maximum and a minimum, then f is continuous.*
2. *Let f be 2-1 (i.e., $|f^{-1}[\{y\}]| \leq 2$ for each $y \in L_0$) and $W = \{y \in L_0 : |f^{-1}[\{y\}]| = 2\}$. Then $\tilde{f} : L_1 \to L_0[W]$, defined by*
$$\tilde{f}(u) = \begin{cases} \langle f(u), 1\rangle & \textit{if } u = \max f^{-1}[\{y\}] \textit{ for some } y \in W, \\ f(u) & \textit{otherwise,} \end{cases}$$
is an order isomorphism; therefore, the LOTS L_1 can be identified with the LOTS $L_0[W]$.

To see (1), use the fact that $f^{-1}[(\leftarrow, y)_{L_0}]$ is equal to $(\leftarrow, \min f^{-1}[\{y\}])_{L_1}$ whenever $\min f^{-1}[(\leftarrow, y)_{L_0}]$ exists. The following is known.

Lemma 2.2 ([1, Problem 3.12.3(a)])**.** *Let L be a LOTS. Then the following are equivalent.*

1. *L is compact.*
2. *Every subset A of L, including $A = \emptyset$, has a least upper bound $\sup_L A$.*
3. *Every subset A of L, including $A = \emptyset$, has a greatest lower bound $\inf_L A$.*

Note that $\sup_L \emptyset = \inf_L L = \min L$ and $\sup_L L = \inf_L \emptyset = \max L$ whenever L is compact. Also note that $(\leftarrow, u)_L = \emptyset$ if and only if $u = \min L$ and, analogously, $(u, \rightarrow)_L = \emptyset$ if and only if $u = \max L$.

Now in the remainder of this section, fix a compact LOTS $L = \langle L, <_L, \lambda(<_L)\rangle$. Set

$$G(L) = \{u \in L : u = \sup_L(\leftarrow, u)_L = \inf_L(u, \rightarrow)_L\},$$

$$G^M(L) = \{u \in G(L) : (\leftarrow, u)_L \neq \emptyset, (u, \rightarrow)_L \neq \emptyset\}.$$

Note that $G^M(L) = G(L) \setminus \{\min L, \max L\}$. Note that if $W \subseteq G^M(L)$, then $\min L = \min L[W]$ and $\max L = \max L[W]$ hold.

Lemma 2.3. *Let L be a compact LOTS and $W \subseteq G^M(L)$. Then the following hold.*

(1) *The LOTS $L[W]$ is compact.*
(2) *The subspace topology $\lambda(<_L) \restriction (L \setminus W)$ on $L \setminus W$ coincides with the subspace topology $\lambda(<_{L[W]}) \restriction (L \setminus W)$.*
(3) *If $L \setminus W$ is dense in L, then it is also dense in $L[W]$.*

Proof. (1) and (2) are straightforward.

(3) Assume that $L \setminus W$ is dense in L and there is a non-empty open set U in $L[W]$ disjoint from $L \setminus W$. Pick $u \in U$. First assume $u \in L$. Then we have $u \in W \subseteq G^M(L)$. Since U is open in $L[W]$, we can pick $v \in L[W]$ with $v <_{L[W]} u$ and $(v, u]_{L[W]} \subseteq U$. When $v \in L$, by $u = \sup_L(\leftarrow, u)_L$, $(v, u)_L$ is non-empty open in L. Thus, $\emptyset \neq (v, u)_L \cap (L \setminus W) \subseteq U \cap (L \setminus W) = \emptyset$, a contradiction. When $v \in W \times \{1\}$, say $v = \langle v', 1\rangle$ for some $v' \in W$. Similarly, as above, $(v', u)_L$ is non-empty open in L; then $\emptyset \neq (v', u)_L \cap (L \setminus W) = (v, u)_{L[W]} \cap (L \setminus W) \subseteq U \cap (L \setminus W) = \emptyset$, a contradiction. Next assume $u \in W \times \{1\}$, say $u = \langle u', 1\rangle$ for some $u' \in W$. We can pick $v \in L[W]$ with $u <_{L[W]} v$ and $[u, v)_{L[W]} \subseteq U$. When $v \in L$, by $u' = \inf_L(u', \rightarrow)_L$, $(u', v)_L$ is non-empty open in L. Thus, $\emptyset \neq (u', v)_L \cap (L \setminus W) = [u, v)_{L[W]} \cap (L \setminus W) \subseteq U \cap (L \setminus W) = \emptyset$, a contradiction. When $v \in W \times \{1\}$, say $v = \langle w, 1\rangle$ for some $w \in W$. Since $u <_{L[W]} v$, we have $u' <_L w$. Similarly, as above, $(u', w)_L$ is non-empty open in L; then $\emptyset \neq (u', w)_L \cap (L \setminus W) = (u, w)_{L[W]} \cap (L \setminus W) \subseteq U \cap (L \setminus W) = \emptyset$, a contradiction. This completes the proof. □

Now we have the following.

Corollary 2.4. *Let L be a compact LOTS and $W \subseteq G^M(L)$. If X is dense in L and $X \subseteq L \setminus W$, then X is also a dense subspace of $L[W]$.*

The following lemma may clarify the structure of $L[W]$.

Lemma 2.5. *Let L be a compact LOTS and $W \subseteq G^M(L)$.*

(1) *If $u, v \in L$ and $u <_L v$ and $(u, v)_L = \emptyset$, then $(u, v)_{L[W]} = \emptyset$.*
(2) *If $u \in G(L)$, then $u = \sup_{L[W]}(\leftarrow, u)_{L[W]} = \sup_{L[W]}(\leftarrow, u)_L$.*
(3) *If $u \in G(L) \setminus W$, then $u = \inf_{L[W]}(u, \rightarrow)_{L[W]} = \inf_{L[W]}(u, \rightarrow)_L$.*

(4) *If* $u \in W$*, then* $\langle u,1\rangle = \min(u,\to)_{L[W]}$*,* $u = \max(\leftarrow,\langle u,1\rangle)_{L[W]}$*,* $u = \sup_{L[W]}(\leftarrow,u)_{L[W]} = \sup_{L[W]}(\leftarrow,u)_L$*, and* $\langle u,1\rangle = \inf_{L[W]}(\langle u,1\rangle,\to)_{L[W]} = \inf_{L[W]}(u,\to)_L$*.*

Proof. (1) Assume $(u,v)_L = \emptyset$ and $(u,v)_{L[W]} \neq \emptyset$. Then $(u,v)_{L[W]}$ is $\{\langle u,1\rangle\}$ with $u \in W \subseteq G^M(L)$. This contradicts $u = \inf_L(u,\to)_L$.

(2) Let $u \in G(L)$. As in the proof of Lemma 2.3, using $u = \sup_L(\leftarrow,u)_L$ for every $v <_{L[W]} u$, one can take $v' \in L$ with $v <_{L[W]} v' <_{L[W]} u$. Then we are done.

(3) Similar to (2).

(4) The first and second are evident. The third follows from (2). The fourth is similar to (2). □

3. The Simplest Linearly Ordered Compactification

In this section, we fix a GO-space $X = \langle X, <_X, \tau\rangle$. We will visualize the simplest linearly ordered compactification (denoted by lX) of X.

First, we present the following lemma.

Lemma 3.1. *Let L be a linearly ordered compactification of a GO-space X.*

(1) *If* $u \in L \setminus X$*, then* $u = \sup_L(\leftarrow,u)_L$ *or* $u = \inf_L(u,\to)_L$*.*
(2) *If* $u \in L$ *and* $u = \sup_L(\leftarrow,u)_L$*, then* $u = \sup_L((\leftarrow,u)_L \cap X)$*.*
(3) *If* $u \in L$ *and* $u = \inf_L(u,\to)_L$*, then* $u = \inf_L((u,\to)_L \cap X)$*.*

To prove the lemma, use the density of X.

Now we describe lX. First, let X_G denote the set of all gaps of the linearly ordered set $\langle X, <_X\rangle$; that is,

$$X_G = \{\langle A,B\rangle : \langle A,B\rangle \text{ is a gap of } X\}.$$

Note that X_G does not depend on its GO-topology τ. We may assume $X \cap X_G = \emptyset$; in fact, this is a theorem of ZFC. Let $X^* = \langle X^*, <_{X^*}, \lambda(<_{X^*})\rangle$ be the LOTS described in §1; that is,

$$X^* = (X^- \times \{-1\}) \cup X \cup (X^+ \times \{1\})$$

with the lexicographic order $<_{X^*}$ under the identification $X = X \times \{0\}$. Our lX is

$$lX = X^* \cup X_G$$

with the order $<_{lX}$, where for $u,v \in lX$, $u <_{lX} v$ is defined by

$$\begin{cases} \bullet\ u,v \in X^* \text{ and } u <_{X^*} v, \\ \bullet\ u = \langle A,B\rangle \in X_G, v = \langle x,i\rangle \in X^* \text{ and } x \in B, \\ \bullet\ u = \langle x,i\rangle \in X^*, v = \langle A,B\rangle \in X_G \text{ and } x \in A, \\ \bullet\ u = \langle A,B\rangle, v = \langle C,D\rangle \in X_G \text{ and } A \subsetneq C, \end{cases}$$

where $\langle x, 0\rangle$ is identified with x. Obviously, $<_{lX}$ extends $<_{X^*}$; therefore, it also extends $<_X$. Also note that if X has no minimum (maximum), then $\langle\emptyset, X\rangle \in X_G$ ($\langle X, \emptyset\rangle \in X_G$) and it is $\min lX$ ($\max lX$).

Define $f : X^* \cup (X^*)_G \to lX$, where $(X^*)_G$ is the set of all gaps in X^*, by

$$f(u) = \begin{cases} u & \text{if } u \in X^* \\ \langle H \cap X, K \cap X\rangle & \text{if } u = \langle H, K\rangle \in (X^*)_G \,. \end{cases}$$

By the density of X in X^*, f is well defined and an order isomorphism with $f \restriction X = 1_X$. Since $X^* \cup (X^*)_G$ is a linearly ordered compactification of X^*, lX is also a linearly ordered compactification of X. We show the following lemma.

Lemma 3.2. *Let X be a GO-space. Then lX is a linearly ordered compactification of X such that $(u, v)_{lX} \neq \emptyset$ for every $u, v \in lX \setminus X$ with $u <_{lX} v$.*

Proof. Let $u, v \in lX \setminus X$ with $u <_{lX} v$. The case $u, v \in X^* \setminus X$ follows from Lemma 1.4(5), so we may assume $u \in lX \setminus X^* = X_G$, say $u = \langle A, B\rangle$. Let us assume $v \in X^*$, say $v = \langle x, i\rangle$. It follows from $u <_{lX} v$ that $x \in B$. Since B has no min, take $x' \in B$ with $x' <_X x$. Then $u <_{lX} x' <_{lX} v$. Next assume $v \in lX \setminus X^*$, say $v = \langle C, D\rangle$. Then $A \subsetneq C$, so taking $x' \in C \setminus A$, we have $u <_{lX} x' <_{lX} v$. □

4. The Structure of Linearly Ordered Compactifications

We fix a linearly ordered set $\langle X, <_X\rangle$. In this section, from the need to distinguish between the topologies τ's on $\langle X, <_X\rangle$, we use the terminology X_τ for expressing the GO-space $\langle X, <_X, \tau\rangle$.

Definition 4.1. $\mathbb{L}_X$ denotes the class of all linearly ordered compactifications of GO-spaces whose underlying linearly ordered set is $\langle X, <_X\rangle$. Also for a GO-space $X_\tau = \langle X, <_X, \tau\rangle$, $\mathcal{L}_{X_\tau}$ denotes the class of all linearly ordered compactifications of X_τ. Note that $\mathbb{L}_X = \bigcup_{\tau \in GT_X} \mathcal{L}_{X_\tau}$, where GT_X is the set of all GO-topologies on $\langle X, <_X\rangle$; see §1.

For $L_0, L_1 \in \mathbb{L}_X$, define $L_0 \leq L_1$ if there is a continuous map $f : L_1 \to L_0$ such that $f \restriction X = 1_X$. Obviously, the order $\leq$ is reflexive and transitive.

First we check the following lemma.

Lemma 4.2. *Let $L_0, L_1 \in \mathbb{L}_X$ and assume that there is a map $f : L_1 \to L_0$ such that $f \restriction X = 1_X$. Then the following are equivalent:*

(1) *f is continuous,*
(2) *f is* 3-1*, order preserving, and onto.*

Proof. (2) $\to$ (1) This follows from Lemma 2.1(1).

(1) $\to$ (2) Assume that f is continuous. Since $X = f[X] \subseteq f[L_1]$ and X is dense in L_0, we have $f[L_1] = L_0$.

Claim 1. f is order preserving.

Proof. Assume $u <_{L_1} u'$ and $f(u') <_{L_0} f(u)$. We will derive a contradiction. Since L_0 is a T_2 GO-space, there are disjoint convex open sets U and U' in L_0 with $f(u) \in U$ and $f(u') \in U'$. Because of the continuity of f, one can take convex open sets V and V' in L_1 with $u \in V$ and $u' \in V'$ and $f[V] \subseteq U$ and $f[V'] \subseteq U'$. Then, obviously, $V \cap V' = \emptyset$. Since X is dense in L_1, one can take $x \in V \cap X$ and $x' \in V' \cap X$. Then by $u <_{L_1} u'$ and the convexity of V and V', we have $x <_X x'$. By $f(u') <_{L_0} f(u)$, the convexity of U and U' and $f(x) \in U$ and $f(x') \in U'$, we have $x' = f(x') <_{L_0} f(x) = x$, a contradiction.

Claim 2. If $u <_{L_1} u'$, $f(u) = f(u')$, and $(u, u')_{L_1} \neq \emptyset$, then $(u, u')_{L_1} = \{x\}$ for some $x \in X$.

Proof. Assuming $u <_{L_1} u'$, $f(u) = f(u')$, and $(u, u')_{L_1} \neq \emptyset$, take x in $(u, u')_{L_1} \cap X$. If $(u, x)_{L_1} \neq \emptyset$ were true, then by taking $x' \in (u, x)_{L_1} \cap X$, we have $f(u) \leq f(x') \leq f(x) \leq f(u')$; thus, $x = f(x) = f(x') = x'$, a contradiction. So we have $(u, x)_{L_1} = \emptyset$; similarly, $(x, u)_{L_1} = \emptyset$.

Claim 3. f is 3-1.

Proof. Assume $u_0 <_{L_1} u_1 <_{L_1} u_2 <_{L_1} u_3$ and $f(u_0) = f(u_1) = f(u_2) = f(u_3)$. It follows from $(u_0, u_2) \neq \emptyset$ and Claim 2 that $(u_0, u_2) = \{u_1\}$ and $u_1 \in X$. Similarly, we have $(u_1, u_3) = \{u_2\}$ and $u_2 \in X$. Now we have $f(u_1) = u_1 < u_2 = f(u_2)$, a contradiction.

This concludes the proof of the lemma. $\square$

Lemma 4.3. *Let $L_0, L_1 \in \mathbb{L}_X$, and for each $i \in 2$, let L_i be a linearly ordered compactification of $X_{\tau_i} = \langle X, <_X, \tau_i\rangle$. Assume that there is a continuous map $f : L_1 \to L_0$ such that $f \restriction X = 1_X$. The following are equivalent:*

(1) *f is* 2-1*,*
(2) $X^+_{\tau_1} \cap X^-_{\tau_1} \subseteq X^+_{\tau_0} \cup X^-_{\tau_0}$.

Proof. (1) $\to$ (2) Assume that there is x in $(X^+_{\tau_1} \cap X^-_{\tau_1}) \setminus (X^+_{\tau_0} \cup X^-_{\tau_0})$. It suffices to see the following.

Claim. $f(\langle x, 1\rangle) = f(\langle x, -1\rangle) = x$.

Proof. It follows from $x < \langle x, 1\rangle \in X^+_{\tau_1} \times \{1\} \subset X^*_{\tau_1}$ that $x = f(x) \leq f(\langle x, 1\rangle)$. If $x < f(\langle x, 1\rangle)$ were true, then using the density of X in L_0, we see $(x, f(\langle x, 1\rangle))_{L_0} = \emptyset$; thus, $(\leftarrow, x]_X \in \tau_0$. On the other hand, by $x \in X^+_{\tau_1}$, $(\leftarrow, x]_X \notin \lambda(<_X)$ holds. Therefore, we have $x \in X^+_{\tau_0}$, a contradiction. So we have $x = f(\langle x, 1\rangle)$; $x = f(\langle x, -1\rangle)$ is similar.

(2) $\to$ (1) Assuming that f is not 2-1, pick $u_0, u_1, u_2 \in L_1$ such that $u_0 <_{L_1} u_1 <_{L_1} u_2$ and $f(u_0) = f(u_1) = f(u_2)$. As in Lemma 4.2(2), we have $(u_0, u_2)_{L_1} = \{u_1\}$ and $u_1 \in X$. By $f \restriction X = 1_X$, we also have $u_0, u_2 \notin X$. That $(\leftarrow, u_1]_X \in \tau_1$ and $[u_1, \to)_X \in \tau_1$ is obvious. By $u_2 \in (u_1, \to)_{L_1}$ and the density of X, we have $(u_1, \to)_X \neq \emptyset$. If $(\leftarrow, u_1]_X \in \lambda(<_X)$ were true, then there is $x \in X$ such that $u_1 <_X x$ and $(u_1, x)_X = \emptyset$. By $u_2 \notin X$ and $(u_1, u_2)_{L_1} = \emptyset$, we have $u_2 <_X x$; thus, $(u_1, x)_{L_1} \neq \emptyset$, a contradiction. Therefore, $(\leftarrow, u_1]_X \notin \lambda(<_X)$ holds; similarly, we have $[u_1, \to)_X \notin \lambda(<_X)$. Now we see $u_1 \in X^+_{\tau_1} \cap X^-_{\tau_1}$. If $u_1 \in X^+_{\tau_0}$ were true, then by $u_1 < \langle u_1, 1\rangle \in X^+_{\tau_0} \times \{1\} \subset X^*_{\tau_0}$ and $(u_1, \langle u_1, 1\rangle)_{L_0} = \emptyset$, we have $f(u_2) = u_1 \in (\leftarrow, \langle u_1, 1\rangle)_{L_0}$. By continuity of f, there is an open neighborhood V of u_2 in L_1 such that $f[V] \subset (\leftarrow, \langle u_1, 1\rangle)_{L_0}$. We may assume $V \subset (u_1, \to)_{L_1}$. Pick $x \in V \cap X$, then $u_2 <_{L_1} x$ and $x = f(x) \leq_{L_0} u_1 <_X x$, a contradiction. Thus, we have $u_1 \notin X^+_{\tau_0}$; similarly, we have $u_1 \notin X^-_{\tau_0}$. □

Applying Lemma 4.3 to $\tau = \tau_0 = \tau_1$, we see the following corollary.

Corollary 4.4. *Let $L_0, L_1 \in \mathcal{L}_{X_\tau}$ for some $\tau \in GT_X$. If there is a continuous map $f : L_1 \to L_0$ such that $f \restriction X = 1_X$, then f is* 2-1*,*

Lemma 4.5. *Let $L_0, L_1 \in \mathbb{L}_X$. Then the following are equivalent:*

(1) *$L_0 \leq L_1$ and $L_1 \leq L_0$;*
(2) *there is a* 1-1 *continuous map $f : L_1 \to L_0$ such that $f \restriction X = 1_X$;*
(3) *there is an order isomorphism $f : L_1 \to L_0$ such that $f \restriction X = 1_X$.*

Proof. (3) $\to$ (1) This follows from the fact that an order isomorphism between LOTS's is a homeomorphism.

(1) $\to$ (2) Let $f : L_1 \to L_0$ and $g : L_0 \to L_1$ be continuous maps with $f \restriction X = 1_X$ and $g \restriction X = 1_X$. Then the combination $g \circ f$ has to be 1_{L_1}; therefore, f is 1-1.

(2) $\to$ (3) Let $f : L_1 \to L_0$ be a 1-1 continuous map with $f \restriction X = 1_X$. It follows from Lemma 4.2 that f is 1-1, order preserving, and onto, which means f is an order isomorphism. □

Note that if $L_0, L_1 \in \mathbb{L}_X$ with $L_0 \leq L_1$ and $L_1 \leq L_0$, then $L_0, L_1 \in \mathcal{L}_{X_\tau}$ for some $\tau \in GT_X$. If one of the equivalents in Lemma 4.5 is satisfied, then we identify L_0 with L_1. Under this identification, we will investigate the structure of the partially ordered sets $\langle \mathbb{L}_X, \leq\rangle$ and $\langle \mathcal{L}_{X_\tau}, \leq\rangle$. Remember that X_G is the set of all gaps of X and $lX_\tau = X^*_\tau \cup X_G$ (in §3, apply $X = X_\tau$), where $X_\tau = \langle X, <_X, \tau\rangle$. Now let X^M_G denote the set of all middle gaps of X; that is,

$$X^M_G = \{\langle A, B\rangle : \langle A, B\rangle \text{ is a middle gap of } X\}.$$

Then $|X_G \setminus X_G^M| \leq 2$, and note that X_G and X_G^M only depend on the linearly ordered set $\langle X, <_X\rangle$. Also remember the definitions of $G(L)$ and $G^M(L)$ for a compact LOTS L in §2; now we apply the results in §2 for $L = lX_\tau$.

Lemma 4.6. *$X_G^M \subseteq G^M(lX_\tau)$ and $X_G \subseteq G(lX_\tau)$ hold.*

Proof. Let $u \in X_G^M$, say $u = \langle A, B\rangle$. Because $A \neq \emptyset$ and $B \neq \emptyset$, we have $(\leftarrow, u)_{lX_\tau} \neq \emptyset$ and $(u, \rightarrow)_{lX_\tau} \neq \emptyset$. Assume $v = \sup_{lX_\tau}(\leftarrow, u)_{lX_\tau} <_{lX_\tau} u$. First assume $v \in X$. Since $v \in A$ and A has no maximum, we can take $x \in A$ with $v <_X x <_{lX_\tau} u$; this contradicts the definition of v. Next assume $v \notin X$. It follows from Lemma 3.2 that $(v, u)_{lX_\tau} \neq \emptyset$, also contradicting the definition of v. Therefore, we have $\sup_{lX_\tau}(\leftarrow, u)_{lX_\tau} = u$. Similarly, we have $\inf_{lX_\tau}(u, \rightarrow)_{lX_\tau} = u$. Now $X_G \subseteq G(lX_\tau)$ is obvious. □

Now for every $W \subseteq X_G^M$, using the notation in §2, we let

$$l_W X_\tau = (lX_\tau)[W].$$

Then $lX_\tau = l_\emptyset X_\tau$. We also let

$$LX_\tau = l_{X_G^M} X_\tau.$$

Later we will see that lX_τ is the minimum and LX_τ is the maximum in $\langle \mathcal{L}_{X_\tau}, \leq\rangle$ and that $lX_{\lambda(<_X)}$ is the minimum and $LX_{\tau(X_R, X_L)}$ is the maximum in $\langle \mathbb{L}_X, \leq\rangle$.

Lemma 4.7. *If $\tau \in GT_X$, then $\mathcal{L}_{X_\tau} = \{l_W X_\tau : W \subseteq X_G^M\}$.*

Proof. The inclusion $\supseteq$ follows from Lemma 4.6 and Corollary 2.4.

To see the inclusion $\subseteq$, let $L \in \mathcal{L}_{X_\tau}$. Define $f : L \to lX_\tau$ by

$$f(u) = \begin{cases} \langle \{x \in X : x <_L u\}, \{x \in X : u <_L x\}\rangle & \text{if } u \in L \setminus X_\tau^*, \\ u & \text{otherwise.} \end{cases}$$

The following claim shows that f is well defined and onto.

Claim 1. $f[L \setminus X_\tau^*] = X_G$.

Proof. To see the inclusion $\subseteq$, let $u \in L \setminus X_\tau^*$, $A = \{x \in X : x <_L u\}$, and $B = \{x \in X : u <_L x\}$. Assume that A has the maximal element x_0; then, by the density of X, $(x_0, u)_L = \emptyset$ holds. If $x_0 \in X_\tau^+$ were true, then we have $u = \langle x_0, 1\rangle \in X_\tau^+ \times \{1\} \subseteq X_\tau^*$ (see Lemma 1.4(1)), a contradiction. Thus, we have $x_0 \notin X_\tau^+$. Because $(\leftarrow, x_0]_X = A \in \tau$, we have $(\leftarrow, x_0]_X \in \lambda(<_X)$. Since $(x_0, \rightarrow)_L \neq \emptyset$ holds (u witnesses this), we have $(x_0, \rightarrow)_X \neq \emptyset$. Thus, there is $z \in X$ with $z >_X x$ and $(x_0, z)_X = \emptyset$. It follows from $(x_0, u)_L = \emptyset$, $u \notin X$, and $z \in X$ that $u <_L z$; therefore, $(x_0, z)_L \neq \emptyset$, and hence $(x_0, z)_X \neq \emptyset$, a contradiction. We have shown that A has no maximum; similarly, B has no minimum. This means $f(u) \in X_G$.

To see the inclusion $\supseteq$, let $w \in X_G$, say $w = \langle A, B \rangle$. Putting $u = \sup_L A$, we see $f(u) = w$.

Claim 2. f is order preserving.

Proof. Let $u, v \in L$ with $u <_L v$. We will see $f(u) \leq_{lX_\tau} f(v)$. By $X^*_\tau \subseteq L$, we may assume $u \notin X^*_\tau$ or $v \notin X^*_\tau$. But in the case $u \notin X^*_\tau$ and $v \notin X^*_\tau$, it is obvious by the definition of f and the claim above. We consider the case $u \notin X^*_\tau$ and $v \in X^*_\tau$. When $v \in X$, by $v \in \{x \in X : u <_L x\}$, we see $f(u) <_{lX} v = f(v)$. When $v = \langle x, 1 \rangle$ for some $x \in X^+_\tau$, we have $u <_L x$; see Lemma 1.4(1). Now we have $f(u) <_{lX_\tau} x <_{lX_\tau} v = f(v)$. When $v = \langle x, -1 \rangle$ for some $x \in X^-_\tau$, by Lemma 1.4(2) and (3), we can take $z \in (u, v)_L \cap X$. Then $f(u) <_{lX_\tau} z <_{lX_\tau} v = f(v)$. The case $u \in X^*_\tau$ and $v \notin X^*_\tau$ is similar.

Claim 3. f is 2-1.

Proof. Because $f \restriction X^*_\tau = 1_{X^*_\tau}$, $f[L \setminus X^*_\tau] = X_G$, and $X^*_\tau \cap X_G = \emptyset$, it suffices to see that $f \restriction (L \setminus X^*_\tau)$ is 2-1. So assume that for some $u_0, u_1, u_2 \in L \setminus X^*_\tau$ with $u_0 < u_1 < u_2$, $f(u_0) = f(u_1) = f(u_2)$ holds. Applying the density of X to $(u_0, u_2)_L$, we can take $x \in (u_0, u_2)_L \cap X$. Then by $u_0 < x < u_2$, we have $f(u_0) < x < f(u_1)$, a contradiction.

Now let $W = \{w \in X_G : |f^{-1}[\{w\}]| = 2\}$. We have the following.

Claim 4. $W \subseteq X^M_G$.

Proof. Let $w \in W$ and we fix $u_0, u_1 \in L \setminus X^*_\tau$ with $u_0 < u_1$ and $w = f(u_0) = f(u_1)$. If $(u_0, u_1)_L \neq \emptyset$ were true, then by taking $x \in (u_0, u_1)_L \cap X$, we have $f(u_0) < x < f(u_1)$ as above, a contradiction. Thus, we have $(u_0, u_1)_L = \emptyset$. By $(\leftarrow, u_1)_L \neq \emptyset$, take $x \in (\leftarrow, u_1)_L \cap X$. Then we have $x < u_0$ for some $x \in X$. Moreover, by $(u_0, \rightarrow)_L \neq \emptyset$, we have $u_0 < y$ for some $y \in X$. This means $w = f(u_0) \in X^M_G$.

Now by Lemma 2.1(2), $\tilde{f} : L \to (lX_\tau)[W] = l_W X_\tau$ is an order isomorphism with $\tilde{f} \restriction X = 1_X$. By Lemma 4.5, we have $L = l_W X_\tau$. And this concludes the proof. $\square$

Lemma 4.8. *If for each $i \in 2$, we let $X_{\tau_i} = \langle X, <_X, \tau_i \rangle$ be a GO-space and $W_i \subseteq X^M_G$, then the following are equivalent:*

(1) $l_{W_1} X_{\tau_1} \geq l_{W_0} X_{\tau_0}$;
(2) $\tau_1 \supseteq \tau_0$ *and* $W_1 \supseteq W_0$.

Proof. Note that $\tau_1 \supseteq \tau_0$ is equivalent to both $X^+_{\tau_1} \supseteq X^+_{\tau_0}$ and $X^-_{\tau_1} \supseteq X^-_{\tau_0}$ (see Proposition 1.3).

(2) $\to$ (1) Let $\tau_1 \supseteq \tau_0$ and $W_1 \supseteq W_0$ and define $f : l_{W_1}X_{\tau_1} \to l_{W_0}X_{\tau_0}$ by

$$f(u) = \begin{cases} x & \text{if } u = \langle x, 1\rangle \text{ for some } x \in X^+_{\tau_1} \setminus X^+_{\tau_0}, \\ x & \text{if } u = \langle x, -1\rangle \text{ for some } x \in X^-_{\tau_1} \setminus X^-_{\tau_0}, \\ c & \text{if } u = \langle c, 1\rangle \text{ for some } c \in W_1 \setminus W_0, \\ u & \text{otherwise.} \end{cases}$$

Obviously, f is 3-1, order preserving, and onto with $f \restriction X = 1_X$. By Lemma 4.2, we have $l_{W_1}X_{\tau_1} \geq l_{W_0}X_{\tau_0}$.

(1) $\to$ (2) Let $f : l_{W_1}X_{\tau_1} \to l_{W_0}X_{\tau_0}$ be a continuous map with $f \restriction X = 1_X$. Since 1_X is a continuous map from X_{τ_1} to X_{τ_0}, we have $\tau_1 \supseteq \tau_0$. It suffices to see $W_1 \supseteq W_0$. So let $c \in W_0$ and say $c = \langle A, B\rangle$, where $\langle A, B\rangle$ is a gap of X with $A \neq \emptyset$ and $B \neq \emptyset$. Since f is onto and $\langle c, 1\rangle \in W_0 \times \{1\} \subseteq l_{W_0}X_{\tau_0}$, there is $u \in l_{W_1}X_{\tau_1}$ with $f(u) = \langle c, 1\rangle$. It follows from $\langle c, 1\rangle \notin X$ that $u \notin X$.

Claim 1. $u \notin X^*_{\tau_1}$.

Proof. Assume $u \in X^*_{\tau_1}$. By $u \notin X$, we have $u \in X^+_{\tau_1} \times \{1\} \cup X^-_{\tau_1} \times \{-1\}$.

First, we consider the case $u \in X^+_{\tau_1} \times \{1\}$, say $u = \langle x, 1\rangle$ for some $x \in X^+_{\tau_1}$. When $x \in A$, take $z \in A$ with $x <_X z$. Then by $u <_{l_{W_1}X_{\tau_1}} z$ (see Lemma 1.4(1)), we have $f(u) \leq f(z) = z < c < \langle c, 1\rangle = f(u)$, a contradiction. When $x \in B$, take $z \in B$ with $z <_X x$. Then by $z <_{l_{W_1}X_{\tau_1}} u$, we have $f(u) = \langle c, 1\rangle < z = f(z) \leq f(u)$, a contradiction.

Next, we consider the case $u \in X^-_{\tau_1} \times \{-1\}$, say $u = \langle x, -1\rangle$ for some $x \in X^-_{\tau_1}$. When $x \in A$, by $u < x$, we have $f(u) \leq f(x) = x < c < \langle c, 1\rangle = f(u)$, a contradiction. When $x \in B$, take $z \in B$ with $z <_X x$. Then by $z <_{l_{W_1}X_{\tau_1}} u$, we have $z = f(z) \leq f(u) = \langle c, 1\rangle < z$, a contradiction.

Claim 2. $u \notin X_G$.

Proof. Assume $u \in X_G$, say $u = \langle C, D\rangle$. If $c < u$ were true, then by taking $x \in C \setminus A$, we have $c < x < u$. Therefore, we have $f(u) = \langle c, 1\rangle < x = f(x) \leq f(u)$, a contradiction. If $u < c$ were true, then by taking $x \in A \setminus C$, we have $u < x < c$. Therefore, we have $\langle c, 1\rangle = f(u) \leq f(x) = x < c < \langle c, 1\rangle$, a contradiction. Thus, $u = c$ holds. Since f is order preserving, continuous, and $f(c) = \langle c, 1\rangle$, there is $v \in l_{W_1}X_{\tau_1}$ such that $v <_{l_{W_1}X_{\tau_1}} c$ and $f[(v, \to)_{l_{W_1}X_{\tau_1}}] \subseteq (c, \to)_{l_{W_0}X_{\tau_0}}$. Since c is a gap and $v < c$, we have $(v, c)_{l_{W_1}X_{\tau_1}} \neq \emptyset$. Take $x \in (v, c)_{l_{W_1}X_{\tau_1}} \cap X$; then we have $f(x) = \langle c, 1\rangle$, a contradiction.

By the claims above and $l_{W_1}X_{\tau_1} = (X^*_{\tau_1} \cup X_G) \cup W_1 \times \{1\}$, we see $u \in W_1 \times \{1\}$, say $u = \langle c', 1\rangle$ with $c' = \langle A', B'\rangle$ for some $c' \in W_1$. The following claim completes the proof.

Claim 3. $c = c'$.

Proof. If $A \subsetneq A'$ were true, then by taking $x \in A' \setminus A$, we have $c < x < c' < \langle c', 1\rangle = u$ in $l_{W_1} X_{\tau_1}$. Now we have $f(u) = \langle c, 1\rangle < x = f(x) \leq f(u)$, a contradiction. If $A' \subsetneq A$ were true, then by taking $x \in A \setminus A'$, we have $c' < x < c$. By $u = \langle c', 1\rangle < x$, we have $f(u) \leq f(x) = x < c < \langle c, 1\rangle = f(u)$, a contradiction. Thus, we see $u = u'$. □

Now we have the following theorem.

Theorem 4.9. *Let $\langle X <_X\rangle$ be a linearly ordered set. Then the following hold:*

(1) *The partially ordered set $\langle \mathbb{L}_X, \leq\rangle$ is order isomorphic to*
$$\langle \mathcal{P}(X_R), \subseteq\rangle \times \langle \mathcal{P}(X_L), \subseteq\rangle \times \langle \mathcal{P}(X_G^M), \subseteq\rangle;$$
therefore, $lX_{\lambda(<_X)}$ is the minimum and $LX_{\tau(X_R,X_L)}$ is the maximum in $\langle \mathbb{L}_X, \leq\rangle$.

(2) *For each $\tau \in GT_X$, the partially ordered set $\langle \mathcal{L}_{X_\tau}, \leq\rangle$ is order isomorphic to*
$$\langle \mathcal{P}(X_G^M), \subseteq\rangle;$$
thus, lX_τ is the minimum and LX_τ is the maximum in $\langle \mathcal{L}_{X_\tau}, \leq\rangle$.

From (2), we see that the structure of $\langle \mathcal{L}_{X_\tau}, \leq\rangle$ does not depend on its topology τ.

Example 4.10. Let $X = \mathbb{R}$ be the LOTS; then $X_R = X_L = \mathbb{R}$ and $X_G^M = \emptyset$. Therefore, $\langle \mathbb{L}_\mathbb{R}, \leq\rangle$ is order isomorphic to $\langle \mathcal{P}(\mathbb{R}), \subseteq\rangle \times \langle \mathcal{P}(\mathbb{R}), \subseteq\rangle$. Since $X_G^M = \emptyset$, each of $\mathbb{R}$, $\mathbb{S}$, and $\mathbb{M}$ has the unique linearly ordered compactification $\mathbb{R} \cup \{-\infty, \infty\}$, $(\mathbb{R} \cup \{-\infty, \infty\}) \cup \mathbb{R} \times \{1\}$, and $(\mathbb{R} \cup \{-\infty, \infty\}) \cup \mathbb{P} \times \{-1, 1\}$, respectively, where $-\infty = \langle \emptyset, \mathbb{R}\rangle$ and $\infty = \langle \mathbb{R}, \emptyset\rangle$ are the end gaps. The minimum in $\langle \mathbb{L}_\mathbb{R}, \leq\rangle$ is $\mathbb{R} \cup \{-\infty, \infty\}$ and the maximum in $\langle \mathbb{L}_\mathbb{R}, \leq\rangle$ is $(\mathbb{R} \times \{-1, 0, 1\}) \cup \{-\infty, \infty\}$, where $\mathbb{R}$ is identified with $\mathbb{R} \times \{0\}$.

Example 4.11. Let $X = \mathbb{Q}$ be the LOTS. Then $X_R = X_L = \mathbb{Q}$. For every middle gap $\langle A, B\rangle$ of $\mathbb{Q}$, assign $\sup_\mathbb{R} A \in \mathbb{P}$. Using this assignment, we may consider $X_G = \mathbb{P} \cup \{-\infty, \infty\}$ and $X_G^M = \mathbb{P}$, where $-\infty$ and ∞ are the end gaps of $\mathbb{Q}$. So $\langle \mathbb{L}_\mathbb{Q}, \leq\rangle$ is order isomorphic to $\langle \mathcal{P}(\mathbb{Q}), \subseteq\rangle \times \langle \mathcal{P}(\mathbb{Q}), \subseteq\rangle \times \langle \mathcal{P}(\mathbb{P}), \subseteq\rangle$. Then $l\mathbb{Q} = l_\emptyset \mathbb{Q} = \mathbb{Q} \cup \mathbb{P} \cup \{-\infty, \infty\}$, which is identified with $\mathbb{R} \cup \{-\infty, \infty\}$, is the minimum in $\langle \mathbb{L}_\mathbb{Q}, \leq\rangle$; $l_\mathbb{P} \mathbb{Q}_{\tau(\mathbb{Q},\mathbb{Q})} = (\mathbb{R} \cup \{-\infty, \infty\} \cup \mathbb{Q} \times \{-1, 1\}) \cup \mathbb{P} \times \{1\}$ is the maximum in $\langle \mathbb{L}_\mathbb{Q}, \leq\rangle$.

Analogously, $\langle \mathbb{L}_\mathbb{P}, \leq\rangle$ is order isomorphic to $\langle \mathcal{P}(\mathbb{P}), \subseteq\rangle \times \langle \mathcal{P}(\mathbb{P}), \subseteq\rangle \times \langle \mathcal{P}(\mathbb{Q}), \subseteq\rangle$.

Example 4.12. Let X_τ be the GO-space $(0, 1) \cup (1, 2) \cup [3, 4) \cup (5, 6]$ with the usual order and the subspace topology τ in $\mathbb{R}$. It has one end gap $0 = \langle \emptyset, X\rangle$. There are two middle gaps $c_0 = \langle (0, 1), (1, 2) \cup [3, 4) \cup (5, 6]\rangle$

and $c_1 = \langle (0,1)\cup(1,2)\cup[3,4),(5,6]\rangle$. Thus, $X_\tau^+ = \emptyset$ and $X_\tau^- = \{3\}$, $X_G = \{0, c_0, c_1\}$ and $X_G^M = \{c_0, c_1\}$. So there are $2^2 = 4$ linearly ordered compactifications of X_τ. With appropriate identifications,

$$lX_\tau = [0,1) \cup (1,2) \cup [3,4) \cup (5,6]) \cup \{\langle 3, -1\rangle\} \cup \{c_0, c_1\}.$$

Identifying $2 = \langle 3, -1\rangle$,

$$\begin{aligned} lX_\tau &= [0,1) \cup \{c_0\} \cup (1,2] \cup [3,4) \cup \{c_1\} \cup (5,6], \\ l_{\{c_0\}}X_\tau &= [0,1) \cup \{c_0, \langle c_0, 1\rangle\} \cup (1,2] \cup [3,4) \cup \{c_1\} \cup (5,6], \\ l_{\{c_1\}}X_\tau &= [0,1) \cup \{c_0\} \cup (1,2] \cup [3,4) \cup \{c_1, \langle c_1, 1\rangle\} \cup (5,6], \\ LX_\tau &= [0,1) \cup \{c_0, \langle c_0, 1\rangle\} \cup (1,2] \cup [3,4) \cup \{c_1, \langle c_1, 1\rangle\} \cup (5,6]. \end{aligned}$$

Moreover, by identifying $c_0 = 1$, $[0,1) \cup \{c_0\} \cup (1,2]$ can be identified with $[0,2]$. Also identifying $c_1 = 4$ and $(5,6] = (4,5]$, $[3,4) \cup \{c_1\} \cup (5,6]$ can be identified with $[3,5]$. Thus, topologically, lX_τ can be considered as $[0,2] \cup [3,5]$. Similarly, we can identify $l_{\{c_0\}}X_\tau = [0,2] \cup [3,5] \cup \{\langle 1,1\rangle\}$, $l_{\{c_1\}}X_\tau = [0,2] \cup [3,5] \cup \{\langle 4,1\rangle\}$, and $l_{\{c_0,c_1\}}X_\tau = [0,2] \cup [3,5] \cup \{\langle 1,1\rangle, \langle 4,1\rangle\}$. Note that $l_{\{c_0\}}X_\tau$ and $l_{\{c_1\}}X_\tau$ are homeomorphic, but are different as linearly ordered compactifications.

Example 4.13. Let $X = (0,1) \cup (1,2) \cup [3,4) \cup (5,6]$ and let $<_X$ be the restriction of the usual order on $\mathbb{R}$, that is, the underlying linearly ordered set of the previous example, so $X_G^M = \{c_0, c_1\}$. Then $\langle \mathbb{L}_X, \leq\rangle$ is order isomorphic to $\langle \mathcal{P}((0,1) \cup (1,2) \cup [3,4) \cup (5,6)), \subseteq\rangle \times \langle \mathcal{P}((0,1) \cup (1,2) \cup [3,4) \cup (5,6]), \subseteq\rangle \times \langle \mathcal{P}(\{c_0, c_1\}), \subseteq\rangle$. The minimum in $\langle \mathbb{L}_X, \leq\rangle$ is $[0,1) \cup \{c_0\} \cup (1,2) \cup [3,4) \cup \{c_1\} \cup (5,6]$, and the maximum in $\langle \mathbb{L}_X, \leq\rangle$ is $(\{\langle 0,0\rangle\} \cup (0,1) \times \{-1,0,1\}) \cup \{c_0, \langle c_0, 1\rangle\} \cup ((1,2)) \times \{-1,0,1\}) \cup ([3,4)) \times \{-1,0,1\}) \cup \{c_1, \langle c_1, 1\rangle\} \cup ((5,6)) \times \{-1,0,1\} \cup \{\langle 6,-1\rangle, \langle 6,0\rangle\})$.

Example 4.14. Let X_τ be a subspace of an ordinal α with the usual order and the subspace topology τ. Taking a large enough ordinal, we may assume α is a successor ordinal, so it is compact. Since the order is a well-order, there are no middle gaps of X_τ, but ∞ can exist. So $X_G^M = \emptyset$; thus, X_τ has the unique linearly ordered compactification. The closure $\mathrm{Cl}_\alpha X_\tau$ of X_τ in α is such a unique one.

Example 4.15. Let $X = \beta$ be an ordinal. Since $X_L = \mathrm{Lim}(\beta)$ and $X_R = X_G^M = \emptyset$, $\langle \mathbb{L}_\beta, \leq\rangle$ is order isomorphic to $\langle \mathcal{P}(\mathrm{Lim}(\beta)), \subseteq\rangle$, where $\mathrm{Lim}(\beta)$ denotes all the limit ordinals in β. Note that if X_τ is as in the previous example, then by enumerating $X_\tau = \{x(\gamma) : \gamma < \beta\}$ with the increasing order for some β, we may consider that the underlying linearly ordered set of X_τ is β.

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Department of Mathematics; Oita University; Oita, 870-1192 Japan
E-mail address: nkemoto@cc.oita-u.ac.jp