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A GENERALIZED DEFINITION OF TOPOLOGICAL ENTROPY

LOUIS BLOCK, JAMES KEESLING, AND LENKA RUCKA

ABSTRACT. Given an arbitrary (not necessarily continuous) function of a topological space to itself, we associate a non-negative extended real number which we call the continuity entropy of the function. In the case where the space is compact and the function is continuous, the continuity entropy of the map is equal to the usual topological entropy of the map. We show that some of the standard properties of topological entropy hold for continuity entropy, but some do not. We show that for piecewise continuous piecewise monotone maps of the interval the continuity entropy agrees with the entropy defined in *Horseshoes and entropy for piecewise continuous piecewise monotone maps* by Michał Misiurewicz and Krystina Ziemian Finally, we show that if f is a continuous map of the interval to itself and g is any function of the interval to itself which agrees with f at all but countably many points, then the continuity entropies of f and g are equal.

1. INTRODUCTION

Topological entropy has become a useful tool for recognizing, quantifying, and classifying the complicated dynamics of continuous maps. Topological entropy was first defined in [1] for a continuous map of a compact topological space to itself. In [9] and [10] an alternate definition was given in the case of a uniformly continuous map of a metric space to itself, and it was shown that this alternate definition coincides with the definition given in [1] in the case of a continuous map of a compact metric space to itself. Another idea which has been explored is to define

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the topological entropy by axioms, as in [3]. With the growing popularity and importance of this tool, many researchers used some type of topological entropy to study the dynamics of maps with discontinuities, especially maps of the interval (see [4], [5], [6], [14], [21], and [22], for example). Also, various definitions of topological entropy for continuous maps of non-compact topological spaces have been investigated (see [12], [13], [16], [17], [19], and [20]).

In this paper we define a type of entropy for an arbitrary function (not necessarily continuous) from a topological space to itself. We call this entropy the continuity entropy of f and denote it by $h_C(f)$. We use the notation $h(g)$ to denote the usual topological entropy of g when g is a continuous map of a compact topological space to itself.

Definition 1.1. Let $f : X \to X$ be a function from a topological space X to itself. We set

$$h_C(f) := \sup\{h(f|_K) : K \in \mathcal{K}(X, f)\},$$

where $\mathcal{K}(X, f)$ is the family of all compact, f-invariant subsets of X such that $f|_K$ is continuous. In the case of $\mathcal{K}(X, f)$ being empty, we put $h_C(f) := 0$.

We have two main results which suggest that the notion of continuity entropy may be useful. The first result states that for a piecewise continuous piecewise monotone function f of the interval to itself, if $h_{MZ}(f)$ denotes the entropy of f as defined in [23], then $h_C(f) = h_{MZ}(f)$. The second result states that if f is a continuous map of the interval to itself and g is a function of the interval to itself which agrees with f at all but countably many points, then $h_C(g) = h_C(f)$. These two results are proved in §3. In §2, we recall some basic properties of the usual topological entropy and consider the extent to which these properties hold for continuity entropy.

2. Properties of Continuity Entropy

We begin this section with some definitions and notation. Let $f : X \to X$ be a function from a topological space X to itself. We let f^0 denote the identity map and, inductively, let $f^n = f \circ f^{n-1}$. A subset $K \subseteq X$ is said to be *f-invariant*, if and only if $f(K) \subseteq K$ and *strictly f-invariant* if and only if $f(K) = K$. A subset M of X is called *f-minimal*, or a *minimal set for f*, if and only if M is a non-empty, closed, f-invariant subset of X which has no non-empty, proper, closed f-invariant subset. By $\omega_f(x)$ we denote the *ω-limit set of x*, i.e., the set of points which are the limit of a subsequence of the sequence $\{x, f(x), f^2(x), ...\}$. We say a point $x \in X$ is *recurrent* if and only if $x \in \omega_f(x)$.

The first result states that topological entropy is an invariant of topological conjugacy. A proof may be found in [1], [2], or [7].

Proposition 2.1. *Let X and Y be compact topological spaces and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. Suppose there is a homeomorphism $\alpha : X \to Y$ such that $\alpha \circ f = g \circ \alpha$. Then $h(f) = h(g)$.*

The same result holds in our setting.

Proposition 2.2. *Let X and Y be topological spaces and let $f : X \to X$ and $g : Y \to Y$ be functions. Suppose there is a homeomorphism $\alpha : X \to Y$ such that $\alpha \circ f = g \circ \alpha$. Then $h_C(f) = h_C(g)$.*

Proof. This follows the previous theorem, as α takes compact f-invariant subsets of X to compact g-invariant subsets of Y and α^{-1} takes compact g-invariant subsets of Y to compact f-invariant subsets of X. □

The next result involves restrictions of a map. The result follows from the proof of Theorem 4 in [1]. A proof also may be found in [7].

Proposition 2.3. *Let X be a compact topological space and let $f : X \to X$ be a continuous map. Suppose that Y is a closed, f-invariant subset of X. Then $h(f) \geq h(f|_Y)$.*

Again, we have a similar result for continuity entropy.

Proposition 2.4. *Let X be a topological space and let $f : X \to X$ be a function. Suppose that B is an f-invariant subset of X. Then $h_C(f) \geq h_C(f|_B)$.*

Proof. Let $K \in \mathcal{K}(B, f|_B)$. Then K is a compact subset of B and $(f|_B)|_K$ is continuous. Hence, K is a compact subset of X and $f|_K$ is continuous. Thus,

$$h((f|_B)|_K) = h(f|_K) \leq h_C(f).$$

Since this is true for each $K \in \mathcal{K}(B, f|_B)$, we have $h_C(f|_B) \leq h_C(f)$. □

The next result appears as [1, Theorem 4].

Proposition 2.5. *Let X be a compact topological space and let $f : X \to X$ be a continuous map. Suppose that $X = \bigcup_{i=1}^{n} X_i$, where each X_i is compact and f-invariant. Then $h(f) = \max\{h(f|_{X_i}) : i = 1, 2, ..., n\}$.*

Again, we have a similar result.

Proposition 2.6. *Let X be a Hausdorff space and let $f : X \to X$ be a function. Suppose that $X = \bigcup_{i=1}^{n} X_i$, where each X_i is compact and f-invariant. Then $h_C(f) = \max\{h_C(f|_{X_i}) : i = 1, 2, ..., n\}$.*

Proof. It follows from Proposition 2.4 that $h_C(f) \geq \max\{h_C(f|_{X_i}) : i = 1, 2, ..., n\}$.

We prove the reverse inequality by contradiction. Suppose that $h_C(f) > \max\{h_C(f|_{X_i}) : i = 1, 2, ..., n\}$. There exists $K \in \mathcal{K}(X, f)$ such that $h(f|_K) > \max\{h_C(f|_{X_i}) : i = 1, 2, ..., n\}$.

Let $D_i = X_i \cap K$ for each $i = 1, 2, ..., n$. Then each D_i is compact and f-invariant. By Proposition 2.5 we have

$$h(f|_K) = \max\{h(f|_{D_i}) : i = 1, 2, ..., n\} \leq \max\{h_C(f|_{X_i}) : i = 1, 2, ..., n\}.$$

This is a contradiction. □

Definition 2.7. Let $f : X \to X$ be a continuous map of a compact metric space to itself. Let $\mathcal{M}(X, f)$ denote the set of all f-invariant, Borel probability measures on X. Let $\mathcal{E}(X, f)$ denote the set of all f-invariant, ergodic, Borel probability measures on X. For $\mu \in \mathcal{M}(X, f)$, the measure theoretic entropy of f denoted $h_\mu(f)$ may be defined; see, e.g., [24] for details.

The following theorem known as the variational principle ([24, p. 188, Theorem 8.6]) relates topological entropy and measure theoretic entropy.

Theorem 2.8. *Let $f : X \to X$ be a continuous map of a compact metric space to itself. Then*

$$h(f) = \sup\{h_\mu(f) : \mu \in \mathcal{M}(X, f)\}.$$

The following is a corollary ([24, p. 190, Corollary 8.6.1]) to the variational principle.

Corollary 2.9. *Let $f : X \to X$ be a continuous map of a compact metric space to itself. Then*

$$h(f) = \sup\{h_\mu(f) : \mu \in \mathcal{E}(X, f)\}.$$

The following result for continuity entropy follows from Corollary 2.9.

Theorem 2.10. *Let $f : X \to X$ be a function from a metric space to itself. If $h_C(f) > 0$, then*

$$h_C(f) = \sup\{\mathcal{S}(K, f) : K \in \mathcal{K}(X, f)\},$$

where

$$\mathcal{S}(K, f) = \sup\{h_\mu(f|_K) : \mu \in \mathcal{E}(K, f|_K)\}.$$

Proof. Let $\lambda = \sup\{\mathcal{S}(K, f) : K \in \mathcal{K}(X, f)\}$. We will show that $h_C(f) = \lambda$.

Let $K \in \mathcal{K}(X, f)$. For any $\mu \in \mathcal{E}(K, f|_K)$, we have $h_\mu(f|_K) \leq h(f|_K) \leq h_C(f)$. Since μ is arbitrary, $S(K, f) \leq h_C(f)$. Since K is arbitrary, $\lambda \leq h_C(f)$.

To prove the reverse inequality, suppose that $0 < \alpha < h_C(f)$. There exists $K \in \mathcal{K}(X, f)$ with $h(f|_K) > \alpha$. Then for some $\mu \in \mathcal{E}(K, f|_K)\}$, we have $h_\mu(f|_K) > \alpha$. This implies that $\alpha < \lambda$. As α is arbitrary, we have $h_C(f) \leq \lambda$. □

Let $R(f)$ denote the set of recurrent points of f. The following result follows from Corollary 2.9 and the fact (noted in [24, p. 157]) that for any $\mu \in \mathcal{M}(X, f)$, we have $\mu(R(f)) = 1$.

Theorem 2.11. *Let $f : X \to X$ be a continuous map of a compact metric space to itself. Then $h(f) = h(f|_{\overline{R(f)}})$.*

In our setting, we have the following.

Theorem 2.12. *Let $f : X \to X$ be a function from a metric space X to itself. Then*

$$h_C(f) = \sup\{h(f|_K)\},$$

where the supremum is taken over all $K \in \mathcal{K}(X, f)$ such that $K \subset \overline{R(f)}$.

Proof. Let $\lambda = \sup\{h(f|_K)\}$, where the supremum is taken over all $K \in \mathcal{K}(X, f)$ such that $K \subset \overline{R(f)}$. It is immediate that $h_C(f) \geq \lambda$. We prove the reverse inequality by contradiction. Suppose that $h_C(f) > \lambda$. Then for some $K \in \mathcal{K}(X, f)$, we have $h(f|_K) > \lambda$. Let D denote the closure of the set of recurrent points of $f|_K$. Then $D \subseteq \overline{R(f)}$, with $D \in \mathcal{K}(X, f)$, and $h(f|_D) = h(f|_K) > \lambda$. This is a contradiction. □

The next result follows from Theorem 2.11.

Theorem 2.13. *Let $f : X \to X$ be a continuous map of a compact metric space to itself. Then $h(f) = h(f|_{X_\infty})$, where $X_\infty = \bigcap_{n \geq 0} f^n(X)$.*

From Theorem 2.13, we obtain the following.

Theorem 2.14. *Let $f : X \to X$ be a function of a metric space to itself. Then $h_C(f) = h(f|_{X_\infty})$, where $X_\infty = \bigcap_{n \geq 0} f^n(X)$. Also,*

$$h_C(f) = \sup\{h(f|_K)\},$$

where the supremum is taken over all $K \in \mathcal{K}(X, f)$ such that $f(K) = K$.

Proof. Let $K \in \mathcal{K}(X, f)$. If $K_\infty = \bigcap_{n \geq 0} f^n(K)$, then $f(K_\infty) = K_\infty$ and $h(f|_{K_\infty}) = h(f|_K)$. □

The following is another property of the usual topological entropy. A proof may be found in [1], [2], or [7].

Theorem 2.15. *Let $f : X \to X$ be a continuous map of a compact topological space to itself, and let $k \in \mathbb{N}$. Then $h(f^k) = k \cdot h(f)$.*

A partial result holds in our setting.

Theorem 2.16. *Let $f : X \to X$ be a function from a topological space to itself. Then for every $k \in \mathbb{N}$, $h_C(f^k) \geq k \cdot h_C(f)$.*

Proof. Since $\mathcal{K}(X, f) \subseteq \mathcal{K}(X, f^k)$ for all $k \in \mathbb{N}$, we have

$$h_C(f^k) = \sup_{M^*} h(f^k|_{M^*}) \geq \sup_{M} h(f^k|_M) = k \cdot \sup_{M} h(f|_M) = k \cdot h_C(f),$$

where $M^* \in \mathcal{K}(X, f^k)$ and $M \in \mathcal{K}(X, f)$. $\square$

Here is an example which shows that equality need not hold in Theorem 2.16.

Proposition 2.17. *For every $k > 1$ there is a function $f : X \to X$ of a compact metric space X with $h_C(f^k) > k \cdot h_C(f)$.*

Proof. First, let us take $k = 2$. Let $X = X_1 \cup X_2$ where X_1 and X_2 are uncountable disjoint compact subspaces of X. Let $\varphi : X_1 \to X_1$ be a homeomorphism with $h(\varphi) > 0$ such that X_1 is φ-minimal. Let $\psi : X_2 \to X_2$ be a continuous map with $h(\psi) = 0$. Let $x_0 \in X_1$, and let $orb_\varphi(x_0) = \{x_n\}_{n=-\infty}^{\infty}$ be the full (two-sided) orbit of x_0 under φ so that $\varphi(x_n) = x_{n+1}$ for every $n \in \mathbb{Z}$. Let $\{y_n\}_{n=-\infty}^{\infty}$ be a countable set consisting of distinct points of X_2.

We define a function $f : X \to X$ by the rule $f(x_n) = y_n$, $f(y_n) = \varphi^2(x_n)$ for all n, and $f = \varphi$ or $f = \psi$ on $X_1 \setminus \{x_n\}$ or $X_2 \setminus \{y_n\}$, respectively. Note that for each $x \in X_1$, we have $f^2(x) = \varphi^2(x)$. Consequently, $h_C(f^2) \geq h(\varphi^2) > 0$. Also, note that $f|_{X_1}$ is not continuous, as there is a dense subset of X_1 which is mapped to X_1 and also a dense subset of X_1 which is mapped to X_2.

We will show that $h_C(f) = 0$. Let $K \in \mathcal{K}(X, f)$. We must show that $h(f|_K) = 0$. This is immediate if $K \subset X_2$, so we may assume that there exists a point $y \in (K \cap X_1)$. We consider two cases. First, suppose that X_1 is φ^2-minimal. Then $y \in K$ implies that $X_1 \subset K$, a contradiction. Finally, suppose that X_1 is not φ^2-minimal. By [8, Lemma 2.1], there is a closed subset M of X_1 with $y \in M$ such that M is φ^2-minimal, and $X_1 = M \cup f(M)$. Again, it follows that $X_1 \subset K$, a contradiction. This completes the proof in the case that $k = 2$.

For $k > 2$, we consider the space X as a disjoint union $X = X_1 \cup X_2 \cup \ldots \cup X_k$, and we use a similar construction. $\square$

The following important result about topological entropy is sometimes called the semiconjugacy property. A proof may be found in [24] or [7].

Theorem 2.18. *Let X and Y be compact spaces and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. Suppose there is a continuous surjective map $\varphi : X \to Y$ such that $\varphi \circ f = g \circ \varphi$. Then $h(f) \geq h(g)$.*

Unfortunately, this theorem fails to hold for continuity entropy.

Proposition 2.19. *The semiconjugacy property does not hold for continuity entropy.*

Proof. Let Y be a compact metric space, and let $g : Y \to Y$ be a continuous map such that Y is g-minimal and $h(g) > 0$. Let $X = Y \times \{1, 2\}$. Then X is compact and metrizable.

Define $f : X \to X$ as follows. Fix $y_0 \in Y$. Set $f(x, 1) = (g(x), 1)$ and $f(x, 2) = (g(x), 2)$ if $x \neq y_0$. Set $f(y_0, 1) = (g(y_0), 2)$ and $f(y_0, 2) = (g(y_0), 1)$. Then f is not continuous at $(y_0, 1)$ and $(y_0, 2)$.

On the other hand, there are no closed, f-invariant, nonempty, proper subsets of X. So $\mathcal{K}(X, f) = \emptyset$ and $h_C(f) = 0$.

Finally, define $\varphi : X \to Y$ by $\varphi(y, k) = y$. So φ is the projection onto the first coordinate. Then φ is a continuous surjective map, and the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow\varphi & & \downarrow\varphi \\ Y & \xrightarrow{g} & Y \end{array}$$

commutes. □

Another property of topological entropy is the following (see [18]).

Theorem 2.20. *Let X and Y be compact spaces and $f : X \to Y$ and $g : Y \to X$ be continuous maps. Then $h(f \circ g) = h(g \circ f)$.*

This property also fails to hold for continuity entropy as we see in our next example.

Proposition 2.21. *There is a compact metric space Z and functions $f, g : Z \to Z$ such that $h_C(f \circ g) > h_C(g \circ f) = 0$.*

Proof. There is a compact metric space Z such that Z is the disjoint union of two compact subspaces X and Y which satisfy the following properties.

(1) X is an infinite compact space, such that there exists a minimal homeomorphism $\varphi : X \to X$ with $h(\varphi) > 0$.
(2) $Y = \{y_n\}_{n=-\infty}^{\infty} \cup \{y_\infty\}$, where $y_n \neq y_m \neq y_\infty$ for $n \neq m$.
(3) $\lim_{n\to-\infty} y_n = \lim_{n\to\infty} y_n = y_\infty$.

Let $x_0 \in X$, and let $orb_\varphi(x_0) = \{x_n\}_{n=-\infty}^{\infty}$ be the full (two-sided) orbit of x_0 under φ so that $\varphi(x_n) = x_{n+1}$ for every $n \in \mathbb{Z}$.

Now we can define functions f and g, such that $(f \circ g)|_X = \varphi$ as follows:

$$f(x) = \begin{cases} x & \text{if } x \in X, \\ x_n & \text{if } x = y_n, n \in \mathbb{Z}, \\ y_\infty & \text{if } x = y_\infty, \end{cases}$$

$$g(x) = \begin{cases} \varphi(x) & \text{if } x \in X \setminus orb_\varphi(x_0), \\ y_{n+1} & \text{if } x = x_n, n \in \mathbb{Z}, \\ y_\infty & \text{if } x \in Y. \end{cases}$$

Now, it is easily checked that $(f \circ g)|_X = \varphi$. Hence,

$$h_C((f \circ g)) \geq h(\varphi) > 0.$$

On the other hand, using the minimality of φ, we see that if K is a compact subset of Z with $(g \circ f)(K) \subseteq K$, then $K \subseteq Y$. Hence, $h_C(g \circ f) = 0$. $\square$

Remark 2.22. A similar example may be easily constructed with $h_C(f \circ g) > h_C(g \circ f) > 0$.

3. Proof of the Main Results

We begin this section by recalling some notation used in [23]. Let I denote the compact interval, and let $\mathcal{P}(I)$ denote the set of all finite partitions of I into disjoint intervals (of any form $[a,b], (a,b), [a,b), or (a,b]$ with $a < b$). A function $f : I \to I$ is said to be *piecewise continuous piecewise monotone* (*PCPM*) if and only if there exists a partition $\mathcal{A} \in \mathcal{P}(I)$, such that f is continuous and (not necessarily strictly) monotone on each interval of this partition. We let $\mathcal{P}(f)$ denote the set of all such partitions.

For a given partition $\mathcal{A} \in \mathcal{P}(f)$, set

$$h(f, \mathcal{A}) = \limsup_{n \to \infty} \frac{1}{n} \log c_n(f, \mathcal{A}),$$

where $c_n(f, \mathcal{A})$ is the number of nonempty sets of the form

$$\bigcap_{i=0}^{n-1} f^{-i}(E_j),$$

where each E_j is an interval in the partition $\mathcal{A}$.

Now, let $f : X \to X$ be a continuous map of a compact metric space to itself. A set $E \subset X$ is said to be (n, ε)-separated if and only if for every $x \neq y \in E$ there is $i \in \{0, 1, \dots, n-1\}$ such that $d(f^i(x), f^i(y)) > \varepsilon$. Let $s_n(f, \varepsilon)$ denote the maximal cardinality of an (n, ε)-separated set in X. For every $\varepsilon > 0$, we set

$$s(f, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(f, \varepsilon) \qquad \text{and} \qquad s(f) = \lim_{\varepsilon \to 0} s(f, \varepsilon).$$

Of course, in this situation we have $h(f) = s(f)$. In fact, in many texts this is how the topological entropy is defined.

Now, let $f : I \to I$ be PCPM. Then $s(f)$ may be defined exactly as above. The following result is proved in [23].

Theorem 3.1. *If $f : I \to I$ is PCPM and $\mathcal{A} \in \mathcal{P}(f)$, then $h(f, \mathcal{A}) = s(f)$.*

We will denote the common value in the previous theorem by $h_{MZ}(f)$. We now prove our first main result.

Theorem 3.2. *If $f : I \to I$ is PCPM, then $h_C(f) = h_{MZ}(f)$.*

Proof. First we show that $h_C(f) \geq h_{MZ}(f)$. We may assume that $h_{MZ}(f) > 0$. Let b be an arbitrary positive real number with $b < h_{MZ}(f)$. Let $\mathcal{A} \in \mathcal{P}(f)$. By [23, Theorem 1], there are positive integers q and n and pairwise disjoint closed intervals $J_1, ..., J_q$, such that

(1) $\frac{1}{n} \log q > b$;
(2) for each $i = 1, \ldots, q$ and $j = 0, \ldots, n-1$, the set $f^j(J_i)$ is contained in the interior of an element of $\mathcal{A}$;
(3) for each $i = 1, \ldots, q$, $f^n(J_i) \supseteq (J_1 \cup \cdots \cup J_q)$.

Set

$$Y = \bigcup_{i=1}^{q} (J_i \cup f(J_i) \cup \cdots \cup f^{n-1}(J_i)).$$

Let X denote the set of $x \in Y$ such that $f^i(x) \in Y$ for each positive integer i. Observe that X is a closed set which is invariant under f, and $f|_X$ is continuous.

Let K denote the set of $x \in (J_1 \cup \cdots \cup J_q)$ such that $f^{tn}(x) \in (J_1 \cup \cdots \cup J_q)$ for each positive integer t. Then $K \neq \emptyset$ and $K \subseteq X \subseteq Y$. Moreover, K is closed and invariant under f^n, and $f^n|_K$ is continuous. We may also observe that K is a disjoint union of the closed sets $K \cap J_1, \ldots K \cap J_q$, and each of these sets is mapped onto K by f^n. It follows that

$$h(f^n|_X) \geq h(f^n|_K) \geq \log q.$$

This implies that

$$h_C(f) \geq h(f|_X) \geq \frac{1}{n} \log q > b.$$

Since b was arbitrary, $h_C(f) \geq h_{MZ}(f)$.

Next, we show $h_C(f) \leq h_{MZ}(f)$. It suffices to show that if X is a closed invariant subset of I such that $f|_X$ is continuous, we have $h(f|_X) \leq h_{MZ}(f)$. It is evident that for any $\varepsilon > 0$ and any positive integer n, if $E \subseteq X$ is an (n, ε)-separated set for $f|_X$, then E is also an (n, ε)-separated set for f. Thus, $s_n(f, \varepsilon) \geq s_n(f|_X, \varepsilon)$. So we have

$$h_{MZ}(f) = s(f) \geq s(f|_X) = h(f|_X). \qquad \square$$

It is shown in [23] that if $f, g : I \to I$ are PCPM maps which differ only on a finite set, then $h_{MZ}(f) = h_{MZ}(g)$. In this case, we have by the previous theorem that $h_C(f) = h_C(g)$.

We will consider the following related question. Suppose that we have a continuous map of a compact metric space, and we change the map at countably many points. Can something be said about the continuity entropy of the two maps? We will give an answer to this question in Theorem 3.9 after some preliminary results.

We will use the following terminology. Let X be a topological space, and let $x \in X$. By a *neighborhood of* x, we mean an open subset of X which contains x. We say that x is a *perfect point of* X if and only if every neighborhood of x contains uncountably many points of X.

Theorem 3.3. *Let X be a topological space which has a countable basis. The set of perfect points of X is a closed set whose complement in X is a countable open set.*

Proof. Let $\mathcal{B}_1$ be a countable basis. Let $\mathcal{B}_2$ be the subset of $\mathcal{B}_1$ which consists of those $B \in \mathcal{B}_1$ which contain only countably many points. Let W be the union of all of the $B \in \mathcal{B}_2$. Then W is a countable open subset of X, and every countable open subset of X is a subset of W.

Let P denote the set of perfect points of X. Then P and W are disjoint subsets of X with $P \cup W = X$. The conclusion follows. $\square$

We will use the following property of ergodic measures, which is a part of [24, p. 27, Theorem 1.5].

Theorem 3.4. *Let $f : X \to X$ be a continuous map of a compact metric space to itself. Let $\mu \in \mathcal{E}(X, f)$ and let B be a Borel set. If $\mu((f^{-1}(B) \setminus B) \cup (B \setminus f^{-1}(B))) = 0$, then either $\mu(B) = 0$ or $\mu(B) = 1$.*

Proposition 3.5. *Let $f : X \to X$ be a continuous map of a compact metric space to itself. Let $\mu \in \mathcal{E}(X, f)$, and suppose that for some $x \in X$, we have $\mu(\{x\}) > 0$. Then x is a periodic point of f, and if D is the forward orbit of x, then $\mu(D) = 1$.*

Proof. It follows from the Poincaré recurrence theorem ([24, p. 26, Theorem 1.4]) that x is a periodic point of f. Let D denote the forward orbit of x. As $D \subset f^{-1}(D)$, it follows that $\mu(f^{-1}(D) \setminus D) = 0$. As μ is ergodic and $\mu(D) > 0$, it follows from Theorem 3.4 that $\mu(D) = 1$. $\square$

Theorem 3.6. *Let X be a compact metric space and let $f : X \to X$ be a continuous map. Let $\mu \in \mathcal{E}(X, f)$ and suppose that $h_\mu(f) > 0$. Let P denote the set of perfect points of X. Then there is a closed, invariant, subset Y of P with $\mu(Y) = 1$.*

Proof. Since $h_\mu(f) > 0$, there does not exist a periodic orbit D with $\mu(D) = 1$. Hence, by Proposition 3.5, we have $\mu(\{x\}) = 0$ for all $x \in X$. Since μ is a measure, we have $\mu(E) = 0$ for any countable subset E of X. Let $A = X \setminus P$ and let $B = \bigcup_{k=1}^{\infty} f^{-k}(A)$. Let $Y = X \setminus B$. Then Y is a closed, invariant, subset of P with $\mu(Y) = 1$. □

We mention the following corollaries.

Corollary 3.7. *Let X be a countable, compact metric space and let $f : X \to X$ be a continuous map. Then $h(f) = 0$.*

Proof. It follows from Theorem 3.6 that for all $\mu \in \mathcal{E}(X, f)$, we have $h_\mu(f) = 0$. By Corollary 2.9, $h(f) = 0$. □

Corollary 3.8. *Let X be an uncountable, compact metric space, and let P denote the set of perfect points of X. Let $f : X \to X$ be a continuous map with $h(f) > 0$. Then $h(f) = \sup\{h(f|Y)\}$ where the supremum is taken over all closed invariant subsets Y of P.*

Proof. Suppose $0 < \beta < h(f)$. By Corollary 2.9, there exists $\mu \in \mathcal{E}(X, f)$, with $h_\mu(f) > \beta$. By Theorem 3.6, there is a closed, invariant, subset Y of P with $\mu(Y) = 1$. Then $h(f|_Y) > \beta$. □

Theorem 3.9. *Let $f : X \to X$ be a continuous map of a compact metric space to itself. Suppose that $g : X \to X$ agrees with f at all but countably many points. Then $h_C(g) \leq h_C(f)$.*

Proof. Proceeding by contradiction, suppose that $h_C(g) > h_C(f)$. There is a $K \in \mathcal{K}(X, g)$ such that $h(g|_K) > h_C(f)$. By Corollary 2.9, there exists $\mu \in \mathcal{E}(K, g|_K)$ such that $h_\mu(g|_K) > h_C(f)$. By Theorem 3.6, there is a closed, g-invariant, subset Y of the set of perfect points of K with $\mu(Y) = 1$. So, we may think of μ as a measure on Y, and we have $h_\mu(g|_Y) > h_C(f)$.

Since $f|_Y$ and $g|_Y$ are continuous, and g agrees with f at all but countably many points, we have $f|_Y = g|_Y$. It follows that Y is f-invariant, $\mu \in \mathcal{E}(Y, f|_Y)$, and

$$h_\mu(f|_Y) = h_\mu(g|_Y) > h_C(f).$$

This is a contradiction as

$$h_C(f) = h(f) \geq h(f|_Y) \geq h_\mu(f|_Y). \qquad \square$$

In general, we need not have equality in the previous theorem.

Example 3.10. *There exists a continuous map $f : X \to X$ of a compact metric space to itself and a function $g : X \to X$ which agrees with f at all but one point such that $h_C(g) < h_C(f)$.*

Proof. Consider $f : X \to X$ such that X is f-minimal and $h(f) > 0$. Fix $x_0 \in X$. Let $g : X \to X$ satisfy $g(x) = f(x)$ for all $x \in X - \{x_0\}$ while $g(x_0) = x_0$. Then $h_C(g) = 0$. □

We will show that if X is the interval, the inequality in Theorem 3.9 can be replaced by equality. To show this, we first prove a result about minimal sets for continuous maps of the interval.

Theorem 3.11. *Let $f : I \to I$ be continuous and let $0 < t < h(f)$. There is an f-minimal set M with $h(f|_M) = t$.*

Proof. Let f and t be given. By a theorem of Michał Misiurewicz (see [21] or [7, p. 215, Theorem VIII.29]), there are positive integers j and q with $\frac{1}{j}\log q > t$ and disjoint intervals $J_1, \ldots, J_q$ such that for each $i = 1, \ldots, q$, we have $(J_1 \cup \cdots \cup J_q) \subseteq f^j(J_i)$. The expression f^j has a q horseshoe (with disjoint intervals) is often used to describe this situation; see, e.g., [2]. Let σ denote the full one-sided shift on q symbols. Since $\log q > jt$, it follows from [15] that there is a σ-minimal set E with $h(\sigma|_E) = jt$.

Now, by a standard argument (see, e.g., [7, p. 35, Proposition II.5]), there is a closed, f^j-invariant subset B of I and a continuous surjective map $\alpha : B \to \Sigma_q$ which is at most two-to-one with $\alpha \circ (f^j|_B) = \sigma \circ \alpha$ (a semiconjugacy).

There is a subset D of $\alpha^{-1}(E)$ such that D is a minimal set for $f^j|_B$. Then the diagram

$$\begin{array}{ccc} D & \overset{f^j|_D}{\to} & D \\ \downarrow\alpha|_D & & \downarrow\alpha|_D \\ E & \overset{\sigma|_E}{\to} & E \end{array}$$

commutes. Since E is a minimal set, $\alpha|_D$ is surjective. Since α is at most two to one, it follows from [9, Theorem 17] that $h(f^j|_D) = h(\sigma|_E) = jt$.

Finally, let $M = \bigcup_{i=0}^{j-1} f^i(D)$. Then M is a minimal set for f. Moreover,

$$h(f^j|_M) = \max\{h(f^j|_D), \cdots, h(f^j|_{f^{j-1}(D)})\},$$

and we have a semiconjugacy

$$\begin{array}{ccc} D & \overset{f^j|_D}{\to} & D \\ \downarrow f & & \downarrow f \\ f(D) & \overset{f^j|_{f(D)}}{\to} & f(D). \end{array}$$

It follows that $h(f^j|_D) \geq h(f^j|_{f(D)})$. Similarly,

$$h(f^j|_{f(D)}) \geq h(f^j|_{f^2(D)}) \geq \cdots \geq h(f^j|_{f^{j-1}(D)}) \geq h(f^j|_{f^j(D)}).$$

But $h(f^j|_{f^j(D)}) = h(f^j|_D)$ as $f^j(D) = D$. It follows that $h(f^j|_D) = h(f^j|_{f(D)}) = \cdots = h(f^j|_{f^{j-1}(D)})$. Thus, $h(f^j|_M) = h(f^j|_D)$, which implies that $h(f|_M) = \frac{1}{j}h(f^j|_M) = \frac{1}{j}h(f^j|_D) = t$. □

We are now ready to prove our second main result and conclude the paper.

Theorem 3.12. *Let $f : I \to I$ be a continuous map of the interval to itself. Suppose that $g : I \to I$ agrees with f at all but countably many points. Then $h_C(f) = h_C(g)$.*

Proof. By Theorem 3.9, it suffices to prove that $h_C(g) \geq h_C(f)$. So we may assume that $h(f) > 0$. Let β satisfy $0 < \beta < h(f)$. Let $D \subset I$ be the set of points x such that $f(x) \neq g(x)$. By Theorem 3.11, there is an uncountable family $\{M_\alpha\}$ of f-minimal sets, such that $h(f|_{M_\alpha}) > \beta$ for each α. Since any two distinct f-minimal sets are disjoint, there exists an f-minimal set M, disjoint from D, with $h(f|_M) > \beta$. So $f|_M = g|_M$ and

$$h_C(g) \geq h(g|_M) = h(f|_M) > \beta.$$

As β was arbitrary, it follows that $h_C(g) \geq h_C(f)$. □

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References

[1] R. L. Adler, A. G. Konheim, and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.

[2] Lluís Alsedà, Jaume Llibre, and Michał Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*. Advanced Series in Nonlinear Dynamics, 5. River Edge, NJ: World Scientific Publishing Co., Inc., 1993.

[3] Lluís Alsedà, Sergiĭ Kolyada, Jaume Llibre, and Lubomír Snoha, *Axiomatic definition of the topological entropy on the interval*, Aequationes Math. **65** (2003), no. 1-2, 113–132.

[4] Christoph Bandt, Gerhard Keller, and Bern Pompe, *Entropy of interval maps via permutations*, Nonlinearity **15** (2002), no. 5, 1595–1602.

[5] Michael Barnsley, Brendan Harding, and Andre Vince, *The entropy of a special overlapping dynamical system*, Ergodic Theory Dynam. Systems **34** (2014), no. 2, 483–500.

[6] Michael Barnsley, Wolfgang Steiner, and Andre Vince, *Critical itineraries of maps with constant slope and one discontinuity*, Math. Proc. Cambridge Philos. Soc. **157** (2014), no. 3, 547–565.

[7] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*. Lecture Notes in Mathematics, 1513. Berlin: Springer-Verlag, 1992.

[8] Louis Block and James Keesling, *A characterization of adding machine maps*, Topology Appl. **140** (2004), no. 2-3, 151–161.

[9] Rufus Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1971), 401–414.

[10] Rufus Bowen, *Periodic points and measures for Axiom A diffeomorphisms*, Trans. Amer. Math. Soc. **154** (1971), 377–397.

[11] Rufus Bowen, *Topological entropy for noncompact sets*, Trans. Amer. Math. Soc. **184** (1973), 125–136.

[12] Jose S. Cánovas, *Topological entropy of continuous transitive maps on the real line*, Dyn. Syst. **24** (2009), no. 4, 473–483.

[13] J. S. Cánovas and J. M. Rodríguez, *Topological entropy of maps on the real line*, Topology Appl. **153** (2005), no. 5-6, 735–746.

[14] Michaela Čiklová, *Dynamical systems generated by functions with connected G_δ graphs*, Real Anal. Exchange **30** (2004/05), no. 2, 617–638.

[15] Christian Grillenberger, *Constructions of strictly ergodic systems. I. Given entropy*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **25** (1972/73), 323–334.

[16] Michael Handel and Bruce Kitchens, *Metrics and entropy for non-compact spaces. With an appendix by Daniel J. Rudolph*, Israel J. Math. **91** (1995), no. 1-3, 253–271.

[17] J. E. Hofer, *Topological entropy for noncompact spaces*, Michigan Math. J. **21** (1974), 235–242.

[18] Sergiĭ Kolyada and Julia Semikina, *On topological entropy: When positivity implies +infinity*, Proc. Amer. Math. Soc. **143** (2015), no. 4, 1545–1558.

[19] Magda Komorníková and Jozef Komorník, *Comparing measure theoretic entropy with topological entropy for noncompact spaces*, Acta Math. Univ. Comenian. **42/43** (1983), 193–195 (1984).

[20] Dominik Kwietniak and Martha Ubik, *Topological entropy of compact subsystems of transitive real line maps*, Dyn. Syst. **28** (2013), no. 1, 62–75.

[21] Michał Misiurewicz, *Horseshoes for mappings of the interval*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **27** (1979), no. 2, 167–169.

[22] Michał Misiurewicz, *Permutations and topological entropy for interval maps*, Nonlinearity **16** (2003), no. 3, 971–976.

[23] Michał Misiurewicz and Krystina Ziemian, *Horseshoes and entropy for piecewise continuous piecewise monotone maps* in From Phase Transitions to Chaos: Topics in Modern Statistical Physics. Ed. Géza Györgyi, Imre Kondor, Lászlo Sasvári, and Tamás Tél. River Edge, NJ: World Sci. Publ., 1992. 489–500.

[24] Peter Walters, *An Introduction to Ergodic Theory.* Graduate Texts in Mathematics, 79. New York-Berlin: Springer-Verlag, 1982.

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