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FINITE GRAPHS HAVE UNIQUE n-FOLD PSEUDO-HYPERSPACE SUSPENSION

ULISES MORALES-FUENTES

Abstract. Let X be a metric continuum. Let n be a positive integer, let $C_n(X)$ be the space of all nonempty closed subsets of X with at most n components, and let $F_1(X)$ be the space of singletons of X. The *n-fold pseudo-hyperspace suspension* of X is the quotient space $C_n(X)/F_1(X)$ and it is denoted by $PHS_n(X)$. In this paper we prove that if X is a finite graph and Y is a continuum such that $PHS_n(X)$ is homeomorphic to $PHS_n(Y)$, then X is homeomorphic to Y. This answers a question by Juan C. Macías.

1. Introduction

A *continuum* is a compact connected metric space. For a continuum X, consider the following set:

$2^X = \{A \subset X : A \text{ is a nonempty closed subset of } X\}$.

Let n be a positive integer; the *n-fold hyperspace* of X, denoted by $C_n(X)$, is the set:

$\{A \in 2^X : \ A \text{ has at most } n \text{ components}\}$;

the *n-fold symmetric product* of X, denoted by $F_n(X)$, is the set:

$\{A \subset X \ : A \text{ has at most } n \text{ points}\}$.

These sets are topologized with the Hausdorff metric which is defined as

$$\mathcal{H}(A,B) = \inf\{\varepsilon > 0 \ : \ A \subset \mathcal{V}_\varepsilon(B) \text{ and } B \subset \mathcal{V}_\varepsilon(A)\},$$

where $\mathcal{V}_\varepsilon(A) = \{x \in X \ : \ d(x,A) < \varepsilon\}$.

Sam B. Nadler, Jr., [21] introduced the hyperspace suspension of a continuum, $HS(X)$, as the quotient space $C_1(X)/F_1(X)$. Later Sergio

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Macías [18] gave a generalization of it, defining the *n-fold hyperspace suspension* of a continuum, $HS_n(X)$, as the quotient space $C_n(X)/F_n(X)$. In 2008, Juan C. Macías [15] introduced the *n-fold pseudo-hyperspace suspension* of a continuum, $PHS_n(X)$, as the quotient space $C_n(X)/F_1(X)$.

For a continuum X, let $\mathcal{S}(X)$ denote one of the hyperspaces 2^X, $F_n(X)$, $C_n(X)$, $HS_n(X)$, or $PHS_n(X)$. X is said to have *unique* $\mathcal{S}(X)$ if the following holds: if Y is a continuum such that $\mathcal{S}(X)$ is homeomorphic to $\mathcal{S}(Y)$, then X is homeomorphic to Y.

The problem of finding conditions on X in order that X has unique $\mathcal{S}(X)$ has been widely studied; see R. Duda [4] and [5], Carl Eberhart and Nadler [6], S. Macías [16], Nadler [20], and Gerardo Acosta [1]. Uniqueness of the hyperspace $C_1(X)$ has been proven for several families of continua:

(1) finite graphs different from an arc and a simple closed curve [4], [5].

(2) hereditarily indecomposable continua ([20, Theorem 0.60]).

(3) indecomposable continua such that all their proper subcontinua are arcs [16].

In [1], Acosta also proved that metric compactifications of the arc $[0, \infty)$, different from an arc, have unique hyperspace $C_1(X)$. S. Macías [17] proved that hereditarily indecomposable continua have unique hyperspace 2^X. Enrique Castañeda and Alejandro Illanes [2] have proven that finite graphs have unique hyperspace $F_n(X)$ for each n. Illanes [12], [13] proved that finite graphs have unique hyperspace $C_n(X)$ for each $n > 1$. It has been proved that dendrites with closed set of end points have unique $C_n(X)$ for each n (see [8], [9], and [11]).

In [7], the authors study the uniqueness of the hyperspace $C_n(X)$ for Peano continua, giving some sufficient and also some necessary conditions for a Peano continuum X to have unique hyperspace $C_n(X)$.

In [10], the authors adopt some of the techniques presented in [7] and prove that finite graphs have unique $HS_n(X)$ for each n. In the present work, we prove that finite graphs have unique $PHS_n(X)$ for each n; we use most of the ideas presented in [10]. Many of the arguments in the proof of our main result are parallel to the arguments of the proof of the main result in [10]; this means that most of the arguments that work for $HS_n(X)$ work for $PHS_n(X)$. However, in working with $PHS_n(X)$, θ_m-graphs need a special treatment in the case where $n \geq 2$.

2. **Definitions**

We use the definitions that are presented in [10] and several are the "$PHS_n(X)$" version of an "$HS_n(X)$"; for the reader's convenience we provide them. We also present some basic definitions.

Let q_X^n denote the quotient map $q_X^n : C_n(X) \to PHS_n(X)$, and let F_X^n denote the element $q_X^n(F_1(X))$.

The topological boundary of U in X will be denoted by $fr_X(U)$. The topological interior of U in X will be denoted by $\mathrm{int}_X(U)$, or simply U° if there is no possible confusion with the underlying topological space in which U lies. And the topological closure of U in X will be denoted by $\mathrm{cl}_X(U)$. The symbol $\mathbb{N}$ will denote the set of positive integers.

Given a finite graph X, a point x of X, and an element n of $\mathbb{N}$, we say that X *has order less than or equal to* n *at* x, written $\mathrm{ord}(x, X) \leq n$, provided that x has a basis $\mathfrak{B}$ of neighborhoods in X such that, for each U in $\mathfrak{B}$, we have that $fr_X(U)$ has cardinality less than or equal to n. We say that X *has order equal to* n *at* x, written $\mathrm{ord}(x, X) = n$, if $\mathrm{ord}(x, X) \leq n$ and $\mathrm{ord}(x, X) \nleq m$ for any natural number m such that $m < n$.

Given a finite graph X, $E(X)$ will denote the set of points of X whose order is one. The elements of $E(X)$ will be called *end points* of X. The set of points of X whose order is greater than or equal to 3 will be denoted by $R(X)$ and its elements will be called *ramification points* of X. Note that $R(X)$ is the set of vertices of X.

Given a finite graph X, a *loop* in X is a simple closed curve S in X such that S has at most one ramification point of X. A *free arc* in X is an arc α with extreme points a and b such that $\alpha - \{a, b\}$ is open in X. A *maximal free arc* is a free arc that is maximal with respect to the inclusion.

Let $m \in \mathbb{N} - \{1, 2\}$. A continuum X will be called the θ_m*-graph* provided that X is a finite graph that can be written as the union of m arcs $J_1, \dots, J_m$, such that there is a two point set $\{v, u\}$ in X, such that the set $\{v, u\}$ is the set of extreme points of each arc J_i, and $J_i \cap J_j = \{v, u\}$ if $j \neq i$.

Let X be a finite graph such that $R(X) \neq \emptyset$; let us define its set of edges in a topological way: $\mathfrak{A}_S(X) = \{J \subset X\ : J$ is a maximal free arc or J is a loop in $X\}$, $\mathfrak{A}_R(X)$ will be the subset of loops of $\mathfrak{A}_S(X)$, and $\mathfrak{A}_E(X)$ will be the subset of $\mathfrak{A}_S(X)$ consisting of arcs with just one ramification point of X.

Now, let $\mathbb{R}^2$ denote the Euclidean plane. Let L_0 be the continuum obtained as $L_0 = D_1 - \mathrm{int}_{\mathbb{R}^2}(D_2)$, where D_1 and D_2 are disks in the plane $\mathbb{R}^2$, D_2 is a proper subset of D_1, and the disks D_1 and D_2 are tangent.

Let X be a finite graph such that $R(X) \neq \emptyset$. Given $J \in \mathfrak{A}_S(X)$, define $\mathcal{E}(J)$ in the following way: If J is an arc, let $\mathcal{E}(J) = C_1(J)$. In the case that J is a loop, let p_J be the only ramification point of X that belongs to J, and define $\mathcal{E}(J) = \{A \in C_1(J) : A = J$ or $A = \{p\}$ for some $p \in J$, or A is a subarc of J such that $p_J \notin A$, or A is a subarc of J such that p_J is one of its extreme points$\}$; from the construction given in [14, Example 5.2], we see that if J is a loop, then $\mathcal{E}(J)$ is homeomorphic to the continuum L_0.

Now let X be a finite graph and let $n \in \mathbb{N}$. Let us define

$$\mathcal{PHL}_n(X) = \{\chi \in PHS_n(X) \ : \ \chi \text{ has a neighborhood } \mathcal{N} \text{ in } PHS_n(X) \text{ such that } \mathcal{N} \text{ is a } 2n\text{-cell}\};$$

$$\partial\mathcal{PHL}_n(X) = \{\chi \in PHS_n(X) \ : \ \chi \text{ has a neighborhood } \mathcal{N} \text{ in } PHS_n(X) \text{ such that } \mathcal{N} \text{ is a } 2n\text{-cell and } \chi \text{ is in the manifold boundary of } \mathcal{N}\};$$

$$\mathcal{PHD}_n(X) = \{\chi \in PHS_n(X) \ : \ \chi \notin \mathcal{PHL}_n(X) \text{ and } \chi \text{ has a basis } \mathcal{B} \text{ of neighborhoods in } PHS_n(X) \text{ such that for each } \mathcal{U} \in \mathcal{B}, \dim(\mathcal{U}) \leq 2n, \text{ and } \mathcal{U} \cap \mathcal{PHL}_n(X) \text{ is arcwise connected}\};$$

$$\mathcal{PHE}_n(X) = \{\chi \in PHS_n(X) \ : \ \dim_\chi[PHS_n(X)] = 2n\}.$$

Let X be a topological space and $m \in \mathbb{N}$. Let $U_1, \ldots, U_m$ be subsets of X. Define

$$\langle U_1, \ldots, U_m \rangle_n = \{A \in C_n(X) \ : \ A \subset U_1 \cup \cdots \cup U_m \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, m\}\};$$

it is known ([14, Theorem 3.1]) that the family of sets of the form $\langle U_1, \ldots, U_m \rangle_n$, where U_i is open in X, is a basis for the topology in $C_n(X)$.

For J, $K \in \mathfrak{A}_S(X)$, let us define

$$\mathcal{PHD}(J, K) = \mathrm{cl}_{PHS_2(X)}(\partial\mathcal{PHL}_2(X) \cap q_X^2(\langle J^\circ, K^\circ \rangle_2)) \cap \mathrm{cl}_{PHS_2(X)}(\partial\mathcal{PHL}_2(X) - q_X^2(\langle J^\circ, K^\circ \rangle_2)).$$

3. Preliminary Results

The proofs of lemmas 3.1 and 3.2 are similar to the proofs of lemmas 2.4 and 2.9(b) of [10], respectively; we omit the details of these proofs.

Lemma 3.1. *Let X be a finite graph such that $R(X) \neq \emptyset$, and let $n \in \mathbb{N}$. Then for each neighborhood $\mathcal{U}$ of F_X^n in $PHS_n(X)$, we have* $\dim[\mathcal{U}] \geq 2n + 1$.

Lemma 3.2. *Let X be a finite graph. If $n \geq 3$ and $R(X) \neq \emptyset$, then $\mathcal{PHD}_n(X) = \{q_X^n(A) \in PHS_n(X) \ : \ A \in C_1(X) - F_1(X)$ and $A \cap R(X) = \emptyset\}$.*

Lemma 3.3. *Let X be a finite graph. Then*

(a) *If $n \geq 3$ and $R(X) \neq \emptyset$, then the components of $\mathcal{PHD}_n(X)$ are the sets of the form $q_X^n(\langle J^\circ \rangle_n \cap C_1(X)) - \{F_X^n\}$, where $J \in \mathfrak{A}_S(X)$.*

(b) *If $R(X) \neq \emptyset$, then the components of $\mathcal{PHE}_n(X)$ are the sets of the form $q_X^n(\langle J_1^\circ, ..., J_m^\circ \rangle_n) - \{F_X^n\}$, where $J_1, ..., J_m \in \mathfrak{A}_S(X)$, and $m \leq n$.*

Proof. (a) By Lemma 3.2, $\mathcal{PHD}_n(X) = \bigcup\{q_X^n(\langle J^\circ\rangle_n \cap C_1(X)) - \{F_X^n\} : J \in \mathfrak{A}_S(X)\}$. It is easy to see that the sets of the form $q_X^n(\langle J^\circ\rangle_n \cap C_1(X)) - \{F_X^n\}$ are arcwise connected and, therefore, connected. Note that the sets of the form $q_X^n(\langle J^\circ\rangle_n \cap C_1(X)) - \{F_X^n\}$ are open in $\mathcal{PHD}_n(X)$ and pairwise disjoint; we conclude that they are the components of $\mathcal{PHD}_n(X)$.

(b) By the main formula in [19] and using Lemma 3.1, we conclude that $\mathcal{PHE}_n(X) = \bigcup\{q_X^n(\langle J_1^\circ, ..., J_m^\circ\rangle_n) - \{F_X^n\} : J_1, ..., J_m \in \mathfrak{A}_S(X)\}$. Using [3, Proposition 2.6], it is easy to show that $\langle J_1^\circ, ..., J_m^\circ\rangle_n - F_1(X)$ is arcwise connected; therefore, the sets of the form $q_X^n(\langle J_1^\circ, ..., J_m^\circ\rangle_n) - \{F_X^n\}$ are connected. The rest of the proof is similar to the proof of (a). □

Lemma 3.4. *Let X be a finite graph such that $R(X) \neq \emptyset$; also let $p \in X$ and let $J \in \mathfrak{A}_S(X)$. Then*

(1) *if J is an arc, then $\{q_X^2(\{p\} \cup A) : A \in \mathcal{E}(J)\}$ is a 2-cell in $PHS_2(X)$;*
(2) *if J is a loop, then $\{q_X^2(\{p\} \cup A) : A \in \mathcal{E}(J)\}$ is homeomorphic to the continuum L_0.*

Proof. Let g be the embedding of $C_1(X)$ into $C_2(X)$ given by $g(A) = \{p\} \cup A$. Since the set $g(\mathcal{E}(J)) \cap F_1(X)$ is either the set $\emptyset$ or the set $\{p\}$, we have that $g(\mathcal{E}(J))/F_1(X)$ is homeomorphic to $\mathcal{E}(J)$; note that in (1), the set $\mathcal{E}(J)$ is a 2-cell, and in (2), it is homeomorphic to the continuum L_0. Now, we finish the proof mentioning that $g(\mathcal{E}(J))/F_1(X)$ is clearly homeomorphic to $\{q_X^2(\{p\} \cup A) : A \in \mathcal{E}(J)\}$. □

Lemma 3.5. *Let X be a finite graph such that $R(X) \neq \emptyset$. Let $J, K \in \mathfrak{A}_S(X)$. Then $\mathcal{PHD}(J,K) = \{q_X^2(\{p\} \cup G) : p \in fr_X(J)$ and $G \in \mathcal{E}(K)$ or $p \in fr_X(K)$ and $G \in \mathcal{E}(J)\}$.*

Proof. Let the right-hand side of the equality of the lemma be denoted by Γ. The proof of $\mathcal{PHD}(J,K) = \Gamma$ is analogous to the proof of Lemma 2.15 in [10], with the exception of the element F_X^2. Each one of the facts, $F_X^2 \in \mathcal{PHD}(J,K)$ and $F_X^2 \in \Gamma$, implies that $1 \leq |J \cap K \cap R(X)| \leq 2$ or $J = K$; this has to be taken into account in proving that if $F_X^2 \in \mathcal{PHD}(J,K)$, then $F_X^2 \in \Gamma$ (and vice versa). So a simple adaptation of the argument needs to be done. Details are left to the reader. □

4. **Models for $\mathcal{PHD}(J,K)$**

For the proof of our main theorem, in the case $n = 2$, explicit models for the set $\mathcal{PHD}(J,K)$ are needed. The topology of $\mathcal{PHD}(J,K)$ depends on what kind of edges J and K are. In all cases, we use Lemma 3.5 in order to write the set $\mathcal{PHD}(J,K)$ as the union of sets whose models have been described previously in Lemma 3.4. In proving uniqueness of $HS_2(X)$

and $C_2(X)$, a similar description, and for an analogous purpose, is made in [10] right after Lemma 2.15 and in [7, Theorem 34], respectively.

In all cases, J and K are elements of $\mathfrak{A}_S(X)$, where X is a finite graph such that $R(X) \neq \emptyset$.

Case A: $J = K$, J is an arc, and $J \notin \mathfrak{A}_E(X)$. By Lemma 3.5, $\mathcal{PHD}(J,J) = \{q_X^2(\{p_1\} \cup A) : A \in \mathcal{E}(J)\} \cup \{q_X^2(\{p_2\} \cup A) : A \in \mathcal{E}(J)\}$, where $J \cap R(X) = \{p_1, p_2\}$. So now, using Lemma 3.4, we see that $\mathcal{PHD}(J,J)$ is the union of two 2-cells whose intersection is the set $\{F_X^2, q_X^2(J), q_X^2(\{p_1, p_2\})\}$. It is easy to see that this set belongs to the manifold boundary of both 2-cells.

Case B: $K = J \in \mathfrak{A}_E(X)$. An analysis similar to that in Case A can be made. Therefore, $\mathcal{PHD}(J,K)$ is a 2-cell.

Case C: $K = J \in \mathfrak{A}_R(X)$. As with Case B, an analysis similar to that in Case A can be made. Therefore, $\mathcal{PHD}(J,K)$ is an L_0 continuum.

For the next cases assume that $J \neq K$.

Case D: J and K are arcs such that none of them belongs to $\mathfrak{A}_E(X)$. Let $\{p_J, q_J\} = J \cap R(X)$ and let $\{p_K, q_K\} = K \cap R(X)$. Let $\mathcal{D}_1 = \{q_X^2(\{p_J\} \cup A) : A \in \mathcal{E}(K)\}$, $\mathcal{D}_2 = \{q_X^2(\{q_J\} \cup A) : A \in \mathcal{E}(K)\}$, $\mathcal{G}_1 = \{q_X^2(\{p_K\} \cup A) : A \in \mathcal{E}(J)\}$, and $\mathcal{G}_2 = \{q_X^2(\{q_K\} \cup A) : A \in \mathcal{E}(J)\}$. Note that since J and K are arcs, we have that $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{G}_1$, and $\mathcal{G}_2$ are 2-cells and $\mathcal{PHD}(J,K) = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{G}_1 \cup \mathcal{G}_2$. Now let us consider the next three subcases.

D.1: $J \cap K = \emptyset$. In this subcase, we have that $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, $\mathcal{D}_1 \cap \mathcal{G}_1 = \{q_X^2(\{p_J, p_K\})\}$, $\mathcal{D}_1 \cap \mathcal{G}_2 = \{q_X^2(\{p_J, q_K\})\}$, $\mathcal{D}_2 \cap \mathcal{G}_1 = \{q_X^2(\{q_J, p_K\})\}$, $\mathcal{D}_2 \cap \mathcal{G}_2 = \{q_X^2(\{q_J, q_K\})\}$, and $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$.

D.2: $J \cap K$ is a one point set; we can consider $p_J = p_K$. In this subcase we see that $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, $\mathcal{D}_1 \cap \mathcal{G}_1 = \{F_X^2\}$, $\mathcal{D}_1 \cap \mathcal{G}_2 = \{q_X^2(\{p_J, q_K\})\}$, $\mathcal{D}_2 \cap \mathcal{G}_1 = \{q_X^2(\{q_J, p_K\})\}$, $\mathcal{D}_2 \cap \mathcal{G}_2 = \{q_X^2(\{q_J, q_K\})\}$, and $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$.

D.3: $J \cap K$ is a two point set; we may assume that $p_J = p_K$ and $q_J = q_K$. In this subcase we see that $\mathcal{D}_1 \cap \mathcal{D}_2 = \{F_X^2, q_X^2(\{p_J, q_K\}), q_X^2(K)\}$, $\mathcal{D}_1 \cap \mathcal{G}_1 = \{F_X^2, q_X^2(\{p_J, q_K\})\}$, $\mathcal{D}_1 \cap \mathcal{G}_2 = \{F_X^2, q_X^2(\{p_J, q_K\})\}$, $\mathcal{D}_2 \cap \mathcal{G}_1 = \{F_X^2, q_X^2(\{p_J, q_K\})\}$, $\mathcal{D}_2 \cap \mathcal{G}_2 = \{F_X^2, q_X^2(\{p_J, q_K\})\}$, and $\mathcal{G}_1 \cap \mathcal{G}_2 = \{F_X^2, q_X^2(\{p_J, q_K\}), q_X^2(J)\}$.

For each of the cases below, a similar analysis can be made to the one in Case D; in each of these cases there are two subcases: (1) $J \cap K = \emptyset$ and (2) $| J \cap K |= 1$. The details are left to the reader.

Case E: $J \notin \mathfrak{A}_E(X) \cup \mathfrak{A}_R(X)$ and $K \in \mathfrak{A}_E(X))$.

Case F: $J \notin \mathfrak{A}_E(X) \cup \mathfrak{A}_R(X)$ and $K \in \mathfrak{A}_R(X)$.

Case G: $\{J, K\} \subset \mathfrak{A}_E(X)$.

Case H: $J \in \mathfrak{A}_E(X)$ and $K \in \mathfrak{A}_R(X)$.

Case I: $\{J, K\} \subset \mathfrak{A}_R(X)$).

Finally, in Figure 1, we present a table with all the possible models for $\mathcal{PHD}_n(J,K)$ when $J \neq K$; note that $\{p_J, q_J, p_K, q_K\} = (J \cup K) \cap R(X)$. They are ordered from left to right and from top to bottom. They correspond to the subcases in cases D, E, F, G, H, and I, respectively. It is important to note that none of the models in Figure 1 is homeomorphic to a model in which $J = K$.

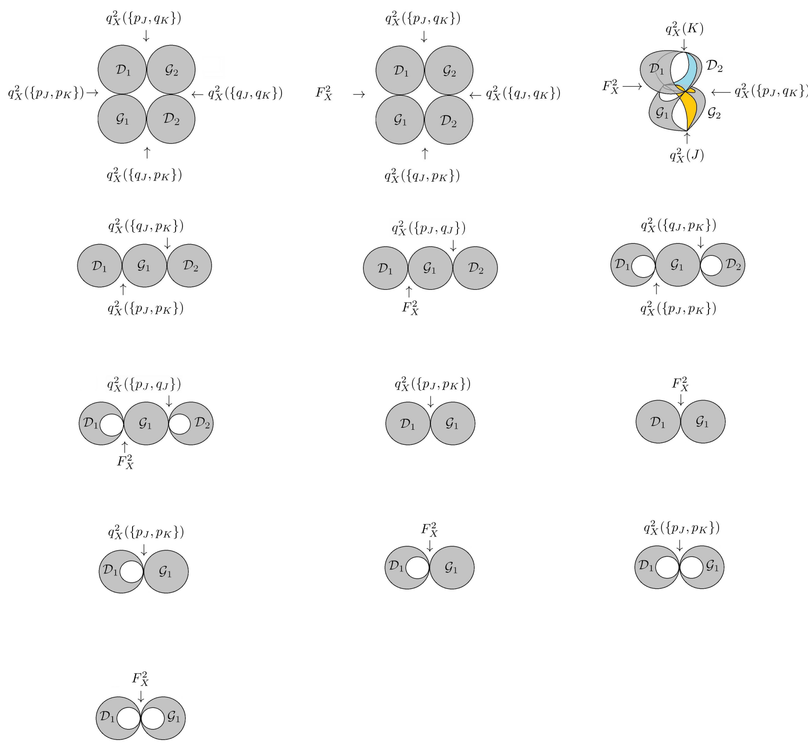

FIGURE 1. All possible models for $\mathcal{PHD}(J,K)$, when $J \neq K$.

5. Main Results

In this section we present a proof for our main result. The first step will be to mention that S. Macías [18] and J. C. Macías [15] have proven that the graphs S^1 and $[0,1]$ have unique n-fold pseudo-hyperspace suspension. Then we prove that for each $m \in \mathbb{N}$, the graph θ_m has unique n-fold pseudo-hyperspace suspension. And finally, we prove that for a graph X, such that $R(X) \neq \emptyset$ and X not homeomorphic to a θ_m-graph for any $m \in \mathbb{N}$, the uniqueness of the n-fold pseudo-hyperspace suspension holds.

Lemma 5.1. *Let X and Y be finite graphs such that $R(X) \neq \emptyset \neq R(Y)$, let $n \in \mathbb{N}$, and let $h : PHS_n(X) \to PHS_n(Y)$ be a homeomorphism. Assume that for each $J \in \mathfrak{A}_S(X)$, there exists $J_h \in \mathfrak{A}_S(Y)$ such that $h(q^n_X(\langle J^\circ \rangle_n \cap C_1(X)) - \{F^n_X\}) \subset q^n_Y(\langle J^\circ_h \rangle_n)$ and $\mathfrak{A}_S(Y) = \{J_h : J \in \mathfrak{A}_S(X)\}$. Then*

(A) *for each $J \in \mathfrak{A}_S(X)$, $h(q^n_X(\langle J^\circ \rangle_n) - \{F^n_X\}) = q^n_Y(\langle J^\circ_h \rangle_n) - \{F^n_Y\}$;*
(B) *for each $J \in \mathfrak{A}_S(X)$, $h^{-1}(q^n_Y(\langle J^\circ_h \rangle_n \cap C_1(Y)) - \{F^n_Y\}) \subset q^n_X(\langle J^\circ \rangle_n) - \{F^n_X\}$;*
(C) *the relation $J \mapsto J_h$ is a bijection between $\mathfrak{A}_S(X)$ and $\mathfrak{A}_S(Y)$.*

Proof. This proof is very similar to the proof of parts (A), (B), and (C) of [10, Theorem 3.1]; therefore, we omit the proof. □

Lemma 5.2. *Let $n \geq 2$ and let X be a finite graph with $R(X) \neq \emptyset$. Then*

$$\left| \bigcap \{cl_{PHS_n(X)}(q^n_X(\langle J^\circ \rangle_n) - \{F^n_X\}) : J \in \mathfrak{A}_S(X)\} \right| = 2$$

if and only if X is homeomorphic to θ_m for some $m \in \mathbb{N}$. In fact,

$$\bigcap \{cl_{PHS_n(\theta_m)}(q^n_{\theta_m}(\langle J^\circ \rangle_n) - \{F^n_{\theta_m}\}) : J \in \mathfrak{A}_S(\theta_m)\} =$$

$$\{F^n_{\theta_m},\ q^n_{\theta_m}(\{v, w\})\},$$

where $\{v, w\}$ is the set of ramification points of θ_m. And if X is not homeomorphic to θ_m for any $m \in \mathbb{N}$, then

$$\bigcap \{cl_{PHS_n(X)}(q^n_X(\langle J^\circ \rangle_n) - \{F^n_X\}) : J \in \mathfrak{A}_S(X)\} = \{F^n_X\}.$$

Proof. First, let us see that if X is the graph θ_m, then

$$\left| \bigcap \{cl_{PHS_n(\theta_m)}(q^n_{\theta_m}(\langle J^\circ \rangle_n) - \{F^n_{\theta_m}\}) : J \in \mathfrak{A}_S(\theta_m)\} \right| = 2.$$

Let v and w be the two ramification points of θ_m, so $R(\theta_m) = \{v, w\}$. Let $J \in \mathfrak{A}_s(\theta_m)$. Because each $\{p\}$, where $p \in J^\circ$, can be approximated by elements in $\langle J^\circ \rangle_n - F_1(\theta_m)$, we have that $\{p\} \in cl_{C_n(\theta_m)}(\langle J^\circ \rangle_n - F_1(\theta_m))$; this implies that $F^n_{\theta_m} \in cl_{PHS_n(\theta_m)}(q^n_{\theta_m}(\langle J^\circ \rangle_n) - \{F^n_{\theta_m}\})$. Also since $n \geq 2$, $\{v, w\}$ can be approximated by elements in $\langle J^\circ \rangle_n - F_1(\theta_m)$. Hence, $\{v, w\} \in cl_{C_n(\theta_m)}(\langle J^\circ \rangle_n - F_1(\theta_m))$; thus,

$$q^n_{\theta_m}(\{v, w\}) \in cl_{PHS_n(\theta_m)}(q^n_{\theta_m}(\langle J^\circ \rangle_n) - \{F^n_{\theta_m}\}).$$

So we have

$$\{F^n_{\theta_m},\ q^n_{\theta_m}(\{v, w\})\} \subset \bigcap \{cl_{PHS_n(\theta_m)}(q^n_X(\langle J^\circ \rangle_n) - \{F^n_{\theta_m}\}) : J \in \mathfrak{A}_S(\theta_m)\}.$$

Now let us assume that

$$\chi \in \bigcap\{cl_{PHS_n(\theta_m)}(q^n_{\theta_m}(\langle J^\circ\rangle_n) - \{F^n_{\theta_m}\}) : J \in \mathfrak{A}_S(\theta_m)\} - \{F^n_{\theta_m},\ q^n_{\theta_m}(\{v,w\})\}.$$

Let $J_0 \in \mathfrak{A}_S(\theta_m)$, and let $\{A_k\}_{k=1}^\infty$ be a sequence of elements of $\langle J_0^\circ\rangle_n$ such that $\lim q^n_{\theta_m}(A_k) = \chi$. Then it is clear that $\lim A_k = A$ for some $A \in \langle J_0\rangle_n$. Since $\chi \neq F^n_{\theta_m}$, $\chi \neq q^n_{\theta_m}(\{v,w\})$, and $q^n_{\theta_m}(A) = \chi$, we have that $A \in C_n(J_0) - (F_1(J_0) \cup \{\{v,w\}\})$. This implies that at least one component of A intersects J_0°. Then $A \cap J_0^\circ \neq \emptyset$. Let $J \in \mathfrak{A}_S(\theta_m)$ be such that $J \neq J_0$. By a similar argument to the one just given, there exists $D \in C_n(J) - (F_1(J) \cup \{\{v,w\}\})$ such that $q^n_\theta(D) = \chi$. Since $\chi \neq F^n_{\theta_m}$, we have that $D = A$. Hence, $J_0^\circ \cap J \neq \emptyset$, which is a contradiction. Thus,

$$\{F^n_X\} \cup \{q^n_X(\{v,w\})\} = \bigcap\{cl_{PHS_n(\theta_m)}(q^n_X(\langle J^\circ\rangle_n) - \{F^n_{\theta_m}\}) : J \in \mathfrak{A}_S(\theta_m)\}.$$

Therefore, $|\bigcap\{cl_{PHS_n(\theta_m)}(q^n_X(\langle J^\circ\rangle_n) - \{F^n_{\theta_m}\}) : J \in \mathfrak{A}_S(\theta_m)\}| = 2$.

Now, we will show that if X is a finite graph such that $R(X) \neq \emptyset$ and it is not the graph θ_m for any $m \in \mathbb{N}$, then

$$\bigcap\{cl_{PHS_n(X)}(q^n_X(\langle J^\circ\rangle_n) - \{F^n_X\}) : J \in \mathfrak{A}_S(X)\} = \{F^n_X\}.$$

Let X be a finite graph such that X is not the graph θ_m for any $m \in \mathbb{N}$. Let $J \in \mathfrak{A}_s(X)$. Since each element $\{p\}$ such that $p \in J^\circ$ can be approximated by elements in $\langle J^\circ\rangle_n - F_1(X)$, we have that $\{p\} \in cl_{C_n(X)}(\langle J^\circ\rangle_n - F_1(X))$.

Hence, $F^n_X \in cl_{PHS_n(X)}(q^n_X(\langle J^\circ\rangle) - \{F^n_X\})$. Therefore,

$$F^n_X \in \bigcap\{cl_{PHS_n(X)}(q^n_X(\langle J^\circ\rangle_n) - \{F^n_X\}) : J \in \mathfrak{A}_S(X)\}.$$

Now let us assume that

$$\chi \in \bigcap\{cl_{PHS_n(X)}(q^n_X(\langle J^\circ\rangle_n) - \{F^n_X\}) : J \in \mathfrak{A}_S(X)\} - \{F^n_X\}.$$

Let $J_0 \in \mathfrak{A}_S(X)$. Then there exists a sequence $\{A_k\}_{k=1}^\infty$ in $\langle J_0^\circ\rangle_n$ such that $\lim q^n_X(A_k) = \chi$. It is clear that $\lim A_k = A$ for some $A \in \langle J_0\rangle_n$. Since $\chi \neq F^n_X$ and $q^n_X(A) = \chi$, we have that $A \in C_n(J_0) - F_1(J_0)$, which implies that at least one component of A intersects J_0°, or that A is the set $R(X) \cap J_0$; note that A, in the latter case, must be a two point set since $A \in C_n(J_0) - F_1(J_0)$, say $A = \{a,b\}$.

Let us assume that we have the first situation: $A \cap J_0^\circ \neq \emptyset$. Let $J \in \mathfrak{A}_S(X)$ be such that $J \neq J_0$. By a similar argument to the one given above, there exists $D \in C_n(J) - F_1(J)$ such that $q^n_X(D) = \chi$. Since $\chi \neq F^n_X$, we have that $D = A$. Hence, $J_0^\circ \cap J \neq \emptyset$, which is a contradiction.

Now let us assume we are in the second situation: $A = \{a,b\}$ with $a \neq b$. Since the only finite graph with more than one vertex in which all its edges share the same set of vertices is a θ_m-graph, we can assure, since X is not a θ_m-graph, that there exists $J \in \mathfrak{A}_S(X)$ such that $J \neq J_0$

and such that $J_0 \cap J \neq \{a,b\}$. By a similar argument as the one given before, there exists $D \in C_n(J) - F_1(J)$ such that $q_X^n(D) = \chi$ and, since $\chi \neq F_X^n$, we have that $D = A$. This implies that $\{a,b\} \in C_n(J) - F_1(J)$. Therefore, J and J_0 have the same set of ramification points of X, which is a contradiction. This means that for a finite graph such that $R(X) \neq \emptyset$ and such that it is not a θ_m-graph for any $m \in \mathbb{N}$, we have

$$\bigcap\{cl_{PHS_n(X)}(q_X^n(\langle J^\circ\rangle_n) - \{F_X^n\}) : J \in \mathfrak{A}_S(X)\} = \{F_X^n\}.$$

Therefore, if X is not the graph θ_m for any $m \in \mathbb{N}$, we have that

$$\mid \bigcap\{cl_{PHS_n(X)}(q_X^n(\langle J^\circ\rangle_n) - \{F_X^n\}) : J \in \mathfrak{A}_S(X)\} \mid = \mid \{F_X^n\} \mid = 1. \qquad \square$$

Theorem 5.3. *Let X be a continuum and let $n \in \mathbb{N}$. If $PHS_n(X)$ is homeomorphic to $PHS_n([0,1])$, then X is homeomorphic to $[0,1]$. And if $PHS_n(X)$ is homeomorphic to $PHS_n(S^1)$, then X is homeomorphic to S^1.*

Proof. For the case $n = 1$, see [18, theorems 5.5 and 5.6]. For the case $n \geq 2$, see [15, Theorem 4.4]. $\square$

Now we will show that the graph θ_m has unique n-fold pseudo-hyperspace suspension.

Theorem 5.4. *If $n, m \in \mathbb{N}$, then the graph θ_m has unique n-fold pseudo-hyperspace suspension.*

Proof. Let Y be a continuum, let $m \in \mathbb{N}$, and let $h : PHS_n(\theta_m) \to PHS_n(Y)$ be a homeomorphism. Then, by [15, Corollary 4.6] and Theorem 5.3, Y is a finite graph such that $R(Y) \neq \emptyset$.

Case 1: $n = 1$.

Since $PHS_1(X)$ is equal to $HS_1(X)$ for any continuum X, the main result in [10] asserts that Y is homeomorphic to θ_m.

Case 2: $n \geq 3$.

We check that θ_m and Y satisfy the hypothesis of Lemma 5.1. Since h is a homeomorphism, $h(\mathcal{PHL}_n(\theta_m)) = \mathcal{PHL}_n(Y)$, which, recalling the definition of $\mathcal{PHD}_n(X)$, implies that $h(\mathcal{PHD}_n(\theta_m)) = \mathcal{PHD}_n(Y)$. Thus, by Lemma 3.3(a), for each $J \in \mathfrak{A}_S(\theta_m)$, there exists $J_h \in \mathfrak{A}_S(Y)$ such that

$$h(q_{\theta_m}^n(\langle J^\circ\rangle_n \cap C_1(\theta_m)) - \{F_{\theta_m}^n\}) = q_Y^n(\langle J_h^\circ\rangle_n \cap C_1(Y)) - \{F_Y^n\} \subset q_Y^n(\langle J_h^\circ\rangle_n) - \{F_Y^n\}.$$

Since there is a one-to-one correspondence between the set of components of $\mathcal{PHD}_n(\theta_m)$ and the set of components of $\mathcal{PHD}_n(Y)$, we obtain that the relation $J \mapsto J_h$ between $\mathfrak{A}_S(\theta_m)$ and $\mathfrak{A}_S(Y)$ is a bijection. Let $R(\theta_m) = \{v, w\}$. Then, by lemmas 5.1 and 5.2,

$$
\begin{aligned}
2 = &| h(\{F^n_{\theta_m},\ q^n_{\theta_m}(\{v,w\})\}) | = \\
&| \bigcap\{cl_{PHS_n(Y)}(h(q^n_{\theta_m}(\langle J^\circ\rangle_n) - \{F^n_{\theta_m}\})) : J \in \mathfrak{A}_S(\theta_m)\} | = \\
&| \bigcap\{cl_{PHS_n(Y)}(q^n_Y(\langle J^\circ_h\rangle_n) - \{F^n_Y\}) : J \in \mathfrak{A}_S(\theta_m)\} | = \\
&| \bigcap\{cl_{PHS_n(Y)}(q^n_Y(\langle J^\circ_h\rangle_n) - \{F^n_Y\}) : J_h \in \mathfrak{A}_S(Y)\} | .
\end{aligned}
$$

Hence, by Lemma 5.2, Y is homeomorphic to the graph θ_m.

Case 3: $n = 2$.

First, we check that for each $J \in \mathfrak{A}_S(\theta_m)$, there exists $J_h \in \mathfrak{A}_S(Y)$ such that $h(q^2_{\theta_m}(\langle J^\circ\rangle_2) - \{F^2_{\theta_m}\}) = q^2_Y(\langle J^\circ_h\rangle_2) - \{F^2_Y\}$.

Since the definition of $\mathcal{PHE}_2(\theta_m)$ is given in terms of topological properties, we have that $h(\mathcal{PHE}_2(\theta_m)) = \mathcal{PHE}_2(Y)$. By Lemma 3.3(b), the components of $\mathcal{PHE}_2(\theta_m)$ are the sets of the form $q^2_{\theta_m}(\langle J^\circ, K^\circ\rangle_2) - \{F^2_{\theta_m}\}$, where $J, K \in \mathfrak{A}_S(\theta_m)$ and the components of $\mathcal{PHE}_2(Y)$ are the sets of the form $q^2_Y(\langle J^\circ, K^\circ\rangle_2) - \{F^2_Y\}$, where $J, K \in \mathfrak{A}_S(Y)$. Therefore, given $J \in \mathfrak{A}_S(\theta_m)$, there exist $J_h, K_h \in \mathfrak{A}_S(Y)$ such that $h(q^2_{\theta_m}(\langle J^\circ\rangle_2) - \{F^2_{\theta_m}\}) = q^2_Y(\langle J^\circ_h, K^\circ_h\rangle_2) - \{F^2_Y\}$. By Lemma 3.1, $F^2_{\theta_m} \notin \partial\mathcal{PHL}_2(\theta_m)$. And since $R(Y) \neq \emptyset$, by Lemma 3.1, $F^2_Y \notin \partial\mathcal{PHL}_2(Y)$. Since the definition of $\partial\mathcal{PHL}_2(\theta_m)$ is given in terms of topological properties, we have that $h(\partial\mathcal{PHL}_2(\theta_m)) = \partial\mathcal{PHL}_2(Y)$. This implies that

$$h(\partial\mathcal{PHL}_2(\theta_m) \cap q^2_{\theta_m}(\langle J^\circ\rangle_2)) = \partial\mathcal{PHL}_2(Y) \cap q^2_Y(\langle J^\circ_h, K^\circ_h\rangle_2)$$

and

$$h(\partial\mathcal{PHL}_2(\theta_m) - q^2_{\theta_m}(\langle J^\circ\rangle_2)) = \partial\mathcal{PHL}_2(Y) - q^2_Y(\langle J^\circ_h, K^\circ_h\rangle_2).$$

Hence, $h(\mathcal{PHD}(J,J)) = \mathcal{PHD}(J_h, K_h)$. Let us assume that $J_h \neq K_h$. Then $\mathcal{PHD}(J,J)$ is homeomorphic to $\mathcal{PHD}(J_h, K_h)$ with $J_h \neq K_h$, which implies that a model in one of the cases A, B, or C is homeomorphic to a model in one of the cases D, E, F, G, H, or I (given in the previous section), clearly, a contradiction. We conclude that $J_h = K_h$. Therefore, $h(q^2_{\theta_m}(\langle J^\circ\rangle_2) - \{F^2_{\theta_m}\}) = q^2_Y(\langle J^\circ_h\rangle_2) - \{F^2_Y\}$, which induces the relation $J \mapsto J_h$. By symmetry, the relation $J \mapsto J_h$ from $\mathfrak{A}_S(\theta_m) \to \mathfrak{A}_S(Y)$ is a bijection.

Now, let $R(\theta_m) = \{v, w\}$. Thus, by Lemma 5.2,

$$
\begin{aligned}
2 = &| h(\{F^2_{\theta_m},\ q^2_{\theta_m}(\{v,w\})\}) | = \\
&| \bigcap\{cl_{PHS_2(Y)}(h(q^2_{\theta_m}(\langle J^\circ\rangle_2) - \{F^2_{\theta_m}\})) : J \in \mathfrak{A}_S(\theta_m)\} | = \\
&| \bigcap\{cl_{PHS_2(Y)}(q^2_Y(\langle J^\circ_h\rangle_2) - \{F^2_Y\}) : J \in \mathfrak{A}_S(\theta_m)\} | = \\
&| \bigcap\{cl_{PHS_2(Y)}(q^2_Y(\langle J^\circ_h\rangle_2) - \{F^2_Y\}) : J_h \in \mathfrak{A}_S(Y)\} | .
\end{aligned}
$$

Therefore, by Lemma 5.2, Y is homeomorphic to the graph θ_m. □

Lemma 5.5. *Let X and Y be finite graphs such that $R(X) \neq \emptyset \neq R(Y)$, and such that X is not a θ_m-graph for any $m \in \mathbb{N}$. Let $n \in \mathbb{N}$, and let $h : PHS_n(X) \to PHS_n(Y)$ be a homeomorphism. Assume that*

for each $J \in \mathfrak{A}_S(X)$, there exists $J_h \in \mathfrak{A}_S(Y)$ such that
$h(q_X^n(\langle J^\circ \rangle_n \cap C_1(X)) - \{F_X^n\}) \subset q_Y^n(\langle J_h^\circ \rangle_n)$
and $\mathfrak{A}_S(Y) = \{J_h \ : \ J \in \mathfrak{A}_S(X)\}$;

then $h(F_X^n) = F_Y^n$. And if we additionally assume that

(1) *if $J \in \mathfrak{A}_R(X)$, then $J_h \in \mathfrak{A}_R(Y)$, and*
(2) *if $J \in \mathfrak{A}_E(X)$, then $J_h \in \mathfrak{A}_E(Y)$,*

then X is homeomorphic to Y.

Proof. First, let us assume that for each $J \in \mathfrak{A}_S(X)$, there exists $J_h \in \mathfrak{A}_S(Y)$ such that $h(q_X^n(\langle J^\circ \rangle_n \cap C_1(X)) - \{F_X^n\}) \subset q_Y^n(\langle J_h^\circ \rangle_n)$ and $\mathfrak{A}_S(Y) = \{J_h \ : \ J \in \mathfrak{A}_S(X)\}$. We will show that $h(F_X^n) = F_Y^n$.

Since X is not a θ_m-graph for any $m \in \mathbb{N}$, by Lemma 5.2, we have that

$$\bigcap \{cl_{PHS_n(X)}(q_X^n(\langle J^\circ \rangle_n) - \{F_X^n\}) : J \in \mathfrak{A}_S(X)\} = \{F_X^n\},$$

and, since the θ_m-graphs have unique n-fold pseudo-hyperspace suspension, a fact proven in Theorem 5.4, we obtain that Y is not a θ_m-graph. Hence, by Lemma 5.2,

$$\bigcap \{cl_{PHS_n(Y)}(q_Y^n(\langle J^\circ \rangle_n) - \{F_Y^n\}) : J \in \mathfrak{A}_S(Y)\} = \{F_Y^n\}.$$

Now, we can use Lemma 5.1(A),(C) to obtain

$$\begin{aligned} & h(\{F_X^n\}) = \\ & \quad \bigcap \{cl_{PHS_n(Y)}(h(q_X^n(\langle J^\circ \rangle_n) - \{F_X^n\})) : J \in \mathfrak{A}_S(X)\} = \\ & \quad \bigcap \{cl_{PHS_n(Y)}(q_Y^n(\langle J_h^\circ \rangle_n) - \{F_Y^n\}) : J_h \in \mathfrak{A}_S(Y)\} = \{F_Y^n\}. \end{aligned}$$

Thus, $h(\{F_X^n\}) = \{F_Y^n\}$.

Now, assume hypotheses (1) and (2). It is easy to see that a construction of a homeomorphism between X and Y can be made in a similar fashion as the one constructed in the proof of [10, Theorem 3.1]; details are left to the reader. □

Theorem 5.6. *If X is a finite graph such that $R(X) \neq \emptyset$, then X has a unique n-fold pseudo-hyperspace suspension.*

Proof. By Theorem 5.4, we assume that X and Y are not homeomorphic to a θ_m-graph for any $m \in \mathbb{N}$. Let Y be a continuum, let $n \in \mathbb{N}$, and let $h : PHS_n(X) \to PHS_n(Y)$ be a homeomorphism. Then, by [15, Corollary 4.6], Y is a finite graph, and Y is such that $R(Y) \neq \emptyset$ (Theorem 5.3).

Case 1: $n = 1$.

Since $PHS_1(X)$ is equal to $HS_1(X)$ for any continuum X, the main result of [10] asserts that Y is homeomorphic to X.

Case 2: $n \geq 3$.

Using Lemma 3.3(a) and Lemma 5.5, the proof of this case is very similar to the proof of [10, Theorem 3.2, Case 1].

Case 3: $n = 2$.

We check that X and Y satisfy the hypothesis of Lemma 5.5.

Let $J \in \mathfrak{A}_S(X)$. By Lemma 3.3(b), there exist $J_h, K_h \in \mathfrak{A}_S(Y)$ such that $h(q_X^2(\langle J^\circ\rangle_2) - \{F_X^2\}) = q_Y^2(\langle J_h^\circ, K_h^\circ\rangle_2) - \{F_Y^2\}$; therefore, giving a similar argument as the one given in the proof of Theorem 5.4, Case 3, we have that $h(\mathcal{PHD}(J, J)) = \mathcal{PHD}(J_h, K_h)$. Now, if we assume $J_h \neq K_h$, then we have that $h(\mathcal{PHD}(J, J)) = \mathcal{PHD}(J_h, K_h)$ with $J_h \neq K_h$; using the models constructed in the past section, we see that this is a contradiction. Therefore, $h(q_X^2(\langle J^\circ\rangle_2) - \{F_X^2\}) = q_Y^2(\langle J_h^\circ\rangle_2) - \{F_Y^2\}$, which induces a relation $J \mapsto J_h$. By symmetry, the relation $J \mapsto J_h$ from $\mathfrak{A}_S(X)$ into $\mathfrak{A}_S(Y)$ is a bijection. Since $h(q_X^2(\langle J^\circ\rangle_2) - \{F_X^2\}) = q_Y^2(\langle J_h^\circ\rangle_2) - \{F_Y^2\}$, it is immediate that $h(q_X^2(\langle J^\circ\rangle_2 \cap C_1(X)) - \{F_X^2\}) \subset q_Y^2(\langle J_h^\circ\rangle_2)$.

So, in order to use Lemma 5.5 to prove that X is homeomorphic to Y, there is only left to show that (1) if $J \in \mathfrak{A}_E(X)$, then $J_h \in \mathfrak{A}_E(Y)$, and (2) if $J \in \mathfrak{A}_R(X)$, then $J_h \in \mathfrak{A}_R(Y)$.

If $J \in \mathfrak{A}_E(X)$, then $\mathcal{PHD}(J, J)$ is homeomorphic to the model in Case B of the past section. Since $h(\mathcal{PHD}(J, J)) = \mathcal{PHD}(J_h, J_h)$, we obtain that $\mathcal{PHD}(J_h, J_h)$ is also homeomorphic to the model in Case B; thus, since none of the models in cases A and C are homeomorphic to the model in case B, we conclude that $J_h \in \mathfrak{A}_E(Y)$. Similarly, if $J \in \mathfrak{A}_R(X)$, then $J_h \in \mathfrak{A}_R(Y)$.

Hence, X, Y, and h satisfy the hypothesis of Lemma 5.5. Therefore, X is homeomorphic to Y. $\square$

So as a consequence of theorems 5.3 and 5.6, we have our main result.

Theorem 5.7. *Let X be a finite graph, then X has unique n-fold pseudo-hyperspace suspension.*

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