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TOPOLOGY PROCEEDINGS

Volume 52, 2018

Pages 235–244

http://topology.nipissingu.ca/tp/

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Electronically published on February 5, 2018

Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 52 (2018)
Pages 235-244

E-Published on February 5, 2018

ON ORDERED TOPOLOGICAL VECTOR SPACES WITH POSITIVE INTERIOR POINTS

KAORI YAMAZAKI

Abstract. Answering a question indicated by Er-Guang Yang in (*Partial answers to some questions on maps to ordered topological vector spaces*, Topology Proc. **50** (2017), 311–317), we show that, for an ordered topological vector space Y with positive interior points, if each non-zero positive element is an order unit, then Y is isomorphic to the real line. We also provide a technique which reduces some vector-valued results to the original real-valued ones by using some Minkowski functionals.

1. Introduction

Throughout this paper, let $\mathbb{R}$ be the set of all real numbers, and $\mathbb{N}$ the set of all natural numbers.

Let us recall some terminology from [1] and [5]. A partially ordered real vector space $(Y, \leq)$ is said to be an *ordered vector space* if the following conditions are satisfied:

(i) $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in Y$,
(ii) $x \leq y$ implies $rx \leq ry$ for all $x, y \in Y$ and all $r \in \mathbb{R}$ with $r \geq 0$.

Let $(Y, \leq)$ be an ordered vector space. Then, $y \in Y$ is *positive* if $\mathbf{0} \leq y$, and the set $\{y \in Y : \mathbf{0} \leq y\}$, called the *positive cone* of Y, is denoted by Y^+. For $y_1, y_2 \in Y$ with $y_1 \leq y_2$, the subspace $(y_1 + Y^+) \cap (y_2 - Y^+)$ of Y, called an *order interval*, is denoted by $[y_1, y_2]$. A topological vector space Y is called an *ordered topological vector space* (*o.t.v.s.*) if Y is an ordered vector space such that the positive cone Y^+ is closed in Y. It is

2010 *Mathematics Subject Classification.* Primary 54C08, 54D20, 46A40.
Key words and phrases. ordered topological vector space, positive interior point, order unit, Minkowski functional, monotonically countably paracompact.
This work was supported by TCUE Special Grant-in-Aid (A) 2017.

known that $e \in Y^+$ is an interior point of Y^+ if and only if $[-e, e]$ is a $\mathbf{0}$-neighborhood [1, Lemma 2.5]. An interior point e of Y^+ is a *positive interior point* of Y if $e > \mathbf{0}$. A point $x \in Y^+$ is called an *order unit* if each $y \in Y$ admits $\lambda > 0$ such that $y \leq \lambda x$. The positive cone Y^+ is *normal* if each $\mathbf{0}$-neighborhood U admits a $\mathbf{0}$-neighborhood V such that $(V + Y^+) \cap (V - Y^+) \subset U$.

Let $f : X \to Y$ be a map from a topological space X into an o.t.v.s. Y. Then f is said to be *lower semi-continuous* (*l.s.c.*) (*upper semi-continuous* (*u.s.c.*), respectively), if for each $x \in X$ and each $\mathbf{0}$-neighborhood U, there exists a neighborhood O_x of x such that $f(O_x) \subset f(x) + U + Y^+$ ($f(O_x) \subset f(x) + U - Y^+$, respectively) (see [2] and [7]). Every continuous map $f : X \to Y$ from a topological space X into an o.t.v.s. Y is l.s.c. and u.s.c., and the converse holds if Y^+ is normal.

The following are obtained in [4, Theorem 25], [3, Theorem 3], and [6, Corollary 3.3]. See [4] for the definition of monotonically countably paracompact spaces. The symbol $(0, \infty)$ stands for the set $\{r \in \mathbb{R} : r > 0\}$.

Theorem 1.1 ([4], [3], [6])**.** *The following statements are equivalent:*

(a) *X is monotonically countably paracompact.*

(b) *There exist operators Φ and Ψ assigning to each u.s.c. function $f : X \to \mathbb{R}$ an l.s.c. function $\Phi(f) : X \to \mathbb{R}$ and a u.s.c. function $\Psi(f) : X \to \mathbb{R}$ with $f \leq \Phi(f) \leq \Psi(f)$ such that $\Phi(f) \leq \Phi(f')$ and $\Psi(f) \leq \Psi(f')$ whenever $f \leq f'$.*

(c) *There exist operators Φ and Ψ assigning to each l.s.c. function $f : X \to (0, \infty)$, a u.s.c. function $\Phi(f) : X \to (0, \infty)$ and an l.s.c. function $\Psi(f) : X \to (0, \infty)$ with $\Psi(f) \leq \Phi(f) \leq f$ such that $\Phi(f) \leq \Phi(f')$ and $\Psi(f) \leq \Psi(f')$ whenever $f \leq f'$.*

In [7], we extend earlier characterizations ([4], [3]) of monotonically countably paracompact spaces X via real-valued functions into ones via some vector-valued maps. For example, it is shown in [7, Theorem 3.1] that $\mathbb{R}$ in Theorem 1.1(b) can be replaced by an o.t.v.s. Y with positive interior points. Namely, for an o.t.v.s. Y with positive interior points, the following condition is equivalent to each of (a), (b), and (c) in Theorem 1.1.

(b′) There exist operators Φ and Ψ assigning to each u.s.c. map $f : X \to Y$ an l.s.c. map $\Phi(f) : X \to Y$ and a u.s.c. map $\Psi(f) : X \to Y$ with $f \leq \Phi(f) \leq \Psi(f)$ such that $\Phi(f) \leq \Phi(f')$ and $\Psi(f) \leq \Psi(f')$ whenever $f \leq f'$.

On the other hand, [7, Question 5.1.3] asks if, for an o.t.v.s. Y with positive interior points, the following condition is equivalent to X being monotonically countably paracompact.

(c′) There exist operators Φ and Ψ assigning to each l.s.c. map $f : X \to Y^+ \setminus \{\mathbf{0}\}$ a u.s.c. map $\Phi(f) : X \to Y^+ \setminus \{\mathbf{0}\}$ and an l.s.c. map $\Psi(f) : X \to Y^+ \setminus \{\mathbf{0}\}$ with $\Psi(f) \leq \Phi(f) \leq f$ such that $\Phi(f) \leq \Phi(f')$ and $\Psi(f) \leq \Psi(f')$ whenever $f \leq f'$.

Recently, Er-Guang Yang [8] showed that (c′) implies X being monotonically countably paracompact and that the converse holds assuming the condition that "each point of $Y^+ \setminus \{\mathbf{0}\}$ is an order unit." Moreover, in [8, Remark 2.3], Yang asks if there exists an o.t.v.s. with positive interior points satisfying this condition other than $\mathbb{R}$. In §2, we answer this question negatively. Namely, we show that if each point of $Y^+ \setminus \{\mathbf{0}\}$ is an order unit, where Y is an o.t.v.s. with positive interior points, then Y is isomorphic to $\mathbb{R}$ as an o.t.v.s.

In §3, by using Minkowski functionals $p_{[-e,e]}$ and p_e, we give alternative proof to some results in [7] and [8], containing the equivalence (b) $\Leftrightarrow$ (b′). This actually provides the technique for reducing many results of maps to o.t.v.s.'s with positive interior points into the original real-valued one.

2. Spaces Y such that Each Point of $Y^+ \setminus \{\mathbf{0}\}$ Is an Order Unit

Let Y be an o.t.v.s. with a positive interior point e. Since each point of $Y^+ \setminus \{\mathbf{0}\}$ is an order unit if and only if each $y \in Y^+ \setminus \{\mathbf{0}\}$ admits $\lambda > 0$ such that $e \leq \lambda y$, the following proposition provides a negative answer to a question in [8, Remark 2.3].

Proposition 2.1. *Let Y be an o.t.v.s. with a positive interior point e. If each $y \in Y^+ \setminus \{\mathbf{0}\}$ admits $\lambda > 0$ such that $e \leq \lambda y$, then Y is isomorphic to $\mathbb{R}$ as an o.t.v.s.*

Proof. Since Y has a positive interior point, it suffices to show that Y is one-dimensional (as a vector space) because the map $j : \mathbb{R} \to Y$ defined by $j(r) = re$ is surjective when Y is one-dimensional, and hence, it is easy to see that j is a natural isomorphism.

`Fact 1.` $Y^+ \setminus \{\mathbf{0}\} \subset \mathrm{int}_Y Y^+$.

Proof. To show this, let $y \in Y^+ \setminus \{\mathbf{0}\}$. From the assumption, take $\lambda > 0$ such that $e \leq \lambda y$; thus, $\frac{e}{\lambda} \leq y$. It follows from $e \in \mathrm{int}_Y Y^+$ that $\frac{e}{\lambda} \in \mathrm{int}_Y Y^+$. Hence, we have

$$y \in \frac{e}{\lambda} + Y^+ \subset \mathrm{int}_Y Y^+ + Y^+ = \mathrm{int}_Y Y^+.$$

`Fact 2.` $Y^+ \setminus \{\mathbf{0}\}$ is a clopen subset of $Y \setminus \{\mathbf{0}\}$.

Proof. For brevity, set $Y' = Y \setminus \{\mathbf{0}\}$. Since the positive cone Y^+ is closed in Y, $Y^+ \setminus \{\mathbf{0}\}$ is closed in $Y \setminus \{\mathbf{0}\} = Y'$. On the other hand,

$Y^+ \setminus \{\mathbf{0}\} \subset \mathrm{int}_Y Y^+ \cap Y'$ holds by Fact 1. Since $\mathrm{int}_Y Y^+ \cap Y'$ is open in Y', it follows that

$$Y^+ \setminus \{\mathbf{0}\} \subset \mathrm{int}_Y Y^+ \cap Y' = \mathrm{int}_{Y'}(\mathrm{int}_Y Y^+ \cap Y') \subset \mathrm{int}_{Y'}(Y^+ \setminus \{\mathbf{0}\}),$$

which shows that $Y^+ \setminus \{\mathbf{0}\}$ is open in Y'.

Since $e \in Y^+ \setminus \{\mathbf{0}\}$, we have that $-e \in Y' \setminus (Y^+ \setminus \{\mathbf{0}\})$. Thus, $Y^+ \setminus \{\mathbf{0}\}$ is a non-empty clopen subset of Y' and whose complement is also non-empty, namely this implies that $Y'(= Y \setminus \{\mathbf{0}\})$ is disconnected. Thus, Y must be one-dimensional. □

3. Around Minkowski Functionals

Let e be a positive interior point of an o.t.v.s. Y. The *Minkowski functional* (*gauge*) *of* $[-e, e]$ is the function

$$p_{[-e,e]} : Y \to \mathbb{R}^+; \quad y \mapsto \inf\{r > 0 : y \in r[-e, e]\},$$

which is known to be a continuous norm (see, e.g., [5] and Proposition 3.1 (1) below). If the positive cone Y^+ is normal, the norm $p_{[-e,e]}$ carries the vector topology of Y; in this case, $\{\frac{1}{n}[-e, e] : n \in \mathbb{N}\}$ is a neighborhood base of $\mathbf{0}$.

Let Y be an o.t.v.s. A subset $A \subset Y$ is *bounded* (*upper-bounded*, respectively) if for each $\mathbf{0}$-neighborhood V, there exists $n \in \mathbb{N}$ such that $A \subset nV$ ($A \subset nV - Y^+$, respectively). If Y^+ has a positive interior point e, A is upper-bounded if and only if $A \subset ne - Y^+$ for some $n \in \mathbb{N}$. If Y^+ is normal and has a positive interior point e, A is bounded if and only if $A \subset [-ne, ne]$ for some $n \in \mathbb{N}$. For o.t.v.s.'s X and Y, a map $f : X \to Y$ is called *bounded preserving* (*upper-bounded preserving*, respectively) if $f(A)$ is bounded (upper-bounded, respectively) in Y for each bounded (upper-bounded, respectively) subset A of X. Also, $f : X \to Y$ is called *order-preserving* if $f(x) \leq f(x')$ whenever $x \leq x'$. An ordered vector space Y is called a (*vector*) *lattice* if each two-point set $\{x, y\}$ has a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$. For a (vector) lattice Y and $y \in Y$, the symbol $|y|$ stands for $y \vee (-y)$.

Proposition 3.1. *Let Y be an o.t.v.s. with a positive interior point e. For the Minkowski functional $p_{[-e,e]}$, the following are valid.*

(1) *$p_{[-e,e]}$ is continuous and bounded preserving.*
(2) *$p_{[-e,e]}(y) \leq p_{[-e,e]}(y')$ holds whenever $\mathbf{0} \leq y \leq y'$.*
(3) *If Y is a lattice, $p_{[-e,e]}(y) \leq p_{[-e,e]}(y')$ holds whenever $|y| \leq |y'|$.*

Proof. (1) The set $[-e, e]$ is a radial, circled, and convex subset of Y; $p_{[-e,e]}$ is a semi-norm (actually, it is a norm); see, e.g., [5, p. 39]. Since $p_{[-e,e]}$ is continuous at $\mathbf{0}$, it follows from [5, II.1.6] that $p_{[-e,e]}$ is continuous. For each bounded set A of Y, there exists $n \in \mathbb{N}$ such that

$A \subset n[-e,e]$, and thus $p_{[-e,e]}(A) \subset [0,n]$. Hence, $p_{[-e,e]}$ is bounded preserving.

(2) Assume $\mathbf{0} \le y \le y'$. If $y' \in r[-e,e]$, then $-re \le \mathbf{0} \le y \le y' \le re$, which shows that $y \in r[-e,e]$. Thus, $p_{[-e,e]}(y) \le p_{[-e,e]}(y')$ holds.

(3) Let Y be a lattice. Since $y \in r[-e,e] \Leftrightarrow -y \in r[-e,e] \Leftrightarrow |y| \in r[-e,e]$, it follows that $p_{[-e,e]}(y) = p_{[-e,e]}(|y|)$ for each $y \in Y$. Since $\mathbf{0} \le |y|$, we have (3) from (2). □

For the Minkowski functional of $e - Y^+$, we use the symbol p_e instead of p_{e-Y^+} for brevity. Namely,

$$p_e : Y \to \mathbb{R}^+;\ y \mapsto \inf\{r > 0 : y \le re\}.$$

Note that $e - Y^+$ is not circled and p_e is not semi-norm. However, p_e has the following good property.

Proposition 3.2. *Let Y be an o.t.v.s. with a positive interior point e. For the Minkowski functional p_e, the following are valid.*

(1) $p_e(y) = 0 \Leftrightarrow y \le \mathbf{0}$.
(2) $p_e(x+y) \le p_e(x) + p_e(y)$ *for each* $x, y \in Y$.
(3) p_e *is continuous, upper-bounded preserving, bounded preserving, and order-preserving.*

Proof. (1) The "if" part is obvious. To show the "only if" part, let $p_e(y) = 0$. Then, $y \le \frac{1}{n}e$ for each $n \in \mathbb{N}$. Since Y is Archimedean (see, e.g., [1, Lemma 2.3] and [5, p. 205]), $y \le \mathbf{0}$.

(2) For each $r_1 > 0$ with $x \le r_1 e$ and each $r_2 > 0$ with $y \le r_2 e$, it follows from $x + y \le (r_1 + r_2)e$ that $p_e(x+y) \le r_1 + r_2$. This provides that $p_e(x+y) \le p_e(x) + p_e(y)$.

(3) To show p_e is continuous at x, let $\varepsilon > 0$. Take a neighborhood $O = x + [-\frac{\varepsilon}{2}e, \frac{\varepsilon}{2}e]$ of x. It follows from (2) that $p_e(x) - p_e(y) \le p_e(x-y) \le \frac{\varepsilon}{2} < \varepsilon$ and $p_e(y) - p_e(x) \le p_e(y-x) \le \frac{\varepsilon}{2} < \varepsilon$ for each $y \in O$, which shows that $p_e(x) - \varepsilon < p_e(y) < p_e(x) + \varepsilon$ for each $y \in O$. Thus, p_e is continuous at x. On the other hand, for an upper bounded set $A \subset Y$, it follows that $A \subset ne - Y^+$ for some $n \in \mathbb{N}$; hence, $p_e(A) \subset [0,n]$, which provides that p_e is upper-bounded preserving. It is similar to show that p_e is bounded preserving and order-preserving. □

For an o.t.v.s. with a positive interior point e, define another map $j_e : \mathbb{R} \to Y; r \mapsto re$. The map j_e is not only continuous (see [8, p. 314]) but also has the following property.

Proposition 3.3. *Let Y be an o.t.v.s. with a positive interior point e. For the map j_e, the following are valid.*

(1) *j_e is continuous, upper-bounded preserving, bounded preserving, and order-preserving.*
(2) *If Y is a lattice, then $|j_e(r)| \leq |j_e(r')|$ holds whenever $|r| \leq |r'|$.*

Proposition 3.4. *Let Y be an o.t.v.s. with a positive interior point e. For the Minkowski functional $p_{[-e,e]}$, p_e, and the map j_e, the following are valid.*

(1) *$|r| = p_{[-e,e]} \circ j_e(r)$ and $r \vee 0 = p_e \circ j_e(r)$.*
(2) *$y, -y \leq j_e \circ p_{[-e,e]}(y)$, and $y \leq j_e \circ p_e(y)$.*

The proofs of propositions 3.3 and 3.4 are easy and omitted.

Let X be a topological space, Y an o.t.v.s., and $f : X \to Y$ a map. f is said to be *locally upper-bounded* (*locally bounded*, respectively) [7] if for each $x \in X$ and each $\mathbf{0}$-neighborhood V, there exist a neighborhood O_x of x and $n \in \mathbb{N}$ such that $f(O_x) \subset nV - Y^+$ ($f(O_x) \subset nV$, respectively). This definition is a natural generalization of local (upper-) boundedness of real-valued functions, where $f : X \to \mathbb{R}$ is *locally upper-bounded* (*locally bounded*, respectively), if for each $x \in X$, there exist a neighborhood O_x of x and $n \in \mathbb{N}$ such that $f(x') < n$ ($|f(x')| < n$, respectively) for each $x' \in O_x$.

The following lemma is essentially proved in [7, Proposition 2.4 and Corollary 2.8].

Lemma 3.5 ([7]). *For a topological space X, an o.t.v.s. Y, and a map $f : X \to Y$, the implications (1) $\Rightarrow$ (2) and (1$'$) $\Rightarrow$ (2$'$) hold.*

(1) *Each $x \in X$ admits a neighborhood O_x of x such that $f(O_x)$ is upper-bounded.*
(1$'$) *Each $x \in X$ admits a neighborhood O_x of x such that $f(O_x)$ is bounded.*
(2) *f is locally upper-bounded.*
(2$'$) *f is locally bounded.*

If, in addition, Y has positive interior points (Y^+ is normal and Y has positive interior points, respectively), (1) $\Leftrightarrow$ (2) ((1$'$) $\Leftrightarrow$ (2$'$), respectively) holds.

Proposition 3.6. *Let X be a topological space, Y an o.t.v.s. with a positive interior point e, and Z an o.t.v.s. Then the following are valid.*

(1) *If $f : X \to Y$ is locally upper-bounded and $g : Y \to Z$ is upper-bounded preserving, then $g \circ f$ is locally upper-bounded.*
(2) *If Y^+ is normal, $f : X \to Y$ is locally bounded, and $g : Y \to Z$ is bounded preserving, then $g \circ f$ is locally bounded.*
(3) *If $f : X \to Y$ is locally bounded, then $p_{[-e,e]} \circ f$ is locally bounded.*

Proof. (1) Let $f : X \to Y$ be locally upper-bounded and $g : Y \to Z$ be upper-bounded preserving, and let $x \in X$. Since f is locally upper-bounded, by Lemma 3.5, there exists a neighborhood O_x of x such that $f(O_x)$ is upper-bounded. Since g is upper-bounded preserving, $g \circ f(O_x)$ is upper-bounded. By applying Lemma 3.5 to $g \circ f$, we have that $g \circ f$ is locally upper-bounded.

(2) is similar to (1).

(3) Let $x \in X$. Since $[-e, e]$ is a $\mathbf{0}$-neighborhood in Y, there exist a neighborhood O_x of x and $n \in \mathbb{N}$ such that $f(O_x) \subset n[-e, e]$. Then $p_{[-e,e]} \circ f(O_x) \subset [0, n]$, which shows that $p_{[-e,e]} \circ f$ is locally bounded. □

Proposition 3.7. *Let X be a topological space and let Y and Z be o.t.v.s.'s. Then, the following are valid.*

(1) *If $f : X \to Y$ is l.s.c. (u.s.c., respectively) and $g : Y \to Z$ is order-preserving l.s.c. (order-preserving u.s.c., respectively), then $g \circ f$ is l.s.c. (u.s.c., respectively).*
(2) *If Y has a positive interior point e and $f : X \to \mathrm{int}_Y Y^+$ is l.s.c., then $p_{[-e,e]} \circ f$ is l.s.c.*

Proof. (1) We show only the case of maps being l.s.c. Let $f : X \to Y$ be l.s.c., $g : Y \to Z$ be order-preserving l.s.c., $x \in X$, and V be a $\mathbf{0}$-neighborhood in Z. Since g is l.s.c., take a $\mathbf{0}$-neighborhood W in Y such that $g(f(x) + W) \subset (g \circ f)(x) + V + Z^+$. Since f is l.s.c., take a neighborhood O_x of x such that $f(O_x) \subset f(x)+W+Y^+$. Then $g\circ f(O_x) \subset g(f(x) + W + Z^+) \subset g(f(x) + W) + Z^+ \subset (g \circ f)(x) + V + Z^+$, where the second inclusion follows from being order-preserving of g. Thus, $g \circ f$ is l.s.c.

(2) Let e be a positive interior point of Y, $f : X \to \mathrm{int}_Y Y^+$ be an l.s.c. map, and $x \in X$. To show $p_{[-e,e]} \circ f : X \to \mathbb{R}$ is l.s.c., let $\varepsilon > 0$. Since $p_{[-e,e]} \circ f(x) > 0$, without loss of generality, we may assume $p_{[-e,e]} \circ f(x) - \varepsilon > 0$. Since $f(x) \in \mathrm{int}_Y Y^+$, there exists a $\mathbf{0}$-neighborhood U such that $f(x) + U \subset Y^+$. Take $\varepsilon' > 0$ such that $\varepsilon' \leq \varepsilon/2$ and $-\varepsilon' e \in U$. Then $f(x) - \varepsilon' e \in Y^+$ and $f(x) + \varepsilon'[-e, e] + Y^+ = f(x) - \varepsilon' e + Y^+ \subset Y^+$. Since $\varepsilon'[-e, e]$ is a $\mathbf{0}$-neighborhood and f is l.s.c., there exists a neighborhood O_x of x such that $f(O_x) \subset f(x) - \varepsilon' e + Y^+$. For each $z \in O_x$, it follows from $\mathbf{0} \leq f(x) - \varepsilon' e \leq f(z)$ that

$$p_{[-e,e]} \circ f(x) - p_{[-e,e]}(\varepsilon' e) \leq p_{[-e,e]}(f(x) - \varepsilon' e) \leq p_{[-e,e]} \circ f(z),$$

where the first inequality follows from $p_{[-e,e]}$ being a semi-norm (see [5, p. 39]), and the second one is due to Proposition 3.1(2). Because $p_{[-e,e]}(\varepsilon' e) = \varepsilon'$, we have that

$$p_{[-e,e]} \circ f(x) - \varepsilon < p_{[-e,e]} \circ f(x) - \varepsilon/2 \leq p_{[-e,e]} \circ f(x) - \varepsilon' \leq p_{[-e,e]} \circ f(z).$$

Hence, $p_{[-e,e]} \circ f$ is l.s.c. □

Propositions 3.2, 3.3, 3.6, and 3.7 provide the following.

Corollary 3.8. *For a topological space X and an o.t.v.s. Y with a positive interior point e, the following are valid.*

(1) *If $f : X \to Y$ is locally upper-bounded (locally bounded, l.s.c., u.s.c, respectively), then $p_e \circ f$ is locally upper-bounded (locally bounded, l.s.c., u.s.c, respectively).*
(2) *If $f : X \to \mathbb{R}$ is locally upper-bounded (locally bounded, l.s.c., u.s.c, respectively), then $j_e \circ f$ is locally upper-bounded (locally bounded, l.s.c., u.s.c, respectively).*

Proof. (1) The case of maps being locally bounded can be proved in a way similar to the proof of Proposition 3.6(3). Other cases follow from propositions 3.2(3), 3.6(1), and 3.7(1).

(2) Note that $\mathbb{R}^+$ is normal. Apply propositions 3.3(1); 3.6(1), (2); and 3.7(1). □

Let Y be an o.t.v.s. with a positive interior point e. We now give a direct proof to (b) $\Leftrightarrow$ (b′) by using the facts of this section.

(b) $\Rightarrow$ (b′): Assume (b) and let Φ and Ψ be operators as in (b). Let $f : X \to Y$ be a u.s.c. map. By Corollary 3.8(1), $p_e \circ f : X \to \mathbb{R}^+$ is u.s.c. Hence, it follows from (b) that $\Phi(p_e \circ f) : X \to \mathbb{R}^+$ is l.s.c. and $\Psi(p_e \circ f) : X \to \mathbb{R}^+$ is u.s.c. Then $j_e \circ \Phi(p_e \circ f) : X \to Y$ is l.s.c. and $j_e \circ \Psi(p_e \circ f) : X \to Y$ is u.s.c. by Corollary 3.8(2). It follows from propositions 3.4(2) and 3.3(1) that $f(x) \leq j_e \circ (p_e \circ f)(x) \leq j_e \circ \Phi(p_e \circ f)(x) \leq j_e \circ \Psi(p_e \circ f)(x)$. Since p_e and j_e are order-preserving (propositions 3.2(3) and 3.3(1)), we can check that $j_e \circ \Phi(p_e \circ f) \leq j_e \circ \Phi(p_e \circ f')$ and $j_e \circ \Psi(p_e \circ f) \leq j_e \circ \Psi(p_e \circ f')$ whenever $f \leq f'$. Thus, the operators assigning to each u.s.c. map f an l.s.c. map $j_e \circ \Phi(p_e \circ f)$ and a u.s.c. map $j_e \circ \Psi(p_e \circ f)$ are required ones in (b′).

(b′) $\Rightarrow$ (b): To prove (b′) $\Rightarrow$ (b), assume (b′) and let Φ and Ψ be operators as in (b′). Let $f : X \to \mathbb{R}$ be a u.s.c. function. By Corollary 3.8(2), $j_e \circ f : X \to Y$ is u.s.c.; it follows from (b′) that $\Phi(j_e \circ f) : X \to Y$ is l.s.c. and $\Psi(j_e \circ f) : X \to Y$ is u.s.c. Then $p_e \circ \Phi(j_e \circ f) : X \to \mathbb{R}^+$ is l.s.c. and $p_e \circ \Psi(j_e \circ f) : X \to \mathbb{R}^+$ is u.s.c. by Corollary 3.8(1). It follows from propositions 3.4(1) and 3.2(3) that $f \leq p_e \circ (j_e \circ f) \leq p_e \circ \Phi(j_e \circ f) \leq p_e \circ \Psi(j_e \circ f)$. Also, by using propositions 3.2(3) and 3.3(1), we can check that $p_e \circ \Phi(j_e \circ f) \leq p_e \circ \Phi(j_e \circ f')$ and $p_e \circ \Psi(j_e \circ f) \leq p_e \circ \Psi(j_e \circ f')$ whenever $f \leq f'$. Thus, the operators assigning to each u.s.c. function f an l.s.c. function $p_e \circ \Phi(j_e \circ f)$ and a u.s.c. function $p_e \circ \Psi(j_e \circ f)$ are required ones in (b).

Let us introduce another example. Let Y be a topological vector space and a vector lattice. Then Y is called a *topological vector lattice* [5] if Y possesses a $\mathbf{0}$-neighborhood base of solid sets A; that is, sets A satisfying that "$x \in A$ and $|y| \le |x|$" imply $y \in A$. Note that the positive cone Y^+ of the topological vector lattice is normal [5, V.7.1].

For a topological vector lattice Y with positive interior points, the following condition (d′) is equivalent to X being monotonically countably paracompact [7, Theorem 3.1]. To show this, it suffices to show that (d′) $\Leftrightarrow$ (d), because it is known that (d) is equivalent to X being monotonically countably paracompact ([4, Theorem 25] and [3, Theorem 2]).

(d) There exists an operator Φ assigning to each locally bounded function $f : X \to \mathbb{R}$ a locally bounded l.s.c. function $\Phi(f) : X \to \mathbb{R}$ with $|f| \le \Phi(f)$ such that $\Phi(f) \le \Phi(f')$ whenever $|f| \le |f'|$.

(d′) There exists an operator Φ assigning to each locally bounded map $f : X \to Y$ a locally bounded l.s.c. map $\Phi(f) : X \to Y$ with $|f| \le \Phi(f)$ such that $\Phi(f) \le \Phi(f')$ whenever $|f| \le |f'|$.

Indeed, to show (d) $\Rightarrow$ (d′), assume (d) and let Φ be an operator in (d). For each locally bounded map $f : X \to Y$, by Proposition 3.6(3) and Corollary 3.8(2), we can check that $j_e \circ \Phi(p_{[-e,e]} \circ f) : X \to Y$ is a locally bounded l.s.c. We have that $|f| \le j_e \circ p_{[-e,e]} \circ f \le j_e \circ \Phi(p_{[-e,e]} \circ f)$ by propositions 3.3(1) and 3.4(2), and that $j_e \circ \Phi(p_{[-e,e]} \circ f) \le j_e \circ \Phi(p_{[-e,e]} \circ f')$ whenever $|f| \le |f'|$ by using propositions 3.1(3) and 3.3(1). Thus, the operator assigning to each locally bounded map $f : X \to Y$ a locally bounded l.s.c. map $j_e \circ \Phi(p_{[-e,e]} \circ f)$ is a required one in (d′).

Similarly, to show (d′) $\Rightarrow$ (d), assume (d′) and let Φ be an operator in (d′). We now define another operator Φ' by assigning to each locally bounded map $g : X \to Y$, $\Phi'(g) : X \to Y$ by $\Phi'(g)(x) = \Phi(g)(x) + e$. For each locally bounded function $f : X \to \mathbb{R}$, it follows from $\mathbf{0} \le |j_e \circ f(x)| \le \Phi(j_e \circ f)(x)$ that $\Phi'(j_e \circ f)(x) = \Phi(j_e \circ f)(x) + e \in Y^+ + e \subset \mathrm{int}_Y Y^+$. Note that, for each locally bounded map $g : X \to Y$, $\Phi'(g)$ is also a locally bounded l.s.c., $|g| \le \Phi'(g)$, and that $\Phi'(g) \le \Phi'(g')$ whenever $|g| \le |g'|$. By propositions 3.6(3) and 3.7(2) and Corollary 3.8(2), we can check that $p_{[-e,e]} \circ \Phi'(j_e \circ f) : X \to \mathbb{R}$ is a locally bounded l.s.c. It follows from propositions 3.1(3) and 3.4(1) that $|f| = p_{[-e,e]} \circ j_e \circ f \le p_{[-e,e]} \circ \Phi'(j_e \circ f)$. Also, by using propositions 3.1(2) and 3.3(2), we have that $p_{[-e,e]} \circ \Phi'(j_e \circ f) \le p_{[-e,e]} \circ \Phi'(j_e \circ f')$ whenever $|f| \le |f'|$. Thus, the operator assigning to each locally bounded function $f : X \to \mathbb{R}$ a locally bounded l.s.c. function $p_{[-e,e]} \circ \Phi'(j_e \circ f)$ is a required one in (d).

Other corresponding equivalences can be proved similarly, for example, [7, Theorem 3.1(2)and conditions on theorems 3.2, 4.1, 4.2, and 4.3]. A similar proof of (c′) $\Rightarrow$ (c) provides an alternative proof to [8, Theorem

2.2]. On the other hand, it should be noted that our technique cannot work on our remaining problem [7, Question 5.1.3], which actually asks if (c) $\Rightarrow$ (c$'$), for the technical reason that "$j_e \circ p_e(y) \leq y$ does not necessarily hold."

Acknowledgment. The author would like to thank the referee for the careful reading of the paper and valuable suggestions. In particular, the referee's comments were very helpful for correcting errors of propositions 3.6 and 3.7 of the first version.

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