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ON SPACES WITH σ-CLOSED-DISCRETE DENSE SETS

RODRIGO R. DIAS AND DÁNIEL T. SOUKUP

ABSTRACT. The main purpose of this paper is to study e-separable spaces, originally introduced by Georges Kurepa as K_0' spaces; we call a space X *e-separable* if and only if X has a dense set which is the union of countably many closed discrete sets. We primarily focus on the behavior of e-separable spaces under products and the cardinal invariants that are naturally related to e-separable spaces. Our main results show that the statement "there is a product of at most $\mathfrak{c}$ many e-separable spaces that fails to be e-separable" is overinsistent with the existence of a weakly compact cardinal.

1. INTRODUCTION

The goal of this paper is to study a natural generalization of separability. Let us call a space X *e-separable* if and only if X has a dense set which is the union of countably many closed discrete sets. The definition is due to Georges Kurepa [18], who introduced this notion as property K_0' in his study of Souslin's problem. Later, e-separable spaces appear in multiple papers related to the study of linearly ordered and GO-spaces [11], [25], [26], [30]. In particular, M. J. Faber [11] showed that e-separable GO-spaces are perfect; however, whether the converse implication is true is famously open: is there, in ZFC, a perfect GO-space (or even just a perfect T_3 space) which is not e-separable? Let us refer the interested reader to a paper of Harold Bennett and David Lutzer [5] for more details and results on this topic.

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Now, our interest lies mainly in studying e-separability with regards to powers and products. Recall that the famous Hewitt–Marczewski–Pondiczery theorem (see[10]) states that the product of at most $\mathfrak{c}$ many separable spaces is again separable. What can we say about e-separable spaces in this matter? Historically, another generalization of separable spaces received more attention: *d-separable spaces*, called K_0 spaces in Kurepa's old notation, i.e., spaces with σ-discrete dense sets. Investigated in great detail (see [1], [2], [17], [21], [28], and [29]), d-separable spaces show very interesting behavior in many aspects, in particular, regarding products. A. V. Arhangel'skiĭ proved in [1] that any product of d-separable spaces is d-separable; in [17], the authors show that for every space X there is a cardinal κ so that X^κ is d-separable. Motivated by these results, one of our main objectives is to understand, as much as possible, the behavior of e-separable spaces under products.

Our paper splits into three main parts. First, we make initial observations on e-separable spaces in §2. Then, in §3, we investigate if the existence of many large closed discrete sets suffices for a space to be e-separable. In particular, we prove that once an infinite power X^κ has a closed discrete set of size $d(X^\kappa)$ (the density of X^κ), then X^κ is e-separable. As a corollary, we show that certain large powers of non-countably-compact spaces are e-separable. Now an interesting open question is whether a countably compact, non-separable space can have an e-separable square.

Next, in §4, we compare two natural cardinal functions: $d(X)$, the size of the smallest dense set in X and with $d_e(X)$, the size of the smallest σ-closed-discrete dense set. In Theorem 4.2, we show that there is a 0-dimensional space X which satisfies $d(X) < d_e(X)$. We show that a similar example can be constructed for d-separable spaces, at least under $\aleph_1 < \mathfrak{c} = 2^{\aleph_0}$; we do not know how to remove this assumption. The section ends with a few interesting open problems.

Our main results are presented in §5. We describe those cardinals κ such that the product of κ many e-separable spaces is e-separable again, and hence present the analogue of the Hewitt–Marczewski–Pondiczery theorem for e-separable spaces. First, note that $2^{\mathfrak{c}^+}$ is not e-separable (as a compact, non-separable space) and so the question of preserving e-separability comes down to products of at most $\mathfrak{c}$ terms. How could it be possible that e-separability is not preserved by small products? The reason must be that there are some large cardinals lurking in the background.

Corollary 5.9. *If the existence of a weakly compact cardinal is consistent with* ZFC*, then so is the statement that there are less than* $\mathfrak{c}$ *many discrete spaces with non-e-separable product.*

Corollary 5.11. *If there is a non-e-separable product of at most* $\mathfrak{c}$ *many e-separable spaces, then there is a weakly compact cardinal in* L.

As we shall see, the proof of these results nicely combines various ideas from topology, set theory, and logic.

Throughout this paper, all spaces are assumed to be T_1. Given a product of discrete spaces $X = \prod\{X_\alpha : \alpha < \lambda\}$ and a function ε satisfying $\mathrm{dom}(\varepsilon) \in [\lambda]^{<\aleph_0}$ and $\varepsilon(\alpha) \in X_\alpha$ for each $\alpha \in \mathrm{dom}(\varepsilon)$, we write

$$[\varepsilon] = \{x \in X : \varepsilon \subseteq x\}.$$

Thus, if $x \in X$ is such that $x \restriction \mathrm{dom}(\varepsilon) = \varepsilon$, then $[\varepsilon]$ is a basic open neighborhood of x in X. We let $D(\kappa)$ denote the discrete space on a cardinal κ.

In general, we use standard notation and terminology consistent with Ryszard Engelking [10].

2. Preliminaries

In this section, after defining our main concept for the paper, we prove a few general facts about e-separable spaces and state some results for later reference.

Definition 2.1. A topological space X is *e-separable* if there is a sequence $(D_n)_{n\in\omega}$ of closed discrete subspaces of X such that $\bigcup_{n\in\omega} D_n$ is dense in X.

We begin with simple observations.

Observation 2.2. Every separable space is e-separable and every e-separable space is d-separable.

Recall the following two well-known cardinal functions: The density of X, denoted by $d(X)$, is the smallest possible size of a dense set in X; the extent of a space X, denoted by $e(X)$, is the supremum of all cardinalities $|E|$ where E is a closed discrete subset of X.

Observation 2.3. Every e-separable space X satisfies $d(X) \le e(X)$; moreover, if $cf(d(X)) > \omega$, then there is a closed discrete set of size $d(X)$ in X. In particular, a countably compact space is e-separable if and only if it is separable.

Example 2.4. *The space $2^{\mathfrak{c}^+}$ is compact and d-separable but not e-separable.*

Proof. By Arhangel'skiĭ [1], d-separability is preserved by products. Also, $2^{\mathfrak{c}^+}$ is not e-separable as $d(2^{\mathfrak{c}^+}) > e(2^{\mathfrak{c}^+}) = \omega$; hence Observation 2.3 can be applied. □

What can we say about metric spaces?

Observation 2.5. Every space with a σ-discrete π-base is e-separable. Hence, every metrizable space is e-separable.

The following result shows that actually a large class of generalized metric spaces is e-separable.

Proposition 2.6. *Every developable space is e-separable.*

Recall that a space X is *developable* if and only if there is a development of X, i.e., a sequence $(\mathscr{G}_n)_{n\in\omega}$ of open covers of X such that for every $x \in X$ and open V containing x, there is an $n \in \omega$ so that $st(x, \mathscr{G}_n) = \bigcup\{U \in \mathscr{G}_n : x \in U\} \subseteq V$.

Proof. It is well known that any developable space is a σ-space, i.e., has a σ-discrete network (see [7, Proposition 1.8]). Clearly, this implies e-separability. □

It is worth comparing the above result with [2, Proposition 2.3], which states that every quasi-developable space is d-separable. A few notable non-e-separable spaces are

- the Michael line, which is a quasi-developable space (see [4]);
- the Alexandrov double circle, which is a hereditarily D-space (see, e.g., [13, Proposition 2.5]).

Finally, for later reference, we would like to state two results on the existence of closed discrete sets in products.

Theorem 2.7 (Łoś [19])**.** *$D(\omega)^{2^\kappa}$ contains a closed discrete set of size κ for every κ less than the first measurable cardinal.*

The above result was first proved in [19], but [16] and [12] are more accessible.

Theorem 2.8 (Mycielski [23])**.** *$D(\omega)^\kappa$ contains a closed discrete set of size κ for every κ less than the first weakly inaccessible cardinal.*

3. Density and Extent for e-Separable Spaces

Our goal now is to elaborate further on the observation that if X is e-separable, then $d(X) \leq e(X)$. In particular, in what context is the implication reversible?

First, note that $d(X) \leq e(X)$ does not imply that there are closed discrete sets of size $d(X)$.

Example 3.1. *There is a σ-closed-discrete (hence, e-separable) space X which contains no closed discrete sets of size $d(X)$.*

Proof. Let $X = \omega_\omega + 1$ and declare all points in ω_ω isolated and let $\{\{\omega_\omega\} \cup A : A \in [\omega_\omega]^{<\aleph_\omega}\}$ form a neighborhood base at ω_ω. □

Next, we show that even a significant strengthening of $d(X) \leq e(X)$ fails to imply e-separability in general.

Example 3.2. *There is a 0-dimensional, non e-separable space X of size ω_1 such that every somewhere dense subset of X contains a closed discrete subset of size ω_1.*

Proof. Let $X = {\omega_1}^{<\omega}$ and declare $U \subseteq X$ to be open if and only if $x \in U$ implies that $\{\alpha < \omega_1 : x^\frown(\alpha) \in U\}$ contains a club. Now, X is a Hausdorff, 0-dimensional, and dense-in-itself space.

Observe that a set $E \subseteq X$ is closed discrete if and only if $\{\alpha < \omega_1 : x^\frown\alpha \in E\}$ is non-stationary for every $x \in X$. This immediately implies that the σ-closed-discrete sets are closed discrete; hence, X cannot be e-separable.

Suppose that $Y \subseteq X$ is dense in a non-empty open set V; $I_x = \{\alpha \in \omega_1 : x^\frown\alpha \in Y\}$ must be stationary for any $x \in V$ and so we can select an uncountable but non-stationary $I \subseteq I_x$. Hence, $\{x^\frown\alpha : \alpha \in I\}$ is an uncountable closed discrete subset of Y. □

Now, let us turn to powers of a fixed space X. Could it be that $d(X^\kappa) \leq e(X^\kappa)$ implies that X^κ is e-separable whenever κ is an infinite cardinal? The answer is negative, at least under the assumption that there are measurable cardinals.

Example 3.3. *If κ is the first measurable cardinal, then $d(\omega^\kappa) = e(\omega^\kappa)$; however, ω^κ is not e-separable.*

Proof. It is clear that $d(\omega^\kappa) = \kappa$; also, $2^\lambda < \kappa$ whenever $\lambda < \kappa$, and so Theorem 2.7 implies that $e(\omega^\kappa) = \kappa$ as well.

If we show that ω^κ has no closed discrete sets of size κ, then ω^κ cannot be e-separable. Suppose that $A = \{x_\alpha : \alpha < \kappa\} \subseteq \omega^\kappa$ and that $\mathcal{U}$ is a

σ-complete non-principal ultrafilter on κ. Note that

$$\kappa = \bigcup_{n\in\omega} \{\alpha < \kappa : x_\alpha(\xi) = n\}$$

for each $\xi < \kappa$. So there is a unique $n \in \omega$ such that $\{\alpha < \kappa : x_\alpha(\xi) = n\} \in \mathcal{U}$. In turn, we can define $y \in \omega^\kappa$ by $y(\xi) = n$ if and only if $\{\alpha < \kappa : x_\alpha(\xi) = n\} \in \mathcal{U}$. It is easy to see that $\{\alpha < \kappa : x_\alpha \in V\} \in \mathcal{U}$ for every open neighborhood V of y, and so $V \cap A$ has size κ. Hence, y is an accumulation point of A. □

However, if we suppose a bit more than $d(X^\kappa) \leq e(X^\kappa)$, then we get the following.

Theorem 3.4. *Let X be any space and let κ be an infinite cardinal. If X^κ contains a closed discrete set of size $d(X^\kappa)$, then X^κ is e-separable.*

The above theorem is an analogue of [17, Theorem 1]: If X^κ has a discrete subspace of size $d(X)$, then X^κ is d-separable. Example 3.3 shows that assuming "X^κ contains a closed discrete set of size $d(X)$" does not imply that X is e-separable.

We will now prove a somewhat technical lemma which immediately implies Theorem 3.4 and will be of use later as well.

Lemma 3.5. *Let X be any space and let κ be an infinite cardinal. Suppose that $D \subseteq X^\kappa$ is dense in X^κ and X^κ contains a closed discrete set of size $|D|$. Then there is a dense set E in X^κ such that*

(1) $|D| = |E|$, $d(D) = d(E)$, *and*
(2) E *is σ-closed-discrete.*

Proof. Pick a countable increasing sequence $(I_n)_{n\in\omega}$ of subsets of κ such that $\kappa = |I_n| = |\kappa \setminus I_n|$ for each $n \in \omega$ and $\kappa = \bigcup_{n\in\omega} I_n$. Fix closed discrete sets E_n of size $|D|$ in $X^{\kappa\setminus I_n}$ and bijections $\varphi_n : D \to E_n$ for each $n \in \omega$.

We define maps $\psi_n : D \to X^\kappa$ by

$$\psi_n(d)(\xi) = \begin{cases} d(\xi), & \text{for } \xi \in I_n, and \\ \varphi_n(d)(\xi), & \text{for } \xi \in \kappa \setminus I_n. \end{cases}$$

Let $E = \bigcup_{n\in\omega} \operatorname{ran}(\psi_n)$. Clearly, $|D| = |E|$ holds.

It is easy to see that E is dense in X^κ: If $[\varepsilon]$ is a basic open set in X^κ, then there is an $n \in \omega$ such that $\operatorname{dom}(\varepsilon) \subseteq I_n$; hence, $\operatorname{ran}(\psi_n) \cap [\varepsilon] \neq \emptyset$.

Next we show (2) by proving that $\operatorname{ran}(\psi_n)$ is closed discrete as well for each $n \in \omega$. Pick any $x \in X^\kappa$. There is a basic open set $[\varepsilon]$ in $X^{\kappa\setminus I_n}$ such that $x\restriction_{\kappa\setminus I_n} \in [\varepsilon]$ and $|[\varepsilon] \cap E_n| \leq 1$. Thus, the basic open set

$\{y \in X^\kappa : \varepsilon \subseteq y\}$ of X^κ, which we (by abuse of notation) also denote by $[\varepsilon]$, satisfies $x \in [\varepsilon]$ and $|[\varepsilon] \cap \operatorname{ran}(\psi_n)| \leq 1$.

Finally, we prove $d(D) = d(E)$. Note that if D_0 is dense in D, then $\bigcup_{n\in\omega} \psi_n{}''D_0$ is dense in E; hence, $d(E) \leq d(D)$. Suppose that $A \in [E]^{<d(D)}$; we want to prove that A is not dense in E. If A is finite, there is nothing to prove. If A is infinite, let

$$D_A = \bigcup_{n\in\omega} \{d \in D : \psi_n(d) \in A\};$$

then D_A cannot be dense in D as $|D_A| \leq |A| < d(D)$.

Fix a basic open set $U = [\varepsilon]$ such that $[\varepsilon] \cap D_A = \emptyset$. There is an $n^* \in \omega$ such that $\operatorname{dom}(\varepsilon) \subseteq I_{n^*}$.

Claim. If $m \geq n^*$, then $[\varepsilon] \cap \{\psi_m(d) : d \in D_A\} = \emptyset$.

To see this, suppose that $m \geq n^*$ and $d \in D_A$. Then $d \restriction I_m = \psi_m(d) \restriction I_m$, $d \notin [\varepsilon]$, and $\operatorname{dom}(\varepsilon) \subseteq I_m$; thus, $\psi_m(d) \notin [\varepsilon]$.

Hence,

$$U \cap \bigcup_{n\in\omega} \psi_n(D_A) \subseteq \bigcup_{n<n^*} \psi_n(D_A);$$

that is, $U \cap \bigcup_{n\in\omega} \psi_n(D_A)$ is closed discrete as each $\psi_n(D_A)$ is closed discrete. However, $A \subseteq \bigcup_{n\in\omega} \psi_n(D_A)$, which shows that $A \cap U$ cannot be dense in U. □

Let us present two corollaries. The aforementioned [17, Theorem 1] implies that X^κ is always d-separable for any $\kappa \geq d(X)$. We know that, say, $[0,1]^\kappa$ is not e-separable when $\kappa \geq \mathfrak{c}^+$ because of Observation 2.3; indeed, $[0,1]^\kappa$ is compact so σ-closed-discrete sets are countable, but $[0,1]^\kappa$ is not separable. However, the following holds.

Corollary 3.6. *Suppose that X is not countably compact. Then $X^{d(X)}$ is e-separable if $d(X)$ is less than the first weakly inaccessible cardinal.*

Proof. Let $\kappa = d(X)$ and note that it suffices to find a closed discrete subspace of X^κ of size $d(X^\kappa)$ by Theorem 3.4. First, note that $d(X^\kappa) = \kappa$. Second, X contains an infinite closed discrete subspace Y since X is not countably compact. So Y^κ is a closed copy of $D(\omega)^\kappa$ in X^κ. Finally, $D(\omega)^\kappa$ does contain a closed discrete set of size κ by Theorem 2.8. □

Corollary 3.7. *Suppose that X is not countably compact. Then $X^{2^{d(X)}}$ is e-separable if $d(X)$ is less than the first measurable cardinal.*

Proof. The proof is the same as for Corollary 3.6, but we are now using Theorem 2.7 instead of Theorem 2.8. □

Interestingly, if X is compact Hausdorff, then X^ω is d-separable already (see [17, Corollary 5]). Furthermore, Justin Tatch Moore [21] shows that there is an L-space X such that X^2 is d-separable. Note that X itself is not d-separable since each discrete subspace of X is countable, but X has uncountable density. Moore's example was improved by Yinhe Peng [24]: There is an L-space X such that X^2 is e-separable. We wonder if the following related question is true.

Problem 3.8. *Is there a non-separable, countably compact X so that X^2 is e-separable?*

4. The Sizes of σ-Discrete Dense Sets

Next, we investigate the size of the smallest σ-discrete dense set in e-separable spaces.

Definition 4.1. For an e-separable space X, we define

$$d_e(X) = \min\{|E| : \text{ E is a dense } \sigma\text{-closed-discrete subset of } X\}.$$

Clearly, $d(X) \leq d_e(X) \leq e(X)$ for any e-separable space X, and next we show that $d(X) = d_e(X)$ fails to hold in general.

Theorem 4.2. *There is a 0-dimensional e-separable space X such that*

$$\mathfrak{c} = d(X) < d_e(X) = e(X) = w(X) = 2^{\mathfrak{c}}.$$

Before beginning the proof, note the following.

Claim 4.3. *Suppose that a space X can be written as $D \cup E$ so that*

(1) *D is dense in X,*
(2) *E is dense and σ-closed-discrete in X,*
(3) *$d(D) < d(E)$, and*
(4) *every $A \in [D]^{\leq e(D)}$ is nowhere dense in X (or equivalently, in D).*

Then X is e-separable and $d(X) < d_e(X)$.

Proof. X is e-separable by (2) and $d(X) \leq d(D)$ by (1). We prove that if $F \in [X]^{\leq d(X)}$ and F is σ-closed-discrete, then F is not dense in X; this proves the claim. Take $F \subseteq X$ as above and note that by (3), there is a non-empty open set $U \subseteq X$ such that $U \cap E \cap F = \emptyset$. As $|F \cap D| \leq e(D)$, $F \cap D$ must be nowhere dense in X. Thus, there is a non-empty open $V \subseteq U$ such that $V \cap F \cap D = \emptyset$. Thus, $V \cap F = \emptyset$ showing that F is not dense. □

Proof of Theorem 4.2. Now, it suffices to construct a 0-dimensional space $X = D \cup E$, satisfying (1)–(4) of Claim 4.3. Let us construct $X = D \cup E \subseteq \omega^{2^{\mathfrak{c}}}$ such that

(i) D is dense in $\omega^{2^{\mathfrak{c}}}$,
(ii) E is dense and σ-closed-discrete in $\omega^{2^{\mathfrak{c}}}$,
(iii) $|D| = \mathfrak{c}$ and $d(E) = 2^{\mathfrak{c}}$, and
(iv) $e(D) = \omega$.

It is trivial to see that (i)–(iii) implies (1)–(3), respectively, while (iv) implies (4) using the fact that $d(\omega^{2^{\mathfrak{c}}}) = \mathfrak{c}$.

First, we construct D. Construct dense subsets $D_n \subseteq n^{2^{\mathfrak{c}}}$ of size $\mathfrak{c}$ which are countably compact for each $n \in \omega$; this can be done by choosing a dense subset $D_n^0 \subseteq n^{2^{\mathfrak{c}}}$ of size $\mathfrak{c}$ and adding accumulation points recursively (ω_1 many times) for all countable subsets. Define $D = \bigcup_{n\in\omega} D_n$. Then D is dense in $\omega^{2^{\mathfrak{c}}}$ as $\bigcup_{n\in\omega} n^{2^{\mathfrak{c}}}$ is dense in $\omega^{2^{\mathfrak{c}}}$ and $e(D) = \omega$ as $e(D_n) = \omega$ for all $n \in \omega$; thus, D satisfies (i), (iv), and the first part of (iii).

Now, we construct E satisfying (ii) and (iii), which finishes the proof. Let $S = \sigma(\omega^{2^{\mathfrak{c}}}) = \{x \in \omega^{2^{\mathfrak{c}}} : |\{\alpha \in 2^{\mathfrak{c}} : x(\alpha) \neq 0\}| < \aleph_0\}$; then $d(S) = 2^{\mathfrak{c}}$ and S is dense in $\omega^{2^{\mathfrak{c}}}$. Recall that $\omega^{2^{\mathfrak{c}}}$ contains a closed discrete set of size $2^{\mathfrak{c}}$ by Theorem 2.8. Now, by applying Lemma 3.5, we find a σ-closed-discrete E which is dense in $\omega^{2^{\mathfrak{c}}}$ and satisfies $d(E) = d(S) = 2^{\mathfrak{c}}$. □

Naturally, one can consider the same problem for d-separable spaces. Let us present an example along the same lines under the assumption $\aleph_1 < \mathfrak{c}$.

Proposition 4.4. *Suppose that $\aleph_1 < \mathfrak{c}$. Then there is a d-separable space X with $d(X) = \aleph_1$ that contains no dense σ-discrete sets of size $\aleph_1$.*

Proof. Moore [20, Theorem 5.4] proves that there is a coloring $c : [\omega_1]^2 \to \omega$ such that for every $n \in \omega$, uncountable pairwise disjoint $A \subseteq [\omega_1]^n$, uncountable $B \subseteq \omega_1$, and $h : n \to \omega$, there exist $a \in A$ and $\beta \in B \setminus \max(a)$ such that $c(a(i), \beta) = h(i)$ for every $i < n$, where $a = \{a(i) : i < n\}$.

Suppose that $D = \{d_n : n \in \omega\}$ is any countable space.

Claim. There is a hereditarily Lindelöf dense subspace $Y \subseteq D^{\omega_1}$ that is not separable.

To prove this, for each $\beta < \omega_1$, we define $y_\beta \in D^{\omega_1}$ as

$$y_\beta(\alpha) = \begin{cases} d_{c(\alpha,\beta)} & \text{if } \alpha < \beta, \\ d_0 & \text{if } \alpha \geq \beta. \end{cases} \tag{4.1}$$

Now let $Y = \{y_\beta : \beta < \omega_1\}$.

We claim that there is an $\alpha < \omega_1$ so that $Y \restriction (\omega_1 \setminus \alpha)$ is dense in $D^{\omega_1 \setminus \alpha} \simeq D^{\omega_1}$. Suppose otherwise; then we can find basic open sets $[\varepsilon_\alpha]$ in $D^{\omega_1\setminus\alpha}$ so that $Y \cap [\varepsilon_\alpha] = \emptyset$. By standard Δ-system arguments, we find $I \in [\omega_1]^{\aleph_1}$, $n \in \omega$, and $h : n \to \omega$ so that $\mathrm{dom}(\varepsilon_\alpha) = \{a_\alpha(i) : i < n\}$ are pairwise disjoint for $\alpha \in I$ and $d_{h(i)} \in \varepsilon_\alpha(a_\alpha(i))$ for each $i < n$. Now, there exist $\alpha \in I$ and $\beta \in \omega_1 \setminus \max(\mathrm{dom}(\varepsilon_\alpha))$ so that $c(a_\alpha(i), \beta) = h(i)$ for all $i < n$. This means that $d_{c(a_\alpha(i),\beta)} \in \varepsilon_\alpha(a_\alpha(i))$ for $i < n$ and so $y_\beta \in [\varepsilon_\alpha]$. This contradicts our assumption.

It is clear that $d(Y \restriction (\omega_1 \setminus \alpha)) = \aleph_1$. It remains to prove that $Y \restriction (\omega_1 \setminus \alpha)$ is hereditarily Lindelöf.

Fix $W \in [\omega_1]^{\aleph_1}$ and, for each $\gamma \in W$, let $[\varepsilon_\gamma]$ be a basic open subset of $D^{\omega_1 \setminus \alpha}$ with $y_\gamma \restriction (\omega_1 \setminus \alpha) \in [\varepsilon_\gamma]$; we may assume that $\max(\mathrm{dom}(\varepsilon_\gamma)) > \gamma$. Suppose, by way of contradiction, that for each $\eta < \omega_1$ we have $\{y_\gamma \restriction (\omega_1 \setminus \alpha) : \gamma \in W\} \nsubseteq \bigcup\{[\varepsilon_\gamma] \colon \gamma \in W \cap \eta\}$. We can then recursively define, for $\zeta < \omega_1$,

- δ_0 as the least element of W;
- $\delta_{\zeta+1}$ as the least $\delta \in W$ satisfying $\delta > \sup_{\eta \leq \zeta} \max(\mathrm{dom}(\varepsilon_{\delta_\eta}))$ and $y_\delta \restriction (\omega_1 \setminus \alpha) \notin \bigcup\{[\varepsilon_\gamma] \colon \gamma \in W \cap (\delta_\zeta + 1)\}$;
- δ_ζ as the least $\delta \in W$ satisfying $\delta > \sup_{\eta < \zeta} \max(\mathrm{dom}(\varepsilon_{\delta_\eta}))$ and $y_\delta \restriction (\omega_1 \setminus \alpha) \notin \bigcup\{[\varepsilon_\gamma] \colon \gamma \in W \cap \sup_{\eta < \zeta} \delta_\eta\}$ if ζ is a limit ordinal.

Again by Δ-system arguments, there exist $r \in [\omega_1 \setminus \alpha]^{<\aleph_0}$, $p : r \to D$, $Z \in [\omega_1]^{\aleph_1}$, $n \in \omega$, and $h : n \to \omega$ satisfying

(i) $r \subseteq \mathrm{dom}(\varepsilon_{\delta_\zeta})$ for all $\zeta \in Z$;
(ii) $\mathrm{dom}(\varepsilon_{\delta_\zeta}) \setminus r = \{a_\zeta(i) : i < n\}$ are pairwise disjoint for $\zeta \in Z$;
(iii) $d_{h(i)} \in \varepsilon_{\delta_\zeta}(a_\zeta(i))$ for each $i < n$; and
(iv) $p \subseteq y_{\delta_\zeta}$ for all $\zeta \in Z$.

Now, there are $\zeta, \zeta' \in Z$ such that $\delta_{\zeta'} \geq \max(\mathrm{dom}(\varepsilon_{\delta_\zeta}))$ and $c(a_\zeta(i), \delta_{\zeta'}) = h(i)$ for all $i < n$. Thus, $d_{c(a_\zeta(i),\delta_{\zeta'})} \in \varepsilon_{\delta_\zeta}(a_\zeta(i))$ for $i < n$, whence $y_{\delta_{\zeta'}} \restriction (\omega_1 \setminus \alpha) \in [\varepsilon_{\delta_\zeta}]$, which is a contradiction since $\delta_{\zeta'} \geq \max(\mathrm{dom}(\varepsilon_{\delta_\zeta})) > \delta_\zeta$ implies $\zeta < \zeta'$ by construction.

Now, by $\mathfrak{c} \geq \aleph_2$, we can pick a countable dense $D \subseteq \omega^{\omega_2}$. Then D^{ω_1} is dense in $(\omega^{\omega_2})^{\omega_1} \simeq \omega^{\omega_2}$. By the claim, there is a dense $Y \subseteq D^{\omega_1}$ such that every discrete subset of Y is countable and, hence, nowhere dense (as Y is non-separable). Now, by Lemma 3.5, there is a dense σ-closed-discrete $E \subseteq \omega^{\omega_2}$ satisfying $d(E) = \aleph_2$, in view of Theorem 2.8 and the fact that, e.g., $\sigma(\omega^{\omega_2}) = \{x \in \omega^{\omega_2} : |\{\alpha \in \omega_2 : x(\alpha) \neq 0\}| < \aleph_0\}$ is a dense subset of ω^{ω_2} with density $\aleph_2$.

Let $X = Y \cup E$. An argument strictly analogous to what is done in Claim 4.3. finishes the proof. $\square$

The assumption $\aleph_1 < \mathfrak{c}$ is somewhat unnatural in Proposition 4.4, but we do not know how to remove it.

Problem 4.5. *Is there a* ZFC *example of a d-separable space X with the property that every σ-discrete dense subset of X has cardinality greater than $d(X)$?*

In particular, we cannot answer the following.

Problem 4.6. *Is there, in* ZFC*, a dense $Y \subseteq 2^{\omega_2}$ of size $\aleph_1$ all of whose σ-discrete subsets are nowhere dense?*

Finally, recall that any compact, e-separable space satisfies $d(X) = d_e(X)$. We do not know if the analogue holds for d-separable spaces.

Problem 4.7. *Is there a σ-discrete dense subset of size $d(X)$ in any compact, d-separable space X?*

5. Preservation Under Products

As mentioned in the introduction, the behavior of separable and d-separable spaces under products and powers is very well described: Separability is preserved by products of size $\leq \mathfrak{c}$ but not bigger; on the other hand, the product of d-separable spaces is always d-separable. Hence, our goal in this section is answering the following natural question: For which cardinals κ is it true that every product of κ many e-separable spaces is e-separable? As noted earlier in Example 2.4, any such κ is at most the continuum.

Let us start with powers of a single e-separable space. We would like to thank Ofelia T. Alas for pointing out the following to us in a private conversation.

Proposition 5.1. *Let X be an e-separable space and $\kappa \leq \mathfrak{c}$. Then the space X^κ is e-separable.*

Proof. Let $(D_n)_{n\in\omega}$ be a sequence of closed discrete subsets of X with $\bigcup_{n\in\omega} D_n$ dense in X. Fix a subspace $Y \subseteq \mathbb{R}$ with $|Y| = \kappa$, and let $\mathcal{B}$ be a countable base for Y. Now consider $T = \bigcup_{n\in\omega}(S_n \times {}^n\omega)$, where $S_n = \{(B_0, \ldots, B_{n-1}) \in {}^n\mathcal{B} : \forall i, j < n\ (i \neq j \Rightarrow B_i \cap B_j \neq \emptyset)\}$ for every $n \in \omega$.

Fix an arbitrary $p \in X$. For each $t = ((B_0, \ldots, B_{n-1}), (k_0, \ldots, k_{n-1})) \in T$, we define E_t to be the set of those $x \in X^Y$ so that there is an $(a_i)_{i<n} \in \prod\{D_{k_i} : i < n\}$ with

$$x(\alpha) = \begin{cases} a_i, & \text{for } \alpha \in B_i \text{ and } i < n, \text{ and} \\ p, & \text{for } \alpha \in Y \setminus \bigcup_{i=0}^{n-1} B_i. \end{cases}$$

It is routine to verify that each E_t is a closed discrete subspace of X^Y and that $\bigcup_{t\in T} E_t$ is dense in X^Y. Since T is countable and $|Y| = \kappa$, it follows that X^κ is e-separable. □

Now, we turn to arbitrary products of e-separable spaces. We will see that the heart of the matter is whether we can find large closed discrete sets in the product of small discrete spaces.

In [22], S. Mrówka introduced a class of cardinals denoted by $\mathcal{M}^*$. We write $\lambda \in \mathcal{M}^*$ if and only if there is a product of λ many discrete spaces $X = \prod\{X_\alpha : \alpha < \lambda\}$ each of size $< \lambda$ so that X has a closed discrete set of size λ. Equivalently, the product $\prod\{D(\nu)^\lambda : \nu \in \lambda \cap \mathrm{Card}\}$ contains a closed discrete set of size λ.

If a cardinal λ is in $\mathcal{M}^*$, then some degree of compactness fails for λ. Let us make this statement precise: Recall that $\mathcal{L}_{\lambda,\omega}$ is the infinitary language which allows conjunctions and disjunctions of $< \lambda$ formulas and universal or existential quantification over finitely many variables. The language $\mathcal{L}_{\lambda,\omega}$ is *weakly compact* by definition if every set of at most λ sentences Σ from $\mathcal{L}_{\lambda,\omega}$ has a model provided that every $S \in [\Sigma]^{<\lambda}$ has a model (see [15, p. 382]).

Theorem 5.2 (Mrówka [22], Chudnovsky [9]). *$\lambda \notin \mathcal{M}^*$ if and only if $\mathcal{L}_{\lambda,\omega}$ is weakly compact.*

Now, as expected, $\lambda \notin \mathcal{M}^*$, or equivalently, the statement "$\mathcal{L}_{\lambda,\omega}$ is weakly compact," has some large cardinal strength. First, we mention two classical results.

Lemma 5.3 ([15, exercises 17.17 and 17.18]). *If $\mathcal{L}_{\lambda,\omega}$ is weakly compact, then λ is weakly inaccessible.*

Lemma 5.4 ([15, Theorem 17.13]). *λ is a weakly compact cardinal if and only if it is strongly inaccessible and $\mathcal{L}_{\lambda,\omega}$ is weakly compact.*

For our current purposes, we can consider the above lemma the definition of weakly compact cardinals. Now, given a weakly compact cardinal λ, we can enlarge the continuum while the language $\mathcal{L}_{\lambda,\omega}$ remains weakly compact.

Theorem 5.5 (Chudnovsky [9], Boos [6]). *If λ is a weakly compact cardinal and $\mathbb{C}_{\lambda^+}$ is the poset for adding λ^+ many Cohen-reals, then $V^{\mathbb{C}_{\lambda^+}} \models$ "$\mathcal{L}_{\lambda,\omega}$ is weakly compact; hence, $\mathfrak{c} \setminus \mathcal{M}^* \neq \emptyset$."*

Finally, in [8], the authors recently showed that a weakly compact cardinal can be recovered from $\mathcal{L}_{\lambda,\omega}$ being weakly compact.

Theorem 5.6 (Hamkins [14], Cody et al. [8]). *If $\mathcal{L}_{\lambda,\omega}$ is weakly compact, then λ is weakly compact in L.*

Now, it is easy to derive our first main result about non-preservation.

Lemma 5.7. *If $\lambda \leq \mathfrak{c}$ and $\lambda \notin \mathcal{M}^*$, then there is a non-e-separable product of λ many discrete spaces.*

Proof. If $\lambda \notin \mathcal{M}^*$, then $\mathcal{L}_{\lambda,\omega}$ is weakly compact and, hence, λ is a regular limit cardinal. Now take discrete spaces X_α of size $< \lambda$ such that

$\sup\{|X_\alpha| : \alpha < \lambda\} = \lambda$. The product $X = \prod\{X_\alpha : \alpha < \lambda\}$ contains no closed discrete subsets of size λ as $\lambda \notin \mathcal{M}^*$. We claim that $d(X) = \lambda$, which follows from the following more general observation.

Observation 5.8. Suppose that $\kappa \leq \mathfrak{c}$ and X_α is discrete for $\alpha < \kappa$. Then $d(\prod\{X_\alpha : \alpha < \kappa\}) = \sup\{|X_\alpha| : \alpha < \kappa\}$.

To prove this observation, simply apply the usual trick appearing in the proof of Proposition 5.1.

Now, we claim that X cannot be e-separable. Indeed, if X is e-separable, then Observation 2.3 implies that X has a closed discrete subset of size $d(X) = \lambda = cf(\lambda) > \omega$; however, this is not the case. □

Hence, we immediately get the following.

Corollary 5.9. *If the existence of a weakly compact cardinal is consistent with* ZFC*, then so is the statement that there is a non-e-separable product of less than* $\mathfrak{c}$ *many discrete spaces.*

Proof. Apply Lemma 5.7 and Theorem 5.5. □

Now, we will obtain that it is also consistent with ZFC that every product of at most $\mathfrak{c}$ many e-separable spaces is e-separable; we will do so by showing that this last statement is implied by the non-existence of weakly compact cardinals in L. It will suffice to prove the following.

Theorem 5.10. *Suppose that* $\lambda \leq \mathfrak{c}$ *is minimal so that there is a family of* λ *many e-separable spaces with non-e-separable product. Then* $\lambda \notin \mathcal{M}^*$ *and so* $\mathcal{L}_{\lambda,\omega}$ *is weakly compact.*

Let us mention that $\mathcal{L}_{\mathfrak{c},\omega}$ is not weakly compact [8] and so $\lambda < \mathfrak{c}$ in the previous theorem. In any case, if $\mathcal{L}_{\lambda,\omega}$ is weakly compact, then λ is weakly compact in L by Theorem 5.6. In turn, we have the following result.

Corollary 5.11. *If there is a non-e-separable product of at most* $\mathfrak{c}$ *many e-separable spaces, then there is a weakly compact cardinal in* L.

By combining corollaries 5.9 and 5.11, we obtain the following.

Corollary 5.12. *The following statements are overinsistent relative to* ZFC:

(a) *there is a product of at most* $\mathfrak{c}$ *many e-separable spaces that fails to be e-separable;*
(b) *there is a weakly compact cardinal.*

Let us now turn to proving Theorem 5.10. We start by reducing the problem to products of discrete spaces again.

Lemma 5.13. *Suppose that $\kappa \leq \mathfrak{c}$. Then the following are equivalent:*

(a) *every product of at most κ many e-separable spaces is e-separable;*
(b) *every product of at most κ many discrete spaces is e-separable.*

Proof. The implication (a)⇒(b) holds trivially. We prove (b)⇒(a).

Let $X = \prod\{X_\alpha : \alpha \in Y\}$, where $Y \subseteq \mathbb{R}$ has cardinality at most κ and each X_α is e-separable. For each $\alpha \in Y$, fix a point $p_\alpha \in X_\alpha$ and a sequence $(E^\alpha_k)_{k\in\omega}$ of closed discrete subsets of X_α with $\overline{\bigcup_{k\in\omega} E^\alpha_k} = X_\alpha$.

Fix a countable base $\mathcal{B}$ for Y and, for each $n \in \omega$, consider

$$S_n = \{(B_i)_{i<n} \in {}^n\mathcal{B} : \forall i, j < n\ (i \neq j \Rightarrow B_i \cap B_j = \emptyset)\};$$

now, for each $t = ((B_i)_{i<n}, (k_0, \ldots, k_{n-1})) \in S_n \times {}^n\omega$, define Y_t to be the set of those $x \in X$ so that

$$x(\alpha) = \begin{cases} x'_\alpha & \text{for some } x'_\alpha \in E^\alpha_{k_i} \text{ for } \alpha \in B_i \text{ and } i < n, \text{ and} \\ p_\alpha, & \text{for } \alpha \in Y \setminus \bigcup_{i=0}^{n-1} B_i. \end{cases}$$

Note that each Y_t is homeomorphic to the product $\prod_{i<n}\prod_{\alpha\in B_i} E^\alpha_{k_i}$. Hence, Y_t is e-separable by (b). Let $(D^t_k)_{k\in\omega}$ be a sequence of closed discrete subsets of Y_t with $\overline{\bigcup_{k\in\omega} D^t_k} = Y_t$. Since each Y_t is closed in X, we have that each D^t_k is a closed discrete subset of X. Finally, as $\bigcup_{n\in\omega}\bigcup_{r\in S_n\times {}^n\omega} Y_t$ is dense in X, it follows that

$$\overline{\bigcup_{n\in\omega}\bigcup_{t\in S_n\times {}^n\omega}\bigcup_{k\in\omega} D^t_k} = X,$$

thus showing that X is e-separable. □

Note that we immediately get the following easy corollary.

Corollary 5.14. *The product of finitely many e-separable spaces is e-separable.*

Second, we show that as long as we take the product of large discrete sets relative to the number of terms, we end up with an e-separable product.

Lemma 5.15. *Let κ be an infinite cardinal. Then the product of at most κ many discrete spaces of cardinality at least κ is e-separable.*

Proof. Let $X = \prod\{X_\alpha : \alpha \in \lambda\}$ where $\lambda \leq \kappa$ and each X_α is a discrete space with cardinality at least κ. We can assume that λ is infinite and that $X_\alpha = |X_\alpha|$ for all $\alpha \in \lambda$.

Define

$$P^i_j = \{(F, p) \in [\lambda]^i \times Fn(\lambda, \kappa) : |p| = j \text{ and } F \cap \mathrm{dom}(p) = \emptyset\}$$

for each $i, j \in \omega$ where $Fn(\lambda, \kappa)$ denotes the set of finite partial functions from λ to κ. Fix an injective function $\varphi : \bigcup_{i,j\in\omega} P^i_j \to \kappa$ such that $\varphi(F, p) > \max(\operatorname{ran}(p))$ for every $(F, p) \in \bigcup_{i,j\in\omega} P^i_j$.

Now, for every $i, j \in \omega$, let E^i_j be the set of all $x \in X$ for which there is $(F, p) \in P^i_j$ satisfying

(1) $x(\xi) \geq \kappa$ for all $\xi \in F$,
(2) $x \in [p]$, and
(3) $x(\xi) = \varphi(F, p)$ for all $\xi \in \lambda \setminus (F \cup \operatorname{dom}(p))$.

It is straightforward to verify that $\bigcup_{i,j\in\omega} E^i_j$ is dense in X. We claim that each E^i_j is a closed discrete subset of X, which will conclude our proof.

From this point on, let $i, j \in \omega$ be fixed.

To see that E^i_j is discrete, pick an arbitrary $x \in E^i_j$ and let this be witnessed by the pair $(F, p) \in P^i_j$. Note that the choice of φ ensures that this (F, p) is unique. Pick any $\eta \in \lambda \setminus (F \cup \operatorname{dom}(p))$ and let

$$V = [x \restriction (\operatorname{dom}(p) \cup F \cup \{\eta\})].$$

Then V is an open neighborhood of x in X satisfying $E^i_j \cap V = \{x\}$.

It remains to show that E^i_j is closed in X. Let then $y \in X \setminus E^i_j$; we must find an open neighborhood V of y in X such that $V \cap E^i_j = \emptyset$. We shall do so by considering several cases.

Case 1: $G = \{\xi \in \lambda : y(\xi) \geq \kappa\}$ has more than i elements. Then we may take any $H \in [G]^{i+1}$ and define $V = [y \restriction H]$.

Case 2: $G = \{\xi \in \lambda : y(\xi) \geq \kappa\}$ has cardinality at most i.

We will split this case in two.

2.1: $\operatorname{ran}(y) \cap \kappa$ is infinite. Then we can take $A \in [\kappa]^{j+2}$ such that $y''A \in [\kappa]^{j+2}$ and define $V = [y \restriction A]$.

2.2: $\operatorname{ran}(y) \cap \kappa$ is finite. Let $\mu = \max(\operatorname{ran}(y) \cap \kappa)$ and $H = \{\xi \in \lambda : y(\xi) < \mu\}$.

We divide this case into three subcases.

2.2.1: $|H| > j$. Pick $H' \in [H]^{j+1}$ and $\beta \in \lambda$ such that $y(\beta) = \mu$. Now take $V = [g \restriction (H' \cup \{\beta\})]$.

2.2.2: $|H| \leq j$ and $\mu \notin \operatorname{ran}(\varphi)$. Let $B \in [\lambda]^{j+1-|H|}$ be such that $y''B = \{\mu\}$ and consider $V = [g \restriction (H \cup B)]$.

2.2.3: $|H| \leq j$ and $\mu \in \operatorname{ran}(\varphi)$. Let $(F, p) \in P^i_j$ be such that $\varphi(F, p) = \mu$ and, as in the previous case, take $B \in [\lambda]^{j+1-|H|}$ satisfying

$y''B = \{\mu\}$. Now define

$$V = [g \restriction (G \cup H \cup B \cup F \cup \mathrm{dom}(p))].$$

In order to get a contradiction, suppose that there is $x \in V \cap E^i_j$ and let $(F', p') \in P^i_j$ witness that $x \in E^i_j$. Since $|H \cup B| = j + 1$ and $x''(H \cup B) = y''(H \cup B) \subseteq \kappa$, we have that $\varphi(F', p') = \max(x''(H \cup B)) = \max(y''(H \cup B)) = \mu$. Hence, $(F', p') = (F, p)$ by injectivity of φ. Now, since $F = \{\xi \in \lambda : x(\xi) \geq \kappa\}$ and $G = \{\xi \in \lambda : y(\xi) \geq \kappa\}$, it follows from $x \restriction (F \cup G) = y \restriction (F \cup G)$ that $F = G$. Similarly, as $H = \{\xi \in \lambda : y(\xi) < \mu\}$ and $\mathrm{dom}(p) = \{\xi \in \lambda : x(\xi) < \mu\}$, it follows from $x \restriction (H \cup \mathrm{dom}(p)) = y \restriction (H \cup \mathrm{dom}(p))$ that $H = \mathrm{dom}(p)$. Thus, the pair $(G, y \restriction H) = (F, p) \in P^i_j$ witnesses that $y \in E^i_j$, a contradiction. $\square$

Finally, we are ready to present the proof of Theorem 5.10.

Proof of Theorem 5.10. Suppose that $\lambda \leq \mathfrak{c}$ is minimal so that there are e-separable spaces X_α such that $X = \prod\{X_\alpha : \alpha < \lambda\}$ is not e-separable. By Lemma 5.13, we can suppose that each X_α is discrete.

Note that

$$X \simeq \prod\{X_\alpha : \alpha < \lambda, |X_\alpha| < \lambda\} \times \prod\{X_\alpha : \alpha < \lambda, |X_\alpha| \geq \lambda\}.$$

By Lemma 5.15, we know that the second term on the right-hand side is e-separable. So, by Corollary 5.14, if X is not e-separable, then $\prod\{X_\alpha : \alpha < \lambda, |X_\alpha| < \lambda\}$ is not e-separable either.

Now, we define $Y_\nu = \prod\{X_\alpha : \alpha < \lambda, |X_\alpha| = \nu\}$ for $\nu \in \lambda \cap \mathrm{Card}$. Note that, by Theorem 5.1, Y_ν is e-separable. Hence, the minimality of λ implies that $I = \{\nu \in \lambda \cap \mathrm{Card} : Y_\nu \neq \emptyset\}$ has size λ; otherwise, $X \simeq \prod\{Y_\nu : \nu \in I\}$ is a smaller non-e-separable product of e-separable spaces. Note that this already shows that $\lambda = \omega_\lambda$.

Let us suppose that $\lambda \in \mathcal{M}^*$; we will arrive at a contradiction shortly. Take a decreasing sequence $(I_n)_{n\in\omega}$ of subsets of I so that $\bigcap\{I_n : n \in \omega\} = \emptyset$ and $\lambda = |I_n| = |I \setminus I_n|$ for each $n \in \omega$. Note that, by Observation 5.8, $d(\prod\{Y_\nu : \nu \in I \setminus I_n\}) = \lambda$.

Claim. $\prod\{Y_\nu : \nu \in I_n\}$ contains a closed discrete set of size λ.

If $\lambda \in \mathcal{M}^*$, then $Z = \prod\{D(\nu)^\lambda : \nu \in \lambda \cap \mathrm{Card}\}$ contains a closed discrete subset of size λ. Hence, it suffices to show that Z embeds into $\prod\{Y_\nu : \nu \in I_n\}$ as a closed subspace. In order to do that, note that the set $\{\nu \in I_n : \nu > \nu_0\}$ has size λ for every $\nu_0 \in \lambda \cap \mathrm{Card}$. Now it is routine to construct the embedding of Z.

Finally, we can apply Lemma 6.1 to see that the product $X = \prod\{Y_\nu : \nu \in I\}$ must be e-separable. This contradicts our initial assumption on X. $\square$

6. Final Remarks and Further Questions

First, referring back to §3, it is natural to ask if we can say something similar to Theorem 3.4 about products. Let us present a result in this direction.

Lemma 6.1. *Suppose that κ is an infinite cardinal and there is a decreasing sequence $(I_n)_{n\in\omega}$ of non-empty subsets of κ with empty intersection such that $\prod\{X_\alpha : \alpha \in I_n\}$ contains a closed discrete subset of size $\delta_n = d(\prod\{X_\alpha : \alpha \in \kappa \setminus I_n\})$ for every $n \in \omega$. Then $X = \prod\{X_\alpha : \alpha < \kappa\}$ is e-separable.*

Proof. Let $X(J)$ denote $\prod\{X_\alpha : \alpha \in J\}$ for $J \subseteq \kappa$. Pick $D_n = \{d^n_\xi : \xi < \delta_n\} \subseteq X(\kappa \setminus I_n)$ dense and let $F_n = \{f^n_\xi : \xi < \delta_n\} \subseteq X(I_n)$ be closed discrete.

Now, for each $n \in \omega$, we define $e^n_\xi \in X$ for $\xi < \delta_n$ by

$$e^n_\xi(\alpha) = \begin{cases} d^n_\xi(\alpha), & \text{for } \alpha \in \kappa \setminus I_n, \text{ and} \\ f^n_\xi(\alpha), & \text{for } \alpha \in I_n. \end{cases}$$

We claim that the set $E_n = \{e^n_\xi : \xi < \delta_n\}$ is closed discrete. This comes from the following observation. Suppose that $E \subseteq \prod\{X_\alpha : \alpha < \kappa\}$ and there is $I \subseteq \kappa$ such that π_I is 1-1 on E and the image $\pi_I{}''E$ is closed discrete in $\prod\{X_\alpha : \alpha \in I\}$. Then E is closed discrete.

Now it is clear that $\bigcup\{E_n : n \in \omega\}$ is a dense and σ-closed-discrete subset of X and the proof is complete. □

Second, recall that if $D(\lambda)$ is the discrete space of size $\lambda \geq \kappa$, then, by Lemma 5.15, $D(\lambda)^\kappa$ is e-separable. Actually, we can say a bit more in this case.

Lemma 6.2. *Let $(\kappa_i)_{i\in I}$ be a sequence of cardinals and consider the product space $X = \prod\{D(\kappa_i) : i \in I\}$. Suppose that the set $\{i \in I : \kappa_i = \kappa\}$ is infinite where $\kappa = \sum_{i\in I} \kappa_i$. Then X has a σ-discrete π-base.*

Proof. Let J be a countable infinite subset of $\{i \in I : \kappa_i = \kappa\}$. Note that $\kappa_j = \sum_{i\in I\setminus\{j\}} \kappa_i$ for all $j \in J$. Now let $\{p^j_n(\alpha) : \alpha \in \kappa_j\}$ be an enumeration of the set

$$\{p \subseteq \bigcup_{i\in I\setminus\{j\}} (\{i\} \times \kappa_i) : p \text{ is a function and } |p| = n\}$$

for every $j \in J$ and $n \in \omega$. Consider

$$A^j_n = \{p^j_n(\alpha) \cup \{(j, \alpha)\} : \alpha \in \kappa_j\};$$

finally, define $\mathcal{V}^j_n = \{[q] : q \in A^j_n\}$.

Note that each $\mathcal{V}_n^j$ is a discrete family: If $a = (a_i)_{i \in I}$ is any point of X, then

$$U = \{(x_i)_{i\in I} \in X : x_j = a_j\}$$

is an open neighborhood of a in X such that

$$\{V \in \mathcal{V}_n^j : V \cap U \neq \emptyset\} = \{V_{p_n^j(a_j)\cup\{(j,a_j)\}}\}.$$

Moreover, $\mathcal{V} = \bigcup_{j\in J}\bigcup_{n\in\omega}\mathcal{V}_n^j$ is a π-base for X, since any non-empty open subset of X is determined by a finite number of coordinates which constitutes a finite subset of $I \setminus \{j\}$ for some $j \in J$. □

Corollary 6.3. *If $\lambda \geq \kappa$, then $D(\lambda)^\kappa$ has a σ-discrete π-base.*

Finally, selection principles (see, e.g., [27]) and selective versions of separability and d-separability (see, e.g., [3], [28]) were proved to be fascinating notions to study. So let us introduce the selective version of e-separability.

Definition 6.4. A topological space X is *E-separable* if, for every sequence of dense sets $(D_n)_{n\in\omega}$ of X, we can select $E_n \subseteq D_n$ so that E_n is closed discrete in X and $\bigcup_{n\in\omega} E_n$ is dense in X.

Note that every space with a σ-discrete π-base is E-separable as well. Let us point out that the example of Theorem 4.2 is an e-separable space which is not E-separable.

We ask the following questions.

Problem 6.5. *Suppose that X is an e-separable space which is the product of discrete spaces. Is X E-separable as well?*

Problem 6.6. *How does E-separability behave under powers and products?*

In particular, let us refer the interested reader to [3] for an in-depth look at the general behavior of D-separability, the selective version of d-separability.

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