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S. García-Ferreira, R. Rojas-Hernández, and
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Department of Mathematics & Statistics
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CATEGORICAL PROPERTIES ON THE HYPERSPACE OF NONTRIVIAL CONVERGENT SEQUENCES

S. GARCÍA-FERREIRA, R. ROJAS-HERNÁNDEZ, AND Y. F. ORTIZ-CASTILLO

Abstract. In this paper, we shall study categorial properties of the hyperspace of all nontrivial convergent sequences $\mathcal{S}_c(X)$ of a Fréchet-Urysohn space X equipped with the Vietoris topology. We mainly prove that $\mathcal{S}_c(X)$ is meager whenever X is a crowded space; as a corollary, we obtain that if $\mathcal{S}_c(X)$ is Baire, then X has a dense subset of isolated points. As an interesting example, $\mathcal{S}_c(\omega_1)$ has the Baire property, where ω_1 carries the order topology. (This answers a question from *The hyperspace of convergent sequences*, Topology Appl. **196** (2015), part B, 795–804.) We can give more examples like this one by proving that the Alexandroff duplicate $\mathcal{A}(Z)$ of a space Z satisfies that $\mathcal{S}_c(\mathcal{A}(Z))$ has the Baire property whenever Z is a Σ-product of completely metrizable spaces and Z is crowded. Also, we show that if $\mathcal{S}_c(X)$ is pseudocompact, then X has a relatively countably compact dense subset of isolated points, every finite power of X is pseudocompact, and every G_δ-point in X must be isolated. We also establish several topological properties of the hyperspace of nontrivial convergent sequences of countable Fréchet-Urysohn spaces with only one non-isolated point.

1. Introduction

All our spaces will be Tychonoff (completely regular and Hausdorff). The letters $\mathbb{P}$ and $\mathbb{N}$ will denote the irrational numbers and the natural numbers, respectively. The positive natural numbers will be denoted by

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$\mathbb{N}^+$. The Greek letter ω stands for the first infinite cardinal number and ω_1 stands for the first uncountable cardinal number endowed with the order topology. If $A, B \subseteq \mathbb{N}$, then $A \subseteq^* B$ means that $A \setminus B$ is finite.

For a topological space X, $\mathcal{CL}(X)$ will denote the set of all nonempty closed subsets of X. For a nonempty family $\mathcal{U}$ of subsets of X let

$$\langle \mathcal{U} \rangle = \{F \in \mathcal{CL}(X) : F \subseteq \bigcup \mathcal{U} \text{ and } F \cap U \neq \emptyset \text{ for every } U \in \mathcal{U}\}.$$

If $\mathcal{U} = \{U_1, \ldots, U_n\}$, in some convenient cases, $\langle \mathcal{U} \rangle$ will be denoted by $\langle U_1, \ldots, U_n \rangle$. A base for the *Vietoris topology* on $\mathcal{CL}(X)$ is the family of all sets of the form $\langle \mathcal{U} \rangle$, where $\mathcal{U}$ runs over all nonempty finite families of nonempty open subsets of X. In the sequel, any subset $\mathcal{D} \subseteq \mathcal{CL}(X)$ will carry the relative Vietoris topology as a subspace of $\mathcal{CL}(X)$. Given $\mathcal{D} \subseteq \mathcal{CL}(X)$ and a nonempty family $\mathcal{U}$ of subsets of X, we let $\langle \mathcal{U} \rangle_{\mathcal{D}} = \langle \mathcal{U} \rangle \cap \mathcal{D}$. For simplicity, if there is no possibility of confusion, we simply write $\langle \mathcal{U} \rangle$ instead of $\langle \mathcal{U} \rangle_{\mathcal{D}}$. All topological notions whose definition is not included here should be understood as in [3].

Some of the most studied hyperspaces on a space X have been

$$\mathcal{K}(X) = \{K \in \mathcal{CL}(X) : K \text{ is compact}\} \text{ and}$$

$$\mathcal{F}(X) = \{F \subseteq X : F \text{ is finite and } F \neq \emptyset\};$$

see, for instance, Valentin Gutev's survey paper [8]. In [13], Jan van Mill, Jan Pelant, and Roman Pol consider the hyperspace consisting of all finite subsets together with all the Cauchy sequences without limit point of a metric space. In a different context, Liang-Xue Peng and Zhi-Fang Guo [14] consider the set $\mathcal{F}_S(X)$ of all convergent sequences of a space X and study the existence of a metric d on the set X such that d metrizes all subspaces of X which belong to $\mathcal{F}_S(X)$; that is, the restriction of d to A generates the subspace topology on A for every $A \in \mathcal{F}_S(X)$ (these kinds of problems have been analyzed in [2]).

The hyperspace of nontrivial convergent sequences was studied in [5] and, more recently, in [6], [9], [10], and [11]. This hyperspace is formally defined as follows:

> Given a space X, a *nontrivial convergent sequence* of X is a subset $S \subseteq X$ such that $|S| = \omega$; S has a unique non-isolated point, denoted by x_S; and $|S \setminus U| < \omega$ for each neighborhood U of x_S. By using this notion, we define $\mathcal{S}_c(X) = \{S \in \mathcal{CL}(X) : S \text{ is a nontrivial convergent sequence}\}$.

It is pointed out in [11] that the family of all subsets of $\mathcal{S}_c(X)$ of the form $\langle \mathcal{U} \rangle$, where $\mathcal{U}$ is a finite family of pairwise disjoint open subsets of X, is a base for the Vietoris topology on $\mathcal{S}_c(X)$. We will refer to this family as the *canonical basis* of $\mathcal{S}_c(X)$ and its elements will be called *canonical open sets*.

Throughout this paper, it is evident why we shall only consider non discrete Fréchet-Urysohn spaces. Thus, the hyperspace $\mathcal{S}_c(X)$ will be always nonempty.

A fundamental task in the study of the hyperspace $\mathcal{S}_c(X)$ is to determine its topological relationship with the base space X and vice versa. For instance, the connection between the connectedness in X and $\mathcal{S}_c(X)$ is explored in [5], [6], and [11]. The category property on $\mathcal{S}_c(X)$ is another of the topological properties studied in [5]; it is proved that $\mathcal{S}_c(X)$ is never a Baire space when the space X is crowded and that $\mathcal{S}_c(X)$ is even meager if, in addition, X is second countable. Following this direction, our main purpose of this article is to continue studying the category properties on $\mathcal{S}_c(X)$.

Section 2 is devoted to studying the hyperspace of countable Freéchet-Urysohn spaces with just one accumulation point. We show that if X is such a space, then $\mathcal{S}_c(X)$ is homeomorphic to the rational numbers if and only if X has a countable base.

Section 3 is devoted to studying categorical properties of the hyperspace of nontrivial convergent sequences. Concerning this topic, in [5, questions 3.4 and 3.5], the authors ask about the meager property in $\mathcal{S}_c(X)$ when X is a metrizable crowded space and about the meager property of the space $\mathcal{S}_c(\omega_1)$. We answer both questions by showing that $\mathcal{S}_c(X)$ is meager whenever X is crowded and that $\mathcal{S}_c(\omega_1)$ has the Baire property. Finally, we prove that if $\mathcal{S}_c(X)$ is pseudocompact, then X has a relatively countably compact dense subset of isolated points, every finite power of X is pseudocompact, and every G_δ-point in X must be isolated. The last result of this paper is that if Z is a Σ-product of completely metrizable spaces and Z is crowded, then $\mathcal{S}_c(\mathcal{A}(Z))$ has the Baire property, where $\mathcal{A}(Z)$ is the Alexandroff duplicate of Z. For further research, we list some open questions related our results.

2. Countable Fréchet-Urysohn Spaces

The main result of this section is to give two non-homeomorphic spaces X and Y such that $\mathcal{S}_c(X)$ and $\mathcal{S}_c(Y)$ are homeomorphic.

Since $\mathcal{S}_c(X)$ is, in general, a dense proper subset of $\mathcal{CL}(X)$, it cannot be compact. However, we can say more.

Lemma 2.1. *For every space X, $\mathcal{S}_c(X)$ is nowhere locally compact.*

Proof. Fix $S \in \mathcal{S}_c(X)$ and let O be a neighborhood of S. Choose a canonical open set $\langle U_1, ..., U_n \rangle$ so that $S \in \langle cl_X(U_1), ..., cl_X(U_n) \rangle \subseteq O$;

the sets $\{cl_X(U_1), ..., cl_X(U_n)\}$ are pairwise disjoint; $\{x_i\} = S \cap cl_X(U_i)$ for each $1 \leq i < n$; and $x_S \in U_n$. Choose a local base $\mathcal{B}$ at x_S and one more $\mathcal{B}_i$ at x_i for each $1 \leq i < n$. Notice that

$$\langle cl_X(U_1 \cap B_1), ..., cl_X(U_{n-1} \cap B_{n-1}), cl_X(U_n \cap B)) \rangle$$

is a closed subset of $\mathcal{S}_c(X)$ for each $(B_1, ..., B_{n-1}, B) \in \mathcal{B}_1 \times \mathcal{B}_2 \times \cdots \mathcal{B}_{n-1} \times \mathcal{B}$. Therefore,

$$\{\langle cl_X(U_1 \cap B_1), ..., cl_X(U_{n-1} \cap B_{n-1}), cl_X(U_n) \cap B) \rangle :$$

$$(B_1, ..., B_{n-1}, B) \in \mathcal{B}_1 \times \mathcal{B}_2 \times \cdots \mathcal{B}_{n-1} \times \mathcal{B}\}$$

is a family of closed subsets of $\langle \mathcal{U} \rangle$ with the finite intersection property. But

$$\bigcap \{\langle cl_X(U_1 \cap B_1), ..., cl_X(U_{n-1} \cap B_{n-1}), cl_X(U_n) \cap B) \rangle :$$

$$(B_1, ..., B_{n-1}, B) \in \mathcal{B}_1 \times \mathcal{B}_2 \times \cdots \mathcal{B}_{n-1} \times \mathcal{B}\} = \emptyset.$$

Therefore, neither $\langle cl_X(U_1), ..., cl_X(U_n) \rangle$ nor $cl_X(O)$ can be compact. $\square$

The space that we are looking for will be countable with only one non-isolated point. To deal with this kind of spaces we shall need the following terminology.

To each free filter $\mathcal{F}$ on $\mathbb{N}$, we associate the space $\xi(\mathcal{F})$ whose underlying set is $\mathbb{N} \cup \{\mathcal{F}\}$; all elements of $\mathbb{N}$ are isolated and the neighborhoods of $\mathcal{F}$ are of the form $A \cup \{\mathcal{F}\}$ where $A \in \mathcal{F}$. It is evident that the space $\xi(\mathcal{F})$ is zero-dimensional and Hausdorff for every free filter $\mathcal{F}$ on $\mathbb{N}$. A free filter $\mathcal{F}$ on $\mathbb{N}$ is said to be a *Fréchet-Urysohn filter* if the space $\xi(\mathcal{F})$ is Fréchet-Urysohn (there are plenty of this kind of filter; see [7]). The simplest example of a Fréchet-Urysohn filter is the *Fréchet filter* $\mathcal{F}_r = \{A \subseteq \mathbb{N} : |\mathbb{N} \setminus A| < \omega\}$, and notice that $\xi(\mathcal{F}_r)$ is a convergent sequence with its limit point. One more example of a Fréchet-Urysohn filter with a countable base is the filter $\mathcal{P} = \{A \subseteq \mathbb{N} : \exists m \in \mathbb{N} \forall n \geq m(P_n \subseteq A)\}$, where $\{P_n : n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$ in infinite subsets. At this point, we can say that these two Fréchet-Urysohn filters are the only ones, up to homeomorphism, with a countable base. These two filters can be characterized as follows.

Theorem 2.2. *Let $\mathcal{F}$ be a Fréchet-Urysohn filter. Then $\mathcal{S}_c(\xi(\mathcal{F}))$ is homeomorphic to $\mathbb{P}$ if and only if $\mathcal{F}$ has a countable base.*

Proof. Necessity. If $\mathcal{S}_c(\xi(\mathcal{F}))$ is homeomorphic to $\mathbb{P}$, then $\mathcal{S}_c(\xi(\mathcal{F}))$ is second countable. Hence, by applying [11, Theorem 6.5], we obtain that $\xi(\mathcal{F})$ is also second countable. Thus, we conclude that the filter $\mathcal{F}$ has a countable base.

Sufficiency. Assume that $\mathcal{B}$ is a countable base of $\mathcal{F}$. It then follows from [11, Corollary 6.15] that the space $\mathcal{S}_c(\xi(\mathcal{F}))$ is separable. Since every open subset of $\xi(\mathcal{F})$ is closed, we may apply [11, Proposition 3.2] to see that the canonical basis of $\mathcal{S}_c(\xi(\mathcal{F}))$ consists of clopen subsets. So we also obtain that $\mathcal{S}_c(\xi(\mathcal{F}))$ is zero-dimensional. In order to see that $\mathcal{S}_c(\xi(\mathcal{F}))$ is homeomorphic to $\mathbb{P}$, in virtue of [12, Theorem 1.9.8], it is enough to show that it is nowhere locally compact and completely metrizable. Indeed, by Lemma 2.1, we know that $\mathcal{S}_c(\xi(\mathcal{F}))$ is nowhere locally compact.

Let us show that $\mathcal{S}_c(\xi(\mathcal{F}))$ is completely metrizable. Enumerate $\mathcal{B}$ as $\{B_n : n \in \mathbb{N}\}$ and, without loss of generality, assume that $B_{n+1} \subseteq B_n$ for each $n \in \mathbb{N}$. Let $E_n = B_n \setminus B_{n+1}$ for each $n \in \mathbb{N}$. Without loss of generality, we may assume that $\{E_n : n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$. Consider the map $f : \xi(\mathcal{F}) \to \mathbb{R}$ defined as $f(m) = 1/2^n$ for $m \in E_n$ and $f(\mathcal{F}) = 0$. Define a metric d on $\xi(\mathcal{F})$ as follows: For $x, y \in \xi(\mathcal{F})$, if $x, y \in E_n$ for some $n \in \mathbb{N}$ and $x \neq y$, then we let $d(x, y) = 1/2^n$, and we define $d(x, y) = |f(x) - f(y)|$ otherwise. Notice that d is a complete metric compatible with the topology of $\xi(\mathcal{F})$. It is well known that the Hausdorff metric induced by d on $\mathcal{CL}(\xi(\mathcal{F}))$ is also complete (see [16]). Thus, we obtain that $\mathcal{CL}(\xi(\mathcal{F}))$ is completely metrizable. To show that $\mathcal{S}_c(\xi(\mathcal{F}))$ is completely metrizable, it is enough to prove that $\mathcal{S}_c(\xi(\mathcal{F}))$ is a G_δ-set in $\mathcal{CL}(\xi(\mathcal{F}))$. For each $B \in \mathcal{B}$, we let $O_B = \{S \in \mathcal{CL}(\xi(\mathcal{F})) : |S \setminus B| < \omega\}$ and note that this set is open in $\mathcal{CL}(\xi(\mathcal{F}))$; indeed,

$$O_B = \bigcup\{\langle\{B\} \cup \{\{x\} : x \in F\}\rangle : F \in \mathcal{F}(\xi(\mathcal{F}) \setminus B) \cup \{\emptyset\}\} \cup \mathcal{F}(\xi(\mathcal{F}) \setminus B).$$

On the other hand, since $\mathcal{F}(\xi(\mathcal{F}))$ is F_σ in $\mathcal{CL}(\xi(\mathcal{F}))$, it follows that $\mathcal{S}_c(\xi(\mathcal{F})) = \bigcap\{O_B : B \in \mathcal{B}\} \setminus \mathcal{F}(\xi(\mathcal{F}))$ is a G_δ-subset of $\mathcal{S}_c(\xi(\mathcal{F}))$. □

Example 2.3. *There are two spaces X and Y such that $\mathcal{S}_c(X)$ is homeomorphic to $\mathcal{S}_c(Y)$, but X is not homeomorphic to Y.*

Proof. By Theorem 2.2, we know that $\mathcal{S}_c(\xi(\mathcal{F}_r))$ is homeomorphic to $\mathcal{S}_c(\xi(\mathcal{P}))$, but it is clear that the spaces $\xi(\mathcal{F}_r)$ and $\xi(\mathcal{P})$ cannot be homeomorphic. □

Both spaces X and Y considered in the previous example have a dense set of isolated points. However, we still do not know any counterexample in the realm of crowded (Fréchet-Urysohn) spaces.

Problem 2.4. *Find two non-homeomorphic crowded spaces X and Y such that $\mathcal{S}_c(X)$ and $\mathcal{S}_c(Y)$ are homeomorphic.*

Addressing Problem 2.4, we would like to make some comments about the following class of sequential, non-Fréchet-Urysohn crowded spaces.

The symbol $FF(\mathbb{N})$ will denote the set of all free filters on $\mathbb{N}$ and $Seq = \bigcup_{n\in N} \mathbb{N}^n$. If $s \in Seq$ and $n \in \mathbb{N}$, then the *concatenation* of s and n is the function $s^\frown n = s \cup \{(dom(s), n)\}$. For a function $\delta : Seq \to FF(\mathbb{N})$, we define a topology τ_δ on Seq by defining $V \in \tau_\delta$ if and only if $\{n \in \mathbb{N} : s^\frown n \in V\} \in \delta(s)$ for every $s \in V$. It is well known that $Seq(\delta) = (Seq, \tau_\delta)$ is an extremally disconnected, zero dimensional Hausdorff space for every function $\delta : Seq \to FF(\mathbb{N})$. Besides, $Seq(\delta)$ is a sequential space provided that $\delta(s)$ is a Fréchet-Urysohn filter for each $s \in Seq$. It is not hard to see that $Seq(\delta)$ cannot be Fréchet-Urysohn.

Question 2.5. *Are $\mathcal{S}_c(Seq(\mathcal{F}_r))$ and $\mathcal{S}_c(Seq(\mathcal{P}))$ homeomorphic?*

After having Theorem 2.2, we shall give next a necessary condition when the space $\mathcal{S}_c(Seq(\mathcal{F}))$ is Baire.

Following A. V. Arhangel'skii [1], we say that $x \in X$ is an α_2-*point* if, for every family $\{S_n : n \in \mathbb{N}\}$ of sequences converging to x, there is $S \in \mathcal{S}_c(X)$ converging to x such that $|S \cap S_n| = \omega$ for all $n \in \mathbb{N}$. We shall say that a filter $\mathcal{F}$ is an α_2-*filter* if every point of the space $\xi(\mathcal{F})$ is an α_2-point.

Lemma 2.6. *Let $\mathcal{F}$ be a Fréchet-Urysohn filter, $S \in \mathcal{S}_c(\xi(\mathcal{F}))$, and $n \in \mathbb{N}$. Then $\mathcal{D}^n_S = \{T \in \mathcal{S}_c(\xi(\mathcal{F})) : |T \cap S| \geq n\}$ is a dense open subset of $\mathcal{S}_c(\xi(\mathcal{F}))$.*

Proof. Fix $S \in \mathcal{S}_c(\xi(\mathcal{F}))$ and $n \in \mathbb{N}$. Assume that $\langle \mathcal{U} \cup \{A\}\rangle$ is a canonical open set of $\mathcal{S}_c(\xi(\mathcal{F}))$, where $\mathcal{U}$ consists of singletons which are elements of $\mathbb{N}$ and $A \in \mathcal{F}$. Since $S \subseteq^* A$, we can easily find $T \in \langle \mathcal{U} \cup \{A\}\rangle$ so that $|T \cap S| \geq n$. To prove that $\mathcal{D}^n_S$ is open, fix $T \in \mathcal{D}^n_S$. Then choose a family of singletons $\mathcal{U}$ consisting of n elements of $T \cap S$. It is then clear that $T \in \langle \mathcal{U} \cup \{\mathbb{N}\}\rangle \subseteq \mathcal{D}^n_S$. $\square$

Theorem 2.7. *Let $\mathcal{F}$ be a Fréchet-Urysohn filter. If the hyperspace $\mathcal{S}_c(\xi(\mathcal{F}))$ is not meager, then $\mathcal{F}$ is an α_2-filter.*

Proof. Let $\{S_m : m \in \mathbb{N}\}$ be a countable subset of $\mathcal{S}_c(\xi(\mathcal{F}))$. According to Lemma 2.6, the set $\mathcal{D}^n_{S_m}$ is open and dense for each $n, m \in \mathbb{N}$. Hence, we can find $T \in \bigcap_{n,m\in\mathbb{N}} \mathcal{D}^n_{S_m}$. It is then clear that $|T \cap S_m| = \omega$ for every $m \in \mathbb{N}$. $\square$

To give another example of a Frechét-Urysohn filter we need the characterization of the Frechét-Urysohn filters given in [15]: A free filter $\mathcal{F}$ is Frechét-Urysohn if and only if there is an AD-family $\mathcal{A}$ maximal with respect to the following properties:

(1) $F \in \mathcal{F}$ if and only if $A \subseteq^* F$ for all $A \in \mathcal{A}$, and
(2) $\mathcal{A}$ is an AD-family.

As a consequence, we have that if $S \in \mathcal{S}_c(\xi(\mathcal{F}))$, then $|S \cap A| = \omega$ for some $A \in \mathcal{A}$. When $\mathcal{F}$ is a Frechét-Urysohn filter and $\mathcal{A}$ is the AD-family given by its characterization, we shall put $\mathcal{F} = \mathcal{F}_{\mathcal{A}}$. If $\mathcal{Q}$ is a partition of $\mathbb{N}$, then $\mathcal{F}_{\mathcal{Q}}$ is the well-known *FAN-filter*. Since the FAN-filter $\mathcal{F}_{\mathcal{Q}}$ cannot be an α_2-filter, it follows from the previous theorem that $\mathcal{S}_c(\xi(\mathcal{F}_{\mathcal{Q}}))$ is meager.

The following problem is then natural.

Problem 2.8. *Determine the Fréchet-Urysohn filters $\mathcal{F}$ on $\mathbb{N}$ for which the space $\mathcal{S}_c(\xi(\mathcal{F}))$ is Baire.*

3. Category in $\mathcal{S}_c(X)$

Theorem 2.2 provides an example of a space X for which $\mathcal{S}_c(X)$ is Baire. In what follows, we shall describe more examples of spaces for which $\mathcal{S}_c(X)$ has this property. To do this we list some easy facts and introduce some useful notation.

Remark 3.1. Let Y be a non discrete subspace of X, then we have that

(a) Y is dense in X if and only if $\mathcal{S}_c(Y)$ is dense in $\mathcal{S}_c(X)$.
(b) Y is open in X if and only if $\mathcal{S}_c(Y)$ is open in $\mathcal{S}_c(X)$.

As a consequence, if Y is open and dense in X, then $\mathcal{S}_c(Y)$ is Baire if and only if $\mathcal{S}_c(X)$ is Baire.

For a space X, $\mathcal{S} \subseteq \mathcal{S}_c(X)$, and $D \subseteq X$, we define

$$\mathcal{G}(\mathcal{S}, D) = \{S \cup F : S \in \mathcal{S} \text{ and } F \in \mathcal{F}(D) \cup \{\emptyset\}\}.$$

The set $\mathcal{G}(\mathcal{S}, X)$ will be simply denoted by $\mathcal{G}(\mathcal{S})$. If $\mathcal{S} \subseteq \mathcal{S}_c(X)$, then we have that $\mathcal{S} \subseteq \mathcal{G}(\mathcal{S}, D)$ for all $D \subseteq X$. The following properties can be easily verified.

(c) If $\mathcal{S} \subseteq \mathcal{S}_c(X)$ is dense, then $\mathcal{G}(\mathcal{S}, D)$ is dense in $\mathcal{S}_c(X)$ for all $D \subseteq X$.
(d) If $Y \subseteq X$ is open and $D \subseteq X$ is discrete, then $\mathcal{G}(\mathcal{S}_c(Y), D)$ is open in $\mathcal{S}_c(X)$.

In the next theorem, we state several conditions on a space X that guarantee the Baire property of the hyperspace $\mathcal{S}_c(X)$. We shall need the following lemmas.

Lemma 3.2. *Let X be a space such that its set D of isolated points is dense in X. If Y is a nonempty open subset of X such that $\mathcal{S}_c(Y)$ is Baire, then $\mathcal{G}(\mathcal{S}_c(Y), D)$ is Baire.*

Proof. Assume that $\{\mathcal{D}_n : n \in \mathbb{N}\}$ is a family of open and dense subsets of $\mathcal{G}(\mathcal{S}_c(Y), D)$. Let $\langle \mathcal{U} \rangle$ be a nonempty canonical open subset of $\mathcal{G}(\mathcal{S}_c(Y), D)$. Without loss of generality, suppose that $\mathcal{U} = \mathcal{U}' \cup \mathcal{U}_D$ where

$\mathcal{U}'$ is a family of pairwise disjoint infinite open subsets of Y and $\mathcal{U}_D$ is a family of singleton sets with elements in D. Set $F = \bigcup \mathcal{U}_D$ and observe that $(S \cap Y) \cup F = S$ for each $S \in \langle \mathcal{U} \rangle$. For each $n \in \mathbb{N}$, define $\mathcal{E}_n = \{S \cap Y : S \in \mathcal{D}_n \cap \langle \mathcal{U} \rangle\}$. It is straightforward to verify that $\mathcal{E}_n$ is open and dense in $\langle \mathcal{U}' \rangle$ for all $n \in \mathbb{N}$. Since $\mathcal{S}_c(Y)$ is Baire and $\langle \mathcal{U}' \rangle$ is a nonempty open subset of $\mathcal{S}_c(Y)$, we can find $T \in \langle \mathcal{U}' \rangle \cap \left(\bigcap\{\mathcal{E}_n : n \in \mathbb{N}\}\right)$. Then we have that $T \cup F \in \langle \mathcal{U} \rangle \cap \left(\bigcap\{\mathcal{E}_n : n \in \mathbb{N}\}\right) \subseteq \langle \mathcal{U} \rangle \cap \left(\bigcap\{\mathcal{D}_n : n \in \mathbb{N}\}\right)$. Since $\langle \mathcal{U} \rangle$ was arbitrary, the space $\mathcal{G}(\mathcal{S}_c(Y), D)$ is Baire. □

We omit the proof of the following well-known results.

Lemma 3.3. *Let X be a space.*

(1) *If $\{V_i : i \in I\}$ is a family of open Baire nonempty open subsets of X, then $\bigcup_{i \in I} V_i$ is also Baire.*
(2) *If X has a family $\mathcal{U}$ consisting of pairwise disjoint nonempty open meager subsets and $\bigcup \mathcal{U}$ is dense in X, then X is also meager.*

Theorem 3.4. *Let X be a space such that its set D of isolated points is dense in X, and let $\{X_\gamma : \gamma \in \Gamma\}$ be a family of clopen subspaces of X. If the following conditions are satisfied*

(i) *the set $\{x_S : S \in \bigcup_{\gamma \in \Gamma} \mathcal{S}_c(X_\gamma)\}$ is dense in $X \setminus D$,*
(ii) *$\mathcal{S}_c(X_\gamma)$ is Baire for each $\gamma \in \Gamma$,*
(iii) *the family $\{\mathcal{G}(\mathcal{S}_c(X_\gamma), D) : \gamma \in \Gamma\}$ is pairwise disjoint, and*
(iv) *$\mathcal{S}_c(X) = \bigcup\{\mathcal{G}(\mathcal{S}_c(X_\gamma)) : \gamma \in \Gamma\}$,*

then $\mathcal{S}_c(X)$ is a Baire space.

Proof. First, we shall prove that $\mathcal{D} = \bigcup\{\mathcal{G}(\mathcal{S}_c(X_\gamma), D) : \gamma \in \Gamma\}$ is dense in $\mathcal{S}_c(X)$. Indeed, let $O = \langle \mathcal{U} \rangle$ be a nonempty canonical open subset of $\mathcal{S}_c(X)$. By condition (i), we may choose $S \in O$ so that $x_S \in X_\gamma$ for some $\gamma \in \Gamma$. Notice that $S \cap X_\gamma \in \mathcal{S}_c(X_\gamma)$. For each $U \in \mathcal{U}$, select a point $d_U \in D \cap U$ and consider the convergent sequence $S_0 = (S \cap X_\gamma) \cup \{d_U : U \in \mathcal{U}\}$. Then we have that $S_0 \in O \cap \mathcal{G}(\mathcal{S}_c(X_\gamma), D)$ and so $\mathcal{D} \cap O \neq \emptyset$. By Remark 3.1(d), we obtain that $\mathcal{D}$ is open in $\mathcal{S}_c(X)$ and $\mathcal{D} = \bigcup\{\mathcal{G}(\mathcal{S}_c(X_\gamma), D) : \gamma \in \Gamma\}$. By condition (ii) and Lemma 3.2, we know that $\mathcal{G}(\mathcal{S}_c(X_\gamma), D)$ is Baire for each $\gamma \in \Gamma$. Thus, Lemma 3.3(1) implies that $\mathcal{D}$ is Baire and so $\mathcal{S}_c(X)$ is also Baire. □

Example 3.5. *The space $\mathcal{S}_c(\omega_1)$ is Baire.*

Proof. Let D be the set of all isolated points of ω_1 and

$$Y = \{\alpha < \omega_1 : \alpha \text{ is a limit ordinal and } (\beta, \alpha) \subseteq D \text{ for some } \beta < \alpha\}.$$

Set $X = Y \cup D$. Since X is open and dense in ω_1, according to Remark 3.1(a),(b), it is enough to show that $\mathcal{S}_c(X)$ is Baire. For each $\alpha \in Y$, pick β_α so that $(\beta_\alpha, \alpha) \subseteq D$, and let $X_\alpha = (\beta_\alpha, \alpha]$. According to Theorem 2.2, we have that the space $\mathcal{S}_c(X_\alpha)$ is Baire for each $\alpha \in Y$. Thus, D and the family $\{X_\alpha : \alpha \in Y\}$ satisfy all the conditions of Theorem 3.4. Therefore, $\mathcal{S}_c(X)$ is a Baire space. $\square$

For the description of our next example, we recall that an *almost disjoint family* $\mathcal{A}$ is an infinite family of infinite subsets of $\mathbb{N}$ such that $|A \cap B| < \omega$ for distinct $A, B \in \mathcal{A}$. The Ψ-*space* associated with an AD-family $\mathcal{A}$, denoted by $\Psi(\mathcal{A})$, is the space whose underlying set is $\mathbb{N} \cup \mathcal{A}$; $\mathbb{N}$ is discrete and, for each $A \in \mathcal{A}$, $(A \setminus F) \cup \{A\}$ with $F \in \mathcal{F}(\mathbb{N})$ is a basic neighborhood of A. It is easy to see that the space $\Psi(\mathcal{A})$ is always first countable and zero dimensional.

Example 3.6. *For every almost disjoint family $\mathcal{A}$, the space $\mathcal{S}_c(\Psi(\mathcal{A}))$ is Baire.*

Proof. Let $D = \mathbb{N}$ and, for every $A \in \mathcal{A}$, define $X_A = A \cup \{A\}$. Theorem 2.2 asserts that the space $\mathcal{S}_c(X_A)$ is Baire for each $A \in \mathcal{A}$. Since D and the family $\{X_A : A \in \mathcal{A}\}$ satisfy all the conditions of Lemma 3.4, we conclude that $\mathcal{S}_c(X)$ is Baire. $\square$

We point out that the spaces just considered above have a dense subset of isolated points. The next result shows that without the presence of isolated points in X, the space $\mathcal{S}_c(X)$ can never have the Baire property ([5, Theorem 3.2] shows that if X is a crowded metric space, then $\mathcal{S}_c(X)$ is not Baire).

Theorem 3.7. *If X is crowded, then $\mathcal{S}_c(X)$ is meager.*

Proof. As X is crowded, we have that the family of all canonical nonempty open subsets $\langle \mathcal{U} \rangle$ of $\mathcal{S}_c(X)$ such that $|\mathcal{U}| \geq 2$ is a base for $\mathcal{S}_c(X)$. Thus, in virtue of Lemma 3.3(2), it suffices to prove that every such open set is meager. Indeed, fix a canonical open set $\langle \mathcal{U} \rangle$ such that $|\mathcal{U}| \geq 2$ and $\langle \mathcal{U} \rangle = \bigcup\{\mathcal{N}(U, n) : U \in \mathcal{U} \text{ and } n \in \mathbb{N}^+\}$, where $\mathcal{N}(U, n) = \{S \in \langle \mathcal{U} \rangle : |S \setminus U| = n\}$ for each $U \in \mathcal{U}$ and $n \in \mathbb{N}^+$. Let us prove that each set $\mathcal{N}(U, n)$ is nowhere dense. Indeed, pick $U \in \mathcal{U}$ and $n \in \mathbb{N}^+$. Let O be a nonempty open set of $\mathcal{S}_c(X)$. Choose a canonical nonempty open set $\langle \mathcal{V} \rangle \subseteq O$ so that $\langle \mathcal{V} \rangle \cap \mathcal{N}(U, n) \neq \emptyset$. Since $|\mathcal{U}| \geq 2$, we can find $U_0 \in \mathcal{U} \setminus \{U\}$ and $V_0 \in \mathcal{V}$ such that $V_0 \cap U_0 \neq \emptyset$. Since X is crowded, we can find a family $\mathcal{W}$ of disjoint nonempty open subsets of X such that $\bigcup \mathcal{W} \subseteq V_0 \cap U_0$ and $|\mathcal{W}| = n + 1$. Then $\langle \mathcal{V} \cup \mathcal{W} \rangle \subseteq \langle \mathcal{V} \rangle \subseteq O$ and $\langle \mathcal{V} \cup \mathcal{W} \rangle \cap \mathcal{N}(U, n) = \emptyset$. So, $\mathcal{N}(U, n)$ is nowhere dense. Therefore, $\langle \mathcal{U} \rangle$ is meager. $\square$

The presence of a dense set of isolated points in the above examples is not a causality, as the next corollary shows.

Corollary 3.8. *If $\mathcal{S}_c(X)$ is Baire, then X must have a dense set of isolated points. In particular, X has the Baire property.*

Proof. Assume that the set of isolated points of X is not dense in X. Thus, we can find a nonempty crowded open set $U \subseteq X$. It then follows that $\mathcal{S}_c(U) = \langle \mathcal{U} \rangle$ is a nonempty meager open subset of $\mathcal{S}_c(U)$, which is a contradiction. Therefore, X must have a dense set of isolated points. $\square$

As we have seen after Theorem 2.7, if $\mathcal{F}_{\mathcal{Q}}$ is the FAN-filter, then $\mathcal{S}_c(\xi(\mathcal{F}_{\mathcal{Q}}))$ is meager. Thus, the hyperspace of nontrivial convergence sequence of a Baire space with a dense set of isolated points is not necessarily Baire. To get a space X with $\mathcal{S}_c(X)$ of the second category but not Baire is now easy. Consider the disjoint union $X = Y \oplus Z$ where $\mathcal{S}_c(Y)$ is Baire and $\mathcal{S}_c(Z)$ is meager. It follows from the equality $\mathcal{S}_c(X) = \mathcal{S}_c(Y) \oplus \mathcal{S}_c(Z) \oplus \langle Y, Z \rangle$ that $\mathcal{S}_c(X)$ is of the second category but not Baire.

Let us remark that $\mathcal{S}_c(X)$ can never be countably compact. In fact, if $S \in \mathcal{S}_c(X)$, then $\{S \setminus F : F \in \mathcal{F}(S \setminus \{x_S\})\}$ converges to $\{x_S\}$ in $\mathcal{CL}(X)$, but $\{x_S\} \notin \mathcal{S}_c(X)$. It follows that $\{S \setminus F : F \in \mathcal{F}(S \setminus \{x_S\})\}$ is closed and discrete in $\mathcal{S}_c(X)$. On the other hand, we have proved that $\mathcal{S}_c(X)$ may have the Baire property. Since pseudocompact implies Baire, it is natural to analyze the pseudocompactness on $\mathcal{S}_c(X)$.

Theorem 3.9. *If $\mathcal{S}_c(X)$ is pseudocompact, then X has a relatively countably compact dense set of isolated points, every finite power of X is pseudocompact, and every G_δ-point in X must be isolated.*

Proof. Assume that $\mathcal{S}_c(X)$ is pseudocompact and let D be the set of isolated points of X. Hence, $\mathcal{S}_c(X)$ has the Baire property and so, by applying Corollary 3.8, we obtain that D is dense in X. We claim that D is relatively countably compact in X. Assume the contrary, that D is not relatively countably compact. Then there exists a countable infinite set $N \subseteq D$ which is clopen in X. For each $F = \{x_1, \ldots, x_n\} \in \mathcal{F}(N)$ consider the clopen set $\mathcal{U}_F = \{X \setminus N, \{x_1\}, \ldots, \{x_n\}\}$ and let $\mathcal{U}_\emptyset = \{X \setminus \mathbb{N}\}$. Notice that $\mathcal{S}_c(X) = \bigoplus\{\langle \mathcal{U}_F \rangle : F \subseteq N \text{ is finite}\}$, but this contradicts the pseudocompactness of $\mathcal{S}_c(X)$. Thus, we have that D is a relatively countably compact subset of X.

Now, we shall verify that X is pseudocompact. Suppose that there exists an infinite family $\{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of X such that $\mathrm{cl}_X(U_{n+1}) \subseteq U_n$ and $\bigcap\{U_n : n \in \mathbb{N}\} = \emptyset$. Since D is discrete, dense, and relatively countably compact in X, X is pseudocompact. Besides, since X is Fréchet, by [3, Theorem 3.10.26] and induction, X^n is pseudocompact for all $n \in \mathbb{N}$.

Finally, assume that X has a non-isolated G_δ-point x. By the regularity of X, we can find a family $\{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of X such that $\mathrm{cl}(U_{n+1}) \subseteq U_n$ and $\bigcap\{U_n : n \in \mathbb{N}\} = \{x\}$. Then we must have that U_n is an infinite sets for all $n \in \mathbb{N}$. So, $\{\langle U_n\rangle : n \in \mathbb{N}\}$ is a family of nonempty open subsets of $\mathcal{S}_c(X)$ such that $\mathrm{cl}(\langle U_{n+1}\rangle) \subseteq \langle U_n\rangle$ and $\bigcap\{\langle U_n\rangle : n \in \mathbb{N}\} = \emptyset$, contradicting the pseudocompactness of $\mathcal{S}_c(X)$. Therefore, the point x is isolated. □

There are plenty of spaces satisfying the conclusions from Theorem 3.9; a concrete example is the Alexandroff duplicate of the Σ-product of ω_1 copies of the discrete space of cardinality two. However, we do not know whether the hyperspace of convergent sequences of this space is pseudocompact. In a more general setting, we have the following question.

Question 3.10. *Is there a space X for which $\mathcal{S}_c(X)$ is pseudocompact?*

We will see in the next theorem that the Alexandroff duplicate will provide examples of spaces with a Baire hyperspace of convergent sequences. For the first example, let us prove a preliminary lemma.

For the next lemma, for a space X, we let $\mathcal{A}(X) = X \times \{0,1\}$ denote its Alexandroff duplicate, where $X \times \{1\}$ is discrete. For each $U \subseteq X$, we define $\widehat{U} = U \times \{0,1\}$.

Lemma 3.11. *If $\mathcal{B}$ is a π-base of a space X and $\mathcal{B}^*$ is a π-base of the set of all non-isolated points of X consisting of non discrete sets, then the family of all canonical sets of the form*

$$\langle \{B\} \cup \mathcal{U}\rangle,$$

where $B \in \mathcal{B}^*$ and $\mathcal{U} \subseteq \mathcal{B}$, is a π-base of $\mathcal{S}_c(\mathcal{A}(X))$ consisting of nonempty open sets.

Proof. Let $\langle \mathcal{V}\rangle$ be a canonical nonempty open subset of $\mathcal{S}_c(\mathcal{A}(X))$. Note that there exists a non discrete set $\mathcal{V}_0 \in \mathcal{V}$. Pick $B \in \mathcal{B}^*$ such that $B \subseteq V_0$ and $B_V \in \mathcal{B}$ such that $B_V \subseteq V$ for each $V \in \mathcal{V} \setminus \{V_0\}$. Then it is clear that $\emptyset \neq \langle \{B\} \cup \mathcal{U}\rangle \subseteq \langle \mathcal{V}\rangle$. □

For the next result, the diameter of a subset A of a metric space (X, d) will be denoted by $\delta(A) := \sup\{d(x, y) : x, y \in A\}$.

Theorem 3.12. *If X is a complete metrizable crowded space, then $\mathcal{S}_c(\mathcal{A}(X))$ has the Baire property.*

Proof. Equip X with a complete compatible metric. Set $\mathcal{B} = \{\{(x,1)\} : x \in X\}$ and $\mathcal{B}^* = \{\widehat{U} : U \subseteq X \text{ is open}\}$. Note that $\mathcal{B}$ is a π-base for $\mathcal{A}(X)$ because X does not have isolated points, and $\mathcal{B}^*$ is a base for $X \times \{0\}$. Suppose that $\{\mathcal{D}_i : i \in \mathbb{N}\}$ is a decreasing sequence of dense open subsets of $\mathcal{S}_c(\mathcal{A}(X))$ and let $\langle\mathcal{U}\rangle$ be a canonical open subset of $\mathcal{S}_c(\mathcal{A}(X))$, where $\mathcal{U} = \{U_1, \ldots, U_k\}$ and $\{U_1, \ldots, U_k\}$ are pairwise disjoint nonempty open subsets of $\mathcal{A}(X)$. By inductively applying Lemma 3.11, we can find a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ in $\mathbb{N}$, a sequence of points $(x_i)_{i\in\mathbb{N}}$, and, for every $i \in \mathbb{N}$, a nonempty open subset W_i of X such that

(a) $cl_X(W_{i+1}) \subseteq W_i$, for every $i \in \mathbb{N}$;
(b) $\delta(W_i) < \frac{1}{2^i}$, for every $i \in \mathbb{N}$;
(c) $\widehat{W}_i \cap \{(x_1, 1), \ldots, (x_{n_i}, 1)\} = \emptyset$, for every $i \in \mathbb{N}$;
(d) $\left\langle \widehat{W}_i, \{(x_1, 1)\}, \ldots, \{(x_{n_i}, 1)\} \right\rangle \subseteq \langle\mathcal{U}\rangle \cap \mathcal{D}_i$, for every $i \in \mathbb{N}$;
(e) $x_j \in W_i$, for all $i, j \in \mathbb{N}$ with $n_i < j$.

It follows from (b) and (e) that $(x_i)_{i\in\mathbb{N}}$ is a Cauchy sequence and since X is complete, there is $x \in \bigcap_{i\in\mathbb{N}} W_i$ such that $x_i \to x$. Consider the sequence $s = \{x\} \cup \{x_i : i \in \mathbb{N}\}$. It is evident from the construction that $s \in \langle\mathcal{U}\rangle \cap \left(\bigcap_{i\in\mathbb{N}} \mathcal{D}_i\right)$. Therefore, $\mathcal{S}_c(\mathcal{A}(X))$ has the Baire property. □

Theorem 3.12 can be generalized as follows.

Theorem 3.13. *If Z is a Σ-product of completely metrizable spaces and Z is crowded, then $\mathcal{S}_c(\mathcal{A}(Z))$ has the Baire property.*

Proof. Assume that $Z = \{x \in X : |\text{suppt}(x)| \leq \omega\}$, where $X = \prod_{i\in I} X_i$ is a product of completely metrizable spaces, $a = (a_i)_{i\in I} \in X$ is a fixed point, and $\text{suppt}(x) := \{i \in I : x_i \neq a_i\}$ for each $x \in X$. For each $i \in I$, we equip X_i with a complete metric. Set $\mathcal{B} = \{\{(x, 1)\} : x \in Z\}$ and note that $\mathcal{B}$ is a π-base for $\mathcal{A}(Z)$ because Z does not have isolated points. Now, for each $n \in \mathbb{N}$, consider the family $\mathcal{B}_n$ of all nonempty canonical open subsets B of Z such that the projection $\pi_i[B]$ has diameter smaller than $\frac{1}{2^n}$ for all $i \in \text{supp}(B)$. (For a canonical open set B of X, we define $\text{supp}(B) = \{i \in I : \pi_i[B] \neq X_i\}$.) It is clear that each $\mathcal{B}_n$ is a base for the space Z consisting of crowded sets. As a consequence, $\widehat{\mathcal{B}}_n = \{\widehat{B} : B \in \mathcal{B}_n\}$ is a family of open non discrete sets which is a π-base at each non-isolated point of $\mathcal{A}(Z)$ for each $n \in \mathbb{N}$. To show that $\mathcal{S}_c(\mathcal{A}(Z))$ is a Baire space, suppose that $\{\mathcal{D}_n : n \in \mathbb{N}\}$ is a family of open dense subsets of $\mathcal{S}_c(\mathcal{A}(Z))$. Fix an arbitrary nonempty canonical open subset $\langle\mathcal{U}\rangle$ of $\mathcal{S}_c(\mathcal{A}(Z))$. Let $\{N_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ in infinite subsets. We will construct recursively, for each $n \in \mathbb{N}$, a set $B_n \in \mathcal{B}_n$ and a finite subset $\mathcal{U}_n$ of $\mathcal{B}$ as follows:

By using Lemma 3.11, we can find a set $B_1 \in \mathcal{B}_1$ and a finite subset $\mathcal{U}_1$ of $\mathcal{B}$ such that $\langle\{\widehat{B}_1\} \cup \mathcal{U}_1\rangle$ is a canonical open set contained in $\langle\mathcal{U}\rangle \cap \mathcal{D}_1$. We may assume that the cardinality of $\mathcal{U}_1$ is at least two. Let $F_1 = \bigcup \mathcal{U}_1$ and $A_1 = \bigcup\{\text{suppt}(x) : (x, 1) \in F_1\}$. Enumerate A_1 as $\{i_m : m \in N_1\}$, repeating the elements of F_1 if necessary. Assume that for each $k \leq n$ we have defined $B_k \in \mathcal{B}_k$ and $\mathcal{U}_k \in \mathcal{B}$ satisfying the corresponding conditions (1)–(4) below. To prepare the next induction step, for every $k \leq n-1$, we define $F_{k+1} = \bigcup(\mathcal{U}_{k+1} \setminus \mathcal{U}_k)$ and enumerate $A_k := \bigcup\{\text{suppt}(x) : (x,1) \in F_k\}$ as $\{i_m : m \in N_k\}$ allowing repetition if necessary. Then, by applying Lemma 3.11 again, we can find a set $B_{n+1} \in \mathcal{B}_{n+1}$ and a finite subset $\mathcal{U}_{n+1}$ of $\mathcal{B}$ so that

(1) $\langle\{\widehat{B}_{n+1}\} \cup \mathcal{U}_{n+1}\rangle$ is a canonical open set contained in $\langle\{\widehat{B}_n\} \cup \mathcal{U}_n\rangle \cap \mathcal{D}_{n+1}$,
(2) $\text{cl}_Z(B_{n+1}) \subseteq B_n$,
(3) $\mathcal{U}_{n+1} \setminus \mathcal{U}_n$ has at least two elements, and
(4) $\{i_m : m \in \left(\bigcup_{k\leq n} N_k\right) \cap n\} \subseteq \text{supp}(B_{n+1})$.

Thus, we have defined B_n and $\mathcal{U}_n$ for each $n \in \mathbb{N}$. It follows from (4) that $A := \bigcup_{n\in\mathbb{N}} A_n = \{i_m : n \in \mathbb{N}\} \subseteq \bigcup_{n\in\mathbb{N}} \text{supp}(B_n) =: C$. On the other hand, we deduce from (1) that if $G_n := \{x : (x,1) \in F_n\}$ for every $n \in \mathbb{N}$, then $G_{n+1} \subseteq B_n$. Fix $i \in C$. Since $\{\text{cl}(\pi_i[B_n]) : n \in \mathbb{N}\}$ is a deceasing sequence of closed subsets of X_i whose diameters converge to 0, there exists a unique point $b_i \in \bigcap_{n\in\mathbb{N}} \text{cl}[\pi_i(B_n)]$. Next we proceed to define $z \in Z$ as

$$z_i = \begin{cases} b_i & \text{if } i \in C \\ a_i & \text{if } i \in I \setminus C. \end{cases}$$

Our desired sequence is $S = \{(z,0)\} \cup \left(\bigcup_{n\in\mathbb{N}} F_n\right)$. Indeed, it is evident from (3) that $\bigcup_{n\in\mathbb{N}} F_n$ is infinite and discrete. Let us see that the sequence $\bigcup_{n\in\mathbb{N}} F_n$ converges to $(z,0)$. First, when $i \in I \setminus C$, the fact that $\bigcup\{\text{suppt}(x) : x \in \bigcup_{n\in\mathbb{N}} G_n\} = A \subseteq C$ implies that $\pi_i[\bigcup_{n\in\mathbb{N}} G_n] = \{a_i\}$, and so $\pi_i[\bigcup_{n\in\mathbb{N}} G_n]$ trivially converges to $a_i = z_i$. Second, when $i \in C$, it is clear from the construction that $\pi_i[\bigcup_{n\in\mathbb{N}} G_n]$ converges to $b_i = z_i$. It then follows that $\bigcup_{n\in\mathbb{N}} G_n$ converges to z. Since Z is crowded, we conclude that $\bigcup_{n\in\mathbb{N}} F_n$ converges to $(z,0)$. □

It is evident that Theorem 3.12 follows directly from Theorem 3.13, but we decided to include the proofs of both since the proof of the former is illustrative and didactic, and after reading it, the proof of the latter will be more understandable.

We list several open questions which are closely related to the results of this article.

Question 3.14. *Is there a space X for which $\mathcal{S}_c(\mathcal{A}(X))$ is pseudocompact?*

Examples 3.5 and 3.6 suggest the following question.

Question 3.15. *Characterize the Baire spaces X for which $\mathcal{S}_c(X)$ is Baire.*

Based on Corollary 3.8, we have formulated the next question.

Question 3.16. *What are the properties of a space X when $\mathcal{S}_c(X)$ is second category?*

Following [4], we say that a space X is *weakly pseudocompact* if it is G_δ-dense in some compactification. We know that every pseudocompact space is weakly pseudocompact and every weakly pseudocompact space is Baire (for the details of these facts, see [4]). Thus, we may weaken Question 3.17.

Question 3.17. *Is there a space X for which $\mathcal{S}_c(X)$ is weakly pseudocompact?*

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(García-Ferreira and Rojas-Hernández) Centro de Ciencias Matemáticas; Universidad Nacional Autónoma de México; Campus Morelia; Apartado Postal 61-;, Santa María, 58089, Morelia; Michoacán, México
E-mail address, García-Ferreira: sgarcia@matmor.unam.mx

E-mail address, Rojas-Hernández: satzchen@yahoo.com.mx

(Ortiz-Castillo) Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, CEP 05508-090, São Paulo, Brazil
E-mail address: jazzerfoc@gmail.com