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ON STIEFEL–WHITNEY CLASSES OF VECTOR BUNDLES OVER REAL STIEFEL MANIFOLDS

PRATEEP CHAKRABORTY AND AJAY SINGH THAKUR

Abstract. In this article, we show that there are at most two integers up to $2(n-k)$, which can occur as the degrees of nonzero Stiefel–Whitney classes of vector bundles over the Stiefel manifold $V_k(\mathbb{R}^n)$. In the case when $n > k(k+4)/4$, we show that if $w_{2^q}(\xi)$ is the first nonzero Stiefel–Whitney class of a vector bundle ξ over $V_k(\mathbb{R}^n)$, then $w_t(\xi)$ is zero if t is not a multiple of 2^q. In addition, we give relations among Stiefel–Whitney classes whose degrees are multiples of 2^q.

1. Introduction

The real Stiefel manifold $V_k(\mathbb{R}^n)$ is the set of all orthonormal k-frames in $\mathbb{R}^n$, and it can be identified with the homogeneous space $SO(n)/SO(n-k)$. The main aim of this article is to study Stiefel–Whitney classes of vector bundles over a real Stiefel manifold.

Recall that the degree of the first nonzero Stiefel–Whitney class of a vector bundle over a CW-complex X is a power of 2 (see, for example, [8, p. 94]). In the case when X is a d-dimensional sphere S^d, M. F. Atiyah and F. Hirzebruch [2, Theorem 1] show that d can occur as the degree of a nonzero Stiefel–Whitney class of a vector bundle over S^d if and only if $d = 1, 2, 4, 8$. The possible Stiefel–Whitney classes of vector bundles over Dold manifold and stunted real projective space are completely determined by R. E. Stong [11] and Ryuichi Tanaka [12], respectively. In this article, we shall

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deal with the case $X = V_k(\mathbb{R}^n)$ and derive certain results on Stiefel–Whitney classes.

Recall from [3] that the reduced cohomology groups $\widetilde{H}^j(V_k(\mathbb{R}^n);\mathbb{Z}_2)$ vanishes for $j < n-k$, so the first possible nontrivial Stiefel–Whitney class of a vector bundle over $V_k(\mathbb{R}^n)$ occurs in degree $n-k$. In [6], it is observed that for a vector bundle ξ over $V_k(\mathbb{R}^n)$, $n > k$, the Stiefel–Whitney class $w_{n-k}(\xi) = 0$ if $n-k \neq 1,2,4,8$ and $w_{n-k+1}(\xi) = 0$ if $n-k = 2,4,8$. We extend this observation to get the following theorem where we show that there are at most two integers up to $2(n-k)$, which can occur as the degrees of nonzero Stiefel–Whitney classes of any vector bundle over $V_k(\mathbb{R}^n)$.

Theorem 1.1. *Let ξ be a vector bundle over $V_k(\mathbb{R}^n), n > k$. Let i be a positive integer with $i \leq 2(n-k)$. Then $w_i(\xi) = 0$ if one of the following conditions is satisfied:*

(1) *$n-k \neq 1,2,4,8$ and $i \neq 2^{\varphi(n-k-1)}$.*
(2) *$n-k = 1,2,4,8$ and $i \neq n-k, 2(n-k)$.*

In Theorem 1.1, $\varphi(m)$ for a non-negative integer m is the number of integers l such that $0 < l \leq m$ and $l \equiv 0,1,2,4 \pmod 8$.

As a corollary (see Corollary 2.1 below) to Theorem 1.1, we observe that if i is the first nonzero Stiefel–Whitney class of a vector bundle ξ over $V_k(\mathbb{R}^n)$, where $n \geq 2k$, and $i \leq n-1$, then i is of the form $2^{\varphi(n-k-1)}$. Now in the next theorem, for a vector bundle over $V_k(\mathbb{R}^n)$, we derive the vanishing of certain Stiefel–Whitney classes whose degrees depend on the degree of the first nonzero Stiefel–Whitney class.

Theorem 1.2. *Let $n > k(k+4)/4$. Let ξ be a vector bundle over $V_k(\mathbb{R}^n)$ with the first non-zero Stiefel Whitney class in degree 2^q. If i is a multiple of 2^q and is written as $i = 2^{q+t_1} + 2^{q+t_2} + \cdots + 2^{q+t_m}$ with $t_j \geq 0$ and $t_j < t_{j+1}$, then $w_i(\xi) = w_{2^{q+t_1}}(\xi) \cdot w_{2^{q+t_2}}(\xi) \cdots w_{2^{q+t_m}}(\xi)$. Further, if i is not a multiple of 2^q, then $w_i(\xi) = 0$.*

Recall from [10] that the *ucharrank*(X) of X is the maximal degree up to which every cohomology class of X is a polynomial in the Stiefel–Whitney classes of a vector bundle over X. The ucharrank of $V_k(\mathbb{R}^n)$ is computed in [6], except for the cases $n-k = 4,8$, in which it is shown that ucharrank$(V_k(\mathbb{R}^n))$ is bounded above by $n-k$. In Example 2.3, we construct a vector bundle ξ over $V_k(\mathbb{R}^n)$ when $n-k = 4,8$, such that $w_{n-k}(\xi) \neq 0$ and, hence, improve the result in [6] to obtain ucharrank$(V_k(\mathbb{R}^n)) = n-k$.

To prove our results, we need the Steenrod algebra action on the mod-2 cohomology ring $H^*(V_k(\mathbb{R}^n);\mathbb{Z}_2)$. Recall from [3, Proposition 9.1 and Proposition 10.3] that the cohomology ring $H^*(V_k(\mathbb{R}^n);\mathbb{Z}_2)$ has a simple system of generators $a_{n-k}, a_{n-k+1}, \dots, a_{n-1}$, where $a_i \in H^i(V_k(\mathbb{R}^n)$ with the following relations:

$$a_i^2 = \begin{cases} a_{2i} & \text{if } 2i \le n-1 \\ 0 & \text{otherwise.} \end{cases}$$

The action of Steenrod algebra is completely determined by knowing that:

$$Sq^i(a_j) = \begin{cases} \binom{j}{i} a_{j+i} & \text{if } j+i \le n-1, \\ \\ 0 & \text{otherwise.} \end{cases}$$

(see [3, §10, Remarques(2)]).

In this article we shall only consider real Stiefel manifold. The cohomology ring will always be with $\mathbb{Z}_2$-coefficients, unless specified otherwise.

2. Proof of Theorem 1.1

We first recall the description, due to Tanaka [12], of Stiefel–Whitney classes of vector bundles over stunted real projective space. For $n > k$, let $P_{n,k}$ be the stunted real projective space obtained from $\mathbb{RP}^{n-1}$ by collapsing the subspace $\mathbb{RP}^{n-k-1}$ to a point. Consider the following cofibration sequence

$$\mathbb{RP}^{n-k-1} \longrightarrow \mathbb{RP}^{n-1} \xrightarrow{g} P_{n,k}.$$

The induced map in cohomology $g^* : H^j(P_{n,k}) \to H^j(\mathbb{RP}^{n-1})$ is an isomorphism when $n-k \le j \le n-1$. Therefore, for any vector bundle ξ over $P_{n,k}$, the Stiefel–Whitney class $w_j(\xi) \neq 0$ if and only if $w_j(g^*(\xi)) \neq 0$. From [1] (see also [12]), we know that the image $g^* : \widetilde{KO}(P_{n,k}) \to \widetilde{KO}(\mathbb{RP}^{n-1})$ is generated by $2^{\varphi(n-k-1)}\gamma$, where γ is the canonical line bundle over $\mathbb{RP}^{n-1}$ and, for a non-negative integer m, $\varphi(m)$ is as defined in the introduction. If we denote the generator of $H^*(\mathbb{RP}^{n-1})$ by t, then, for any integer d, the total Stiefel–Whitney class of the element $d2^{\varphi(n-k-1)}\gamma$ in the image of g^* is given as

$$w(d2^{\varphi(n-k-1)}\gamma) = (1+t)^{d2^{\varphi(n-k-1)}} = (1+t^{2^{\varphi(n-k-1)}})^d.$$

Therefore, the nonzero Stiefel–Whitney classes of any vector bundle ξ over $P_{n,k}$ can occur only in degrees $r2^{\varphi(n-k-1)}$ for some integer r.

To prove Theorem 1.1, we shall use the following observation. For a non-negative integer m, we note that if $m \equiv 1, 2, 3, 4, 5 \pmod 8$, then $\varphi(m) = [m/2] + 1$, and if $m \equiv 0, 6, 7 \pmod 8$, then $\varphi(m) = [m/2]$. From here we can conclude that for a positive integer m, we have $2^{\varphi(m-1)} \ge m$, and the equality holds only if $m = 1, 2, 4, 8$.

Proof of Theorem 1.1. Recall (see [4]) that there is a cellular embedding $f : P_{n,k} \hookrightarrow V_k(\mathbb{R}^n)$ such that the cellular pair $(V_k(\mathbb{R}^n), P_{n,k})$ is $2(n-k)$ connected (see [4, Proposition 1.3]). Hence, the induced map in cohomology $f^* : H^j(V_k(\mathbb{R}^n)) \to H^j(P_{n,k})$ is injective for $j \leq 2(n-k)$. Therefore, for a vector bundle ξ over $V_k(\mathbb{R}^n)$, the Stiefel–Whitney class $w_j(\xi) \neq 0$ if and only if $w_j(f^*(\xi)) \neq 0$ when $n-k \leq j \leq 2(n-k)$. By the description of Stiefel–Whitney classes of vector bundles over $P_{n,k}$, as discussed above, it follows that $w_j(\xi) = 0$ if $n-k \leq j \leq \min\{n-1, 2(n-k)\}$ and $j \neq r2^{\varphi(n-k-1)}$ for any integer r. Since $2^{\varphi(n-k-1)} \geq (n-k)$ and the equality holds only if $n-k = 1,2,4,8$, the only multiples of $2^{\varphi(n-k-1)}$ that can occur between $(n-k)$ and $2(n-k)$ are $2^{\varphi(n-k-1)}$ and $2^{\varphi(n-k-1)+1}$. Moreover, both of these multiples will occur in this range only when $n-k = 1,2,4,8$. Now the proof of the theorem follows if $2(n-k) \leq n-1$. If $n-1 < 2(n-k)$, then the injectivity of the map f^* gives $H^j(V_{n,k}) = 0$ and, hence, $w_j(\xi) = 0$ for $n-1 < j \leq 2(n-k)$. This completes the proof. □

If we assume $n \geq 2k$, then $n-1 < 2(n-k)$. Then the proof of the following corollary follows from Theorem 1.1 and from the fact that $2^{\phi(m-1)} = m$ if $m = 1,2,4,8$. The following corollary will be used in the proof of Theorem 1.2.

Corollary 2.1. *Let $V_k(\mathbb{R}^n)$ be a Stiefel manifold with $n \geq 2k$. Then $w_i(\xi) = 0$ for $i \leq n-1$ and $i \neq 2^{\varphi(n-k-1)}$ for any vector bundle ξ over $V_k(\mathbb{R}^n)$.*

If we fix k and vary n, then we have the following corollary.

Corollary 2.2. *Let k be fixed. Then, except for finitely many values of n, the Stiefel–Whitney classes $w_i(\xi) = 0$ for $i \leq n-1$ and any vector bundle ξ over $V_k(\mathbb{R}^n)$.*

Proof. The proof follows from Corollary 2.1 by using the fact that $n-1 < 2^{\varphi(n-k-1)}$ except for finitely many values of n. □

In view of Theorem 1.1, it will be interesting to know whether there exists a vector bundle ξ over $V_k(\mathbb{R}^n)$ such that $w_{2^{\varphi(n-k-1)}}(\xi) \neq 0$. We have the complete answer when $2^{\varphi(n-k-1)} = n-k$. We observed in the above proof that $2^{\varphi(n-k-1)} = n-k$ if and only if $n-k = 1,2,4,8$. For the case $n-k = 1,2$, we first note that there is a one-to-one correspondence between the set of isomorphism classes of real line bundles over a CW-complex X and the group $H^1(X;\mathbb{Z}_2)$ via the map which sends a line bundle L to Stiefel–Whitney class $w_1(L)$. Similarly, there is a one-to-one correspondence between the set of isomorphism classes of complex line bundles over a CW-complex X and the group $H^2(X;\mathbb{Z})$ via the map which sends a complex line bundle L to Chern class $c_1(L)$. Now, in the case when $n-k=1,2$, the existence of a vector bundle ξ such that $w_{n-k}(\xi)\neq 0$ is a consequence of the fact that $H^1(V_k(\mathbb{R}^{k+1});\mathbb{Z}_2)\neq 0$, and that the mod-2 reduction map $H^2(V_k(\mathbb{R}^{k+2});\mathbb{Z})\to H^2(V_k(\mathbb{R}^{k+2});\mathbb{Z}_2)$ is the projection map $\mathbb{Z}\to\mathbb{Z}_2$ (see [6]). In the following example, when $n-k=4,8$, we construct a vector bundle ξ over $V_k(\mathbb{R}^n)$ such that $w_{n-k}(\xi)\neq 0$.

Example 2.3. Let $\alpha: Spin(n)\to V_k(\mathbb{R}^n)$ be the principal $Spin(n-k)$-bundle over $V_k(\mathbb{R}^n)=Spin(n)/Spin(n-k)$. If $\widetilde{RO}Spin(n-k)$ and $\widetilde{R}Spin(n-k)$ are the reduced real and complex representation rings, respectively, then we have the following commutative diagram:

(2.1)
$$\begin{tikzcd}
\widetilde{RO}Spin(n-k) \arrow[r] \arrow[d] & \widetilde{KO}(Spin(n)/Spin(n-k)) \arrow[dl, "f^*"] \\
\widetilde{KO}(Spin(n-k+1)/Spin(n-k)). &
\end{tikzcd}$$

Here, $f: S^{n-k}=Spin(n-k+1)/Spin(n-k)\to Spin(n)/Spin(n-k)$ is the natural inclusion. When $n-k=8$, the map $\widetilde{RO}Spin(8)\to\widetilde{KO}(S^8)$ in (2.1) is surjective (see [7, p. 195]) and, hence, the map f^* is surjective. If $[\xi]\in\widetilde{KO}(\mathbb{S}^8)$ is the class of the Hopf bundle over S^8, then there exists a bundle η over $V_k(\mathbb{R}^n)$ such that $f^*([\eta])=[\xi]$. Since $w_8(\xi)\neq 0$, we have $w_8(\eta)\neq 0$.

Next, when $n-k=4$, we use the following diagram:

(2.2)
$$\begin{tikzcd}
\widetilde{R}Spin(n-k) \arrow[r] \arrow[d] & \widetilde{K}(Spin(n)/Spin(n-k)) \arrow[dl, "f^*"] \\
\widetilde{K}(Spin(n-k+1)/Spin(n-k)). &
\end{tikzcd}$$

The map $\widetilde{R}Spin(4)\to\widetilde{K}(S^4)$ in (2.2) is surjective ([7, p. 195]). Using the fact that the Hopf bundle ξ over S^4 is a complex vector bundle with $w_4(\xi)\neq 0$, we proceed as above to conclude that there exists a complex vector bundle η over $V_k(\mathbb{R}^n)$ such that the Stiefel–Whitney class $w_4(\eta_{\mathbb{R}})$ of the underlying real bundle $\eta_{\mathbb{R}}$ is nonzero.

3. Proof of Theorem 1.2

Recall the description of the cohomology ring $H^*(V_k(\mathbb{R}^n))$ as in the introduction. Because of the relations among the generators a_{n-k}, a_{n-k+1},

$\dots, a_{n-1}$, any nonzero cohomology class $x \in H^j(V_k(\mathbb{R}^n))$ can be written as

$$x = \sum a_{i_1} \cdot a_{i_2} \cdots a_{i_r}$$

such that $i_t < i_{t+1}$. If a monomial $a_{i_1} \cdot a_{i_2} \cdots a_{i_r}$ in the above summand represents a nonzero cohomology class, then we have

$$(n-k) + (t-1) \leq \deg a_{i_t} \leq n - 1 - r + t.$$

This implies that

$$\sum_{t=1}^{r} (n-k) + (t-1) \leq \sum_{t=1}^{r} a_{i_t} \leq \sum_{t=1}^{r} n - 1 - r + t.$$

Hence, $r(n-k) + r(r-1)/2 \leq j \leq r(n-1) - r(r-1)/2$.

For $0 \leq p \leq k$, we define T_p as the set $\{j \in \mathbb{N} : p(n-k) + p(p-1)/2 \leq j \leq p(n-1) - p(p-1)/2\}$. Therefore, by the above discussion, we have the following lemma.

Lemma 3.1. *If $x = a_{i_1} \cdot a_{i_2} \cdots a_{i_r}$ with $i_t < i_{t+1}$ represents a nonzero cohomology class of $V_k(\mathbb{R}^n)$, then $\deg x \in T_r$.*

If we assume $n > k(k+4)/4$, then, in the following lemma, we give an upper bound for the length of each T_p.

Lemma 3.2. *Let $n > k(k+4)/4$. Then $|r_1 - r_2| < n - k$ for any p and $r_1, r_2 \in T_p$.*

Proof. For any $r_1, r_2 \in T_p$, we have $|r_1 - r_2| \leq p(n-1) - p(p-1)/2 - p(n-k) - p(p-1)/2 = p(k-p)$. The maximum value of the set $\{p(k-p) : 1 \leq p \leq k\}$ is $k^2/4$ if k is even and is $(k^2-1)/4$ if k is odd. Since $n > k(k+4)/4$ if and only if $n - k > k^2/4$, we have $|r_1 - r_2| < n - k$. $\square$

In the following lemma, we derive some results involving binomial coefficients which we shall use in the proof of Theorem 1.2.

Lemma 3.3. *Let s be an odd number and $r \leq 2^t$. Then the binomial coefficient*

(1) $\binom{2^t s + r - 1}{r}$ *is even if and only if $r \neq 0, 2^t$ and*

(2) $\binom{2^t s - 1}{2^{t+1}}$ *is odd if $s \equiv 3 \pmod 4$.*

Proof. To prove (1), we note that if $r \neq 0$, then

$$\binom{2^t s + r - 1}{r} = \left(\frac{2^t s}{r}\right) \left(\prod_{l=1}^{[(r-1)/2]} \frac{2^t s + 2l}{2l}\right) \left(\prod_{l=1}^{[r/2]} \frac{2^t s + 2l - 1}{2l - 1}\right).$$

Now it is easy to see that the third product in the right-hand side of the above equality can be written as ratios of two odd integers. Further,

$(2^t s/r)$ can be written as a ratio of two odd integers if and only if $r = 2^t$. We next show that the product

$$\prod_{l=1}^{[(r-1)/2]} \frac{2^t s + 2l}{2l}$$

is a ratio of odd integers. Let $2l = 2^{t_1} s_1$ where s_1 is an odd integer and $t_1 < t$. Then

$$\frac{2^t s + 2l}{2l} = \frac{2^t s + 2^{t_1} s_1}{2^{t_1} s_1} = \frac{2^{t-t_1} s + s_1}{s_1}$$

is a ratio of odd integers. From here we conclude (1).

Next, we prove (2). We first note that

$$\binom{2^t s - 1}{2^{t+1}} = \left(\prod_{l=1}^{2^t} \frac{2^t s - 2l}{2l} \right) \left(\prod_{l=1}^{2^t} \frac{2^t s - 2l + 1}{2l - 1} \right).$$

Now, if $l \neq 2^{t-1}$ or 2^t, then

$$\frac{2^t s - 2l}{2l}$$

can be written as a ratio of two odd integers. On the other hand, if $l = 2^{t-1}$ and 2^t, then the product

$$\left(\frac{2^t s - 2^t}{2^t} \right) \left(\frac{2^t s - 2^{t+1}}{2^{t+1}} \right) = (s-1)(s-2)/2,$$

which is an odd number because $s \equiv 3 \pmod 4$. This completes the proof of (2), and the proof of the lemma is complete. □

We now prove Theorem 1.2. Before starting the proof, we first note that the hypothesis $n > k(k+4)/4$ implies that $n \geq 2k$. This follows because $n \geq [k(k+4)/4] + 1 \geq 2k$.

Proof of Theorem 1.2. Let $i = 2^{q+t_1} + 2^{q+t_2} + \cdots + 2^{q+t_m}$ with $t_j \geq 0$ and $t_j < t_{j+1}$. If i is a power of 2 (i.e., when $m = 1$) or $H^i(V_k(\mathbb{R}^n)) = 0$, then the first statement of the theorem follows easily. Next, we assume that $m > 1$ and $H^{2^q r}(V_k(\mathbb{R}^n)) \neq 0$. By Wu's formula, we get

$$\begin{aligned} Sq^{2^{q+t_1}}(w_{i-2^{q+t_1}}(\xi)) &= \sum_{r=0}^{2^{q+t_1}} \binom{i - 2^{q+t_1+1} + r - 1}{r} w_{2^{q+t_1}-r}(\xi) \cdot w_{i-2^{q+t_1}+r}(\xi) \\ &= w_{2^{q+t_1}}(\xi) \cdot w_{i-2^{q+t_1}}(\xi) + w_i(\xi). \end{aligned}$$

The last equality above follows by Lemma 3.3(1).

Next, we prove that the left-hand side of the above equation is zero. For this, it is enough to prove that if $x = a_{i_1} \cdot a_{i_2} \cdots a_{i_p}$, with $i_j < i_{j+1}$, is a nonzero cohomology class of degree $i - 2^{q+t_1}$, then the Steenrod square $Sq^{2^{q+t_1}}(x) = 0$. For this, first note that

$$Sq^{2^{q+t_1}}(x) = Sq^{2^{q+t_1}}(a_{i_1} \cdot a_{i_2} \cdots a_{i_p}) = \sum_{l_1+\cdots+l_p=2^{q+t_1}} Sq^{l_1}(a_{i_1}) \cdots Sq^{l_p}(a_{i_p}).$$

We shall show that each summand in the right-hand side of the above equation is zero.

As the monomial $a_{i_1} \cdot a_{i_2} \cdots a_{i_p}$ represents a nonzero cohomology class, it follows, by Lemma 3.1, that its degree $i - 2^{q+t_1} \in T_p$. If a summand $Sq^{l_1}(a_{i_1}) \cdots Sq^{l_p}(a_{i_p})$ is nonzero, then, for all j, we have $l_j + i_j \leq n-1$ and $Sq^{l_j}(a_{i_j}) = a_{i_j+l_j}$. Moreover, since $n \geq 2k$, we have $a_{i_j}^2 = 0$ for all j, and this will imply that $l_{j_1} + i_{j_1} \neq l_{j_2} + i_{j_2}$ for $j_1 \neq j_2$. Hence,

$$p(n-k) + p(p-1)/2 \leq \sum_{j=1}^{p} i_j + l_j = i \leq p(n-1) - p(p-1)/2.$$

This implies that $i \in T_p$. Since $i - 2^{q+t_1}$ also belongs to T_p, the difference, $i - (i - 2^{q+t_1}) = 2^{q+t_1} \geq 2^q \geq n-k$, gives a contradiction to Lemma 3.2; hence, we conclude that $Sq^{2^{q+t_1}}(x) = 0$. This proves that $w_i(\xi) = w_{2^{q+t_1}}(\xi) \cdot w_{i-2^{q+t_1}}(\xi)$. The proof of the first statement follows by induction on m.

Now we prove the last statement of the theorem by applying induction on the set $\{i : i \text{ is not a multiple of } 2^q\}$. If $2^q = 1$, then the last statement is vacuously true. If $2^q > 1$, then $w_1(\xi) = 0$ since the first non-zero Stiefel Whitney class occurs in degree 2^q. This will serve as the base of the induction. To apply the induction, we assume that if $j < i$ and j is not a multiple of 2^q, then $w_j(\xi) = 0$. Now we shall prove for i, where i is not a multiple of 2^q. If $i < 2^q$, then $w_i(\xi) = 0$ by hypothesis. Next, assume that $i > 2^q$, $H^i(V_k(\mathbb{R}^n) \neq 0$, and i is not a multiple of 2^q. We can write i as $i = 2^t s$ where s is odd, $s \geq 3$, and $t < q$. Applying Lemma 3.3(1) in Wu's formula, we get

$$\begin{aligned} Sq^{2^t}(w_{2^t(s-1)}(\xi)) &= \sum_{r=0}^{2^t} \binom{2^t(s-2)+r-1}{r} w_{2^t-r}(\xi) \cdot w_{2^t(s-1)+r}(\xi) \\ &= w_{2^t}(\xi) \cdot w_{2^t(s-1)}(\xi) + w_{2^t s}(\xi). \end{aligned}$$

Therefore, $Sq^{2^t}(w_{2^t(s-1)}(\xi)) = w_{2^t s}(\xi)$ since $w_{2^t}(\xi) = 0$. If $2^t(s-1)$ is not a multiple of 2^q or $H^{2^t(s-1)}(V_k(\mathbb{R}^n)) = 0$, then by induction, we have $w_i(\xi) = 0$. Now assume that $H^{2^t(s-1)}(V_k(\mathbb{R}^n)) \neq 0$ and $2^t(s-1)$ is a multiple of 2^q. Let $2^t(s-1) = 2^{q+t_1} + 2^{q+t_2} + \cdots + 2^{q+t_m}$ with $t_j < t_{j+1}$. We have the following two cases:

Case 1: $m > 1$. Since $2^t(s-1)$ is a multiple of 2^q, by the first statement of the theorem, we have

$$
\begin{aligned}
w_i(\xi) &= Sq^{2^t}(w_{2^{q+t_1}}(\xi)\cdot w_{2^{q+t_2}}(\xi)\cdots w_{2^{q+t_m}}(\xi)) \\
&= \sum_{l_1+\cdots l_m=2^t} Sq^{l_1}(w_{2^{q+t_1}}(\xi))\cdots Sq^{l_m}(w_{2^{q+t_m}}(\xi)).
\end{aligned} \tag{3.1}
$$

Now observe that for each j, we have $l_j \le 2^t < 2^q$ and, hence, the Steenrod square

$$
\begin{aligned}
Sq^{l_j}(w_{2^{q+t_j}}(\xi)) &= \sum_{r=0}^{l_j}\binom{2^{q+t_j}+r-l_j-1}{r} w_{l_j-r}(\xi)\cdot w_{2^{q+t_j}+l_j}(\xi) \\
&= \binom{2^{q+t_j}-1}{l_j} w_{2^{q+t_j}+l_j}(\xi)\,.
\end{aligned}
$$

The last equality in the above equation is because $w_{l_j-r}(\xi)=0$ for $r>0$ since $0<l_j-r<2^q$. Since $m>1$, we have $2^{q+t_j}<2^t(s-1)$. Therefore, $2^{q+t_j}+l_j<2^ts$. Further, if, for some j, we have $l_j>0$, then $2^{q+t_j}+l_j$ is not a multiple of 2^q and, hence, by induction, $w_{2^{q+t_j}+l_j}(\xi)=0$. Now observe that in each summand on the right-hand side of equation (3.1), there is at least one l_j such that $l_j>0$. Therefore,

$$
w_{2^ts}(\xi) = \sum_{l_1+\cdots l_m=2^t} Sq^{l_1}(w_{2^{q+t_1}}(\xi))\cdots Sq^{l_m}(w_{2^{q+t_m}}(\xi)) = 0.
$$

Case 2: $m=1$. In this case, $2^t(s-1)=2^{q+t_1}$ for some $t_1\ge 0$. Thus, $s-1$ is a power of 2. First, we consider the case when $s\ge 5$. Here we observe that $2^{t+1}<2^ts-2^{t+1}$. Since $s-1\equiv 0 \pmod 4$, by Lemma 3.3(2), we have

$$
\begin{aligned}
Sq^{2^{t+1}}(w_{2^ts-2^{t+1}}(\xi)) &= \sum_{r=0}^{2^{t+1}}\binom{2^t(s-4)+r-1}{r} w_{2^{t+1}-r}(\xi)\cdot w_{2^ts-2^{t+1}+r}(\xi) \\
&= w_{2^{t+1}}(\xi)\cdot w_{2^ts-2^{t+1}}(\xi) + \binom{2^t(s-2)-1}{2^{t+1}} w_{2^ts}(\xi) \\
&= w_{2^{t+1}}(\xi)\cdot w_{2^ts-2^{t+1}}(\xi) + w_{2^ts}(\xi)\,.
\end{aligned}
$$

Because $2^ts-2^{t+1}=2^t(s-1)-2^t=2^{q+t_1}-2^t=2^t(2^{q-t+t_1}-1)$, we have that 2^ts-2^{t+1} is not a multiple of 2^q and, hence, by induction, $w_{2^ts-2^{t+1}}(\xi)=0$. Therefore, $w_{2^ts}(\xi)=0$. Next, we deal with the case $s=3$. In this case, we observe that $t=q-1$ and $t_1=0$. Thus, we obtain, by Lemma 3.3(1),

$$
\begin{aligned}
Sq^{2^{q-1}}(w_{2^q}(\xi)) &= \sum_{r=0}^{2^{q-1}}\binom{2^q+r-2^{q-1}-1}{r} w_{2^{q-1}-r}\cdot w_{2^q+r} \\
&= \binom{2^{q-1}+2^{q-1}-1}{2^{q-1}} w_{2^{q-1}3}(\xi) \\
&= w_i(\xi)\,.
\end{aligned}
$$

If $2^q \in T_1$, then, by Lemma 3.1, $w_{2^q}(\xi) = a_j$ for some j since $w_{2^q}(\xi) \neq 0$. Therefore, $w_i(\xi) = Sq^{2^{q-1}}(a_j)$. So $w_i(\xi) = 0$ if $i > n-1$. If $i \leq n-1$, then $w_i(\xi)$ is again zero since $w_{2^q}(\xi)$ is the only nonzero Stiefel–Whitney class up to degree $n-1$ (see Corollary 2.1). Next, we assume that $2^q \notin T_1$. Because $w_{2^q}(\xi) \neq 0$, we have that $2^q \in T_p$ for some p such that $p \geq 2$. To prove $w_{2^{q-1}3}(\xi) = 0$, we consider the Steenrod square operation

$$(3.2) \qquad Sq^{2^{q-1}}(a_{i_1} \cdot a_{i_2} \cdots a_{i_p}) = \sum_{l_1+\cdots+l_p=2^{q-1}} Sq^{l_1}(a_{i_1}) \cdots Sq^{l_p}(a_{i_p})$$

on a monomial $a_{i_1} \cdot a_{i_2} \cdots a_{i_p}$ such that $i_j < i_{j+1}$, which represents a nonzero cohomology class of degree 2^q. By Lemma 3.1, the degree of the monomial 2^q is in T_p. If a summand $Sq^{l_1}(a_{i_1}) \cdots Sq^{l_p}(a_{i_p})$ is nonzero, then, for all j, we have $l_j + i_j \leq n-1$ and $Sq^{l_j}(a_{i_j}) = a_{i_j+l_j}$. Moreover, since $n \geq 2k$, we have $l_{j_1} + i_{j_1} \neq l_{j_2} + i_{j_2}$ for $j_1 \neq j_2$. Hence,

$$p(n-k) + p(p-1)/2 \leq \sum_{j=1}^{p} i_j + l_j = 2^{q-1}3 \leq p(n-1) - p(p-1)/2.$$

This implies that $2^{q-1}3 \in T_p$. Because $p \geq 2$, we have $2^q \geq 2(n-k)+1$, and this implies that $2^{q-1} > n-k$. Therefore, $2^{q-1}3 - 2^q = 2^{q-1} > n-k$, a contradiction to Lemma 3.2. This shows that each summand in the right-hand side of equation (3.2) is zero. Hence, $w_{2^{q-1}3}(\xi) = 0$ if $p \geq 2$. $\square$

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