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CLASSIFICATION OF SIMPLY-CONNECTED TRIVALENT 2-DIMENSIONAL STRATIFOLDS

J. C. GÓMEZ-LARRAÑAGA, F. GONZÁLEZ-ACUÑA, AND WOLFGANG HEIL

Abstract. Trivalent 2-stratifolds are a generalization of 2-manifolds in that there are disjoint simple closed curves where locally three sheets meet. We obtain a classification of trivalent 1-connected 2-stratifolds in terms of their associated labeled graphs.

1. Introduction

A closed 2-*stratifold* is a 2-dimensional cell complex X that contains a collection of finitely many simple closed curves, the components of the 1-skeleton $X^{(1)}$ of X, such that $X - X^{(1)}$ is a 2-manifold and a neighborhood of each point in a component C of $X^{(1)}$ consists of $n \geq 3$ sheets. 2-stratifolds X are a more restricted class than multibranched surfaces, studied by Shosaka Matsuzaki and Makoto Ozawa in [9], and trivalent 2-stratifolds (defined in §2) are a more restrictive class than foams, which have been studied by J. Scott Carter [2] and Mikhail Khovanov [8]. Foams include special spines S that occur as spines of (closed) 3-manifolds M (see, for example, [10] and [12]). Thus, $\pi_1(M) \cong \pi_1(S)$ for some special spine S. But there are significant differences between the fundamental groups of 2-stratifolds and 3-manifolds: 3-manifold groups are residually finite, but every Baumslag–Solitar group (some Hopfian, others non-Hopfian) can be realized as the fundamental group of a 2-stratifold. Also, with the exception of lens spaces and connected sums, closed 3-manifolds are determined by their fundamental groups (see, for example, [1]), but there are infinitely many non-homeomorphic 2-stratifolds with the same fundamental group. However, it can be shown that fundamental groups

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of 2-stratifolds have solvable word problem [6]. Furthermore, closed 3-manifolds that have 2-stratifolds as spines are classified in [7].

2-stratifolds arise in the study of categorical invariants of 3-manifolds. For example, if $\mathcal{G}$ is a non-empty family of groups that is closed under subgroups, one would like to determine which (closed) 3-manifolds have $\mathcal{G}$-category equal to 3. In [3], it is shown that such manifolds have a decomposition into three compact 3-submanifolds H_1, H_2, and H_3, where the intersection of $H_i \cap H_j$ (for $i \neq j$) is a compact 2-manifold and each H_i is $\mathcal{G}$-contractible (i.e., the image of the fundamental group of each connected component of H_i in the fundamental group of the 3-manifold is in the family $\mathcal{G}$). The nerve of this decomposition, which is the union of all the intersections $H_i \cap H_j$ ($i \neq j$), is a closed 2-stratifold and determines whether the $\mathcal{G}$-category of the 3-manifold is 2 or 3.

A 2-stratifold is essentially determined by its associated bipartite labeled graph (defined in §2), and a presentation for its fundamental group can be read off from the labeled graph. Thus, the question arises when a labeled graph determines a simply connected 2-stratifold. In [4], it is shown that a necessary condition is that the underlying graph must be a tree; if the graph is linear, then sufficient and necessary conditions on the labeling are given, and if the graph is trivalent (defined in §2), an algorithm on the labeled graph is developed for determining whether the graph determines a simply connected 2-stratifold. In [5], an algorithm is given that decides whether a given labeled graph (not necessarily trivalent) determines a 2-stratifold that is homotopy equivalent to S^2.

The main result of this paper (Theorem 3.6) is a classification of all trivalent labeled graphs that represent simply connected 2-stratifolds.

2. Properties of the Graph of a 2-Stratifold

We first review the basic definitions and some results given in [4] and [5]. A 2-*stratifold* is a compact, Hausdorff space X that contains a closed (possibly disconnected) 1-manifold $X^{(1)}$ with empty boundary as a closed subspace with the following property: Each point $x \in X^{(1)}$ has a neighborhood homeomorphic to $\mathbb{R} \times CL$ where CL is the open cone on L for some (finite) set L of cardinality > 2 and $X - X^{(1)}$ is a (possibly disconnected) 2-manifold.

A component $C \approx S^1$ of $X^{(1)}$ has a regular neighborhood $N(C) = N_p(C)$ that is homeomorphic to $(Y \times [0,1])/(y,1) \sim (h(y),0)$, where Y is the closed cone on the discrete space $\{1,2,...,d\}$ and $h : Y \to Y$ is a homeomorphism whose restriction to $\{1,2,...,d\}$ is the permutation $p : \{1,2,...,d\} \to \{1,2,...,d\}$. The space $N_p(C)$ depends only on the conjugacy class of $p \in S_d$ and, therefore, is determined by a partition of d. A component of $\partial N_p(C)$ corresponds then to a summand of the partition determined by p. Here, the neighborhoods $N(C)$ are chosen sufficiently small so that, for disjoint components C and C' of $X^{(1)}$, $N(C)$ is disjoint from $N(C')$. The components of $\overline{N(C) - C}$ are called the *sheets* of $N(C)$.

For a given 2-stratifold $(X, X^{(1)})$, there is an associated bipartite graph $G = G(X, X^{(1)})$ embedded in X as follows: In each component C_j of $X^{(1)}$, choose a black vertex b_j. In the interior of each component W_i of $M = \overline{X - \cup_j N(C_j)}$, choose a white vertex w_i. In each component S_{ij} of $W_i \cap N(C_j)$, choose a point y_{ij}, an arc α_{ij} in W_i from w_i to y_{ij}, and an arc β_{ij} from y_{ij} to b_j in the sheet of $N(C_j)$ containing y_{ij}. An edge e_{ij} between w_i and b_j consists of the arc $\alpha_{ij} * \beta_{ij}$. For a fixed i, the arcs α_{ij} are chosen to meet only at w_i.

We label the graph G by assigning to a white vertex W its genus g of W and by labeling an edge S by k, where k is the summand of the partition p corresponding to the component S of $\partial N_p(C)$ where $S \subset \partial N_p(C)$. (Here we use Walter D. Neumann's convention [11] of assigning negative genus g to nonorientable surfaces.) Note that the partition p of $\partial N_p(C)$ corresponding to a black vertex b is determined by the labels of the adjacent edges of b. If G is a tree, then the labeled graph determines X uniquely.

If G is a bipartite labeled graph corresponding to the 2-stratifold X, we let $X_G = X$ and $G_X = G$.

If G is not a tree, then one needs to assign an additional $+$ or $-$ sign to non-terminal edges incident to non-negative labeled vertices in order for the labeled graph G_X to determine uniquely a 2-stratifold X_G (see [4]). However, this ambiguity does not affect the arguments in the present paper; therefore, our edge labels are always positive.

An example is shown in Figure 1, where the labels n on the arrows indicate that the corresponding boundary curve of W_i is attached to C_j under the map $z \to z^n$. A white vertex on the graph corresponding to a surface of genus 0 is not labeled. The fundamental group has a presentation with generators b_1, b_2, b_3, x, y, a_1, a_2, q_1, q_2, and q_3 and relations $a_1 a_2 a_1^{-1} a_2^{-1} q_1 q_2 q_3 = 1$, $b_1^2 = 1$, $b_1^3 = x^2$, $b_1^4 = q_1$, $b_2^5 = q_2$, and $b_3^6 = q_3$.

The following is shown in [4].

Proposition 2.1. *There is a retraction $r : X \to G_X$.*

Here, we may assume that for a black vertex b corresponding to a component C of $X^{(1)}$, $r^{-1}(b) = N(C)$, and for a white vertex w corresponding to a component W of $\overline{X - \cup_j N(C_j)}$, $r^{-1}(st(w)) = int(W)$ where $st(w)$ is the open star of w in G_X.

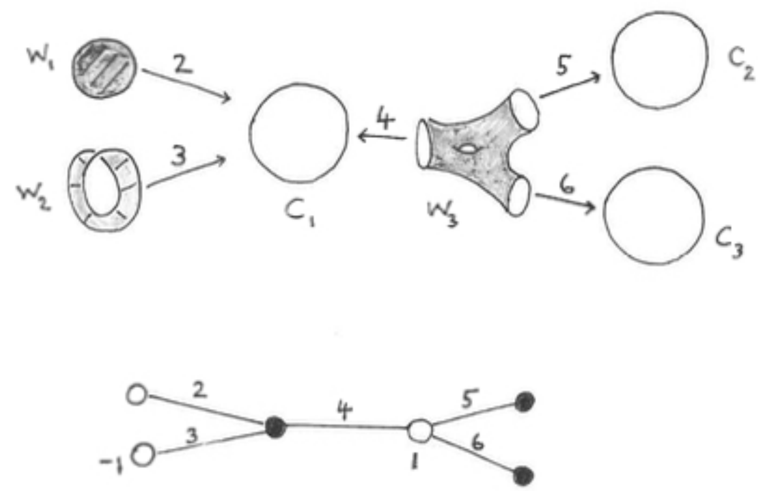

FIGURE 1. A 2-stratifold with its labeled graph

Proposition 2.2. *If X is simply connected, then G_X is a tree, all white vertices of G_X have genus 0, and all terminal vertices are white.*

The proof uses Proposition 2.1 and the following pruning construction.

2.1. Pruning at a subgraph.

Let Γ be a subgraph of $G = G_X$ and let $st(\Gamma)$ be the star of Γ in G. Denote by $\hat{\Gamma}$ the union of Γ and the labeled edges (with their vertices) of $st(\Gamma) - \Gamma$ which are adjacent to a black vertex of Γ. The 2-stratifold $X_{\hat{\Gamma}}$ is obtained from X_G as follows: For the retraction $r : X_G \to G$, delete the components of $r^{-1}(G - \hat{\Gamma})$ and for a white vertex w of $\hat{\Gamma} \cap \overline{G - \hat{\Gamma}}$, fill in the boundary curves of $r^{-1}(w) \cap r^{-1}(\overline{G - \hat{\Gamma}})$ with disks. Note that there is a quotient map $X_G \to X_{\hat{\Gamma}}$. See Figure 2.

Definition 2.3. P is a pruned subgraph of G if $P = \hat{\Gamma}$ for some subgraph Γ of G.

Remark 2.4. For a connected pruned subgraph P of a connected graph G, the quotient map $X_G \to X_P$ induces surjections $\pi(X_G) \to \pi(X_P)$ and $H_1(X_G; \mathbb{Z}_2) \to H_1(X_P; \mathbb{Z}_2)$. Therefore, if P is a pruned subgraph of G and X_G is simply connected, then X_P is simply connected.

Example 2.5. Let B be a set of black vertices of G and let $st(B)$ be the open star of B in G. Then $\Gamma = \hat{\Gamma}$ for each component Γ of $G - st(B)$.

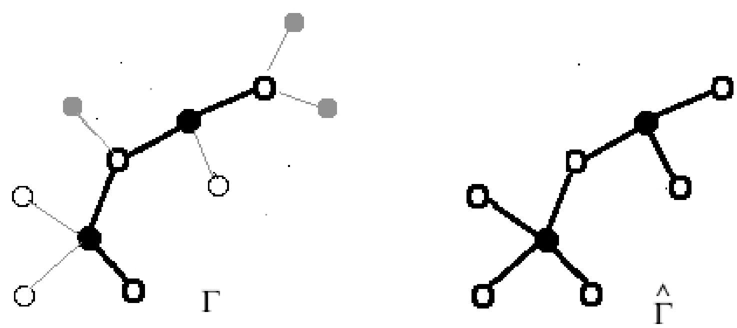

FIGURE 2. G, Γ and $\hat{\Gamma}$

Thus, Γ is a pruned subgraph of G and if X_G is simply connected, then X_Γ is simply connected.

In this paper we consider trivalent graphs. A 2-stratifold X and its labeled bicolored graph G_X are *trivalent* if the sum of the edge weights adjacent to any black vertex is 3. This means that a neighborhood of a point of a component C of the 1-skeleton $X^{(1)}$ has three sheets, so the permutation $p : \{1,2,3\} \to \{1,2,3\}$ of the regular neighborhood $N(C) = N_\pi(C)$ has partition 1+1+1 or 1+2 or 3. We also allow a graph consisting of a single white vertex to be a trivalent graph.

The fundamental group of X_G when G_X is a linear graph has been computed in [4]. For trivalent graphs we have the following lemma.

Lemma 2.6 ([4, Lemma 4]). *If all terminal edges of a trivalent graph G (with a non-zero number of edges) have label 2, then $H_1(X;\mathbb{Z}_2) \neq 0$.*

One of the main results in [4, Theorem 5] is that a trivalent 2-stratifold is simply connected if and only if $H_1(X;\mathbb{Z}_2) = H_1(X;\mathbb{Z}_3) = 0$. The second condition is needed only to insure that G_X has no terminal black vertices; therefore, we get the following theorem.

Theorem 2.7. *Let X be a trivalent 2-stratifold such that all terminal vertices are white. Then X is simply connected if and only if $H_1(X;\mathbb{Z}_2) = 0$.*

3. CLASSIFICATION OF SIMPLY CONNECTED TRIVALENT 2-STRATIFOLDS

The building blocks for constructing labeled trivalent graphs for 1-connected 2-stratifolds are called $b12$-trees and $b111$-trees.

Definition 3.1. (1) The $b12$-*tree* is the bipartite tree consisting of one black vertex incident to two edges, one of label 1 and the other of label 2, and two terminal white vertices, each of genus 0.

(2) The $b111$-*tree* is the bipartite tree consisting of one black vertex incident to three edges, each of label 1, and three terminal white vertices, each of genus 0.

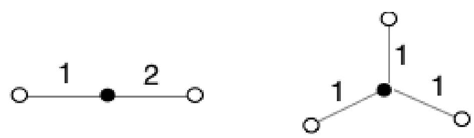

FIGURE 3. $b12$-tree and $b111$-tree

First, we consider special trivalent trees that do not contain $b111$-subtrees, which we call $(2,1)$-collapsible trees.

A $(2,1)$-*collapsible tree* is constructed from a rooted tree T (which may consist of only one vertex) with root r (a vertex of T) as follows: color with white and label 0 the vertices of T, take the barycentric subdivision $sd(T)$ of T, color with black the new vertices (the barycenters of the edges of T) and, finally, label an edge e of $sd(T)$ with 2 (1, respectively) if the distance from e to the root r is even (odd, respectively). This labeled $sd(T)$ is the $(2,1)$-collapsible tree determined by (T,r). Note that we allow a one-vertex tree (with white vertex) as a $(2,1)$-collapsible tree.

A typical example is shown in Figure 4, where regions enclosed by the dashed curves are $(2,1)$-collapsible trees, gray (blue, if viewed electronically) vertices are roots.

Lemma 3.2. *Let X_G be a trivalent 1-connected 2-stratifold. Then after removing the open stars of all black vertices of degree 3, the components $\Gamma_1,\dots,\Gamma_n$ are $(2,1)$-collapsible trees. Furthermore, for each black vertex b of degree 3, at least one of its (three white) neighbors is the root of some Γ_i.*

Proof. From Example 2.5, each Γ_i is a pruned subgraph of G and so $\pi(X_{\Gamma_i}) = 1$. Therefore, for the first part of the proposition, it suffices to show that if Γ has no black vertices of degree 3 and X_Γ is simply connected, then Γ is a $(2,1)$-collapsible tree.

Thus, assume Γ is not a single vertex graph and has no black vertices of degree 3, and X_Γ is simply connected. By Proposition 2.2, Γ is a tree with all white vertices of genus 0 and all terminal vertices white. Then

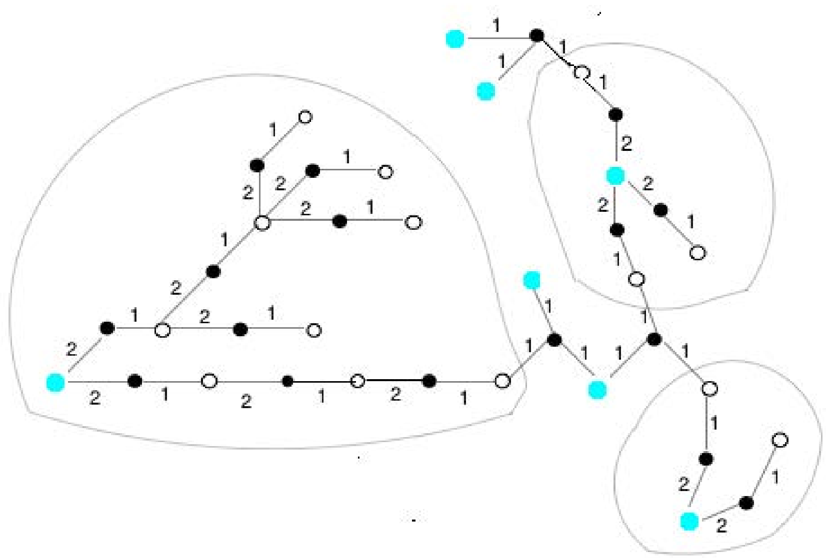

FIGURE 4

by Lemma 2.6, Γ contains a $b12$-subgraph L with terminal edge of label 1. Let w be the white vertex of L which is not a terminal vertex of Γ. Let $G_0 = (\Gamma - L) \cup \{w\}$, let G_i' $(i = 1, \dots, m)$ be the components of $G_0 - \{w\}$, and let $G_i = G_i' \cup \{w\}$. Then X_{G_i} is a pruned subgraph of Γ and, therefore, simply connected for $i = 0, \dots, m$. By induction on the number of vertices, each G_i is a $(2,1)$-collapsible tree with a root r_i.

If $r_i = w$ for each $i = 1, \dots, m$, then Γ is a $(2,1)$-collapsible tree with root w.

If $r_i \neq w$, the label on the edge e_i of G_i incident to w is 1. It follows that there is at most one r_i not equal to w; otherwise, if $r_i \neq w \neq r_j$, the union of the edges and vertices of the simple path in $G_i \cup G_j$ from r_i to r_j is a pruned linear subgraph $\Gamma' = G(2, 1, \dots, 1, 1, 2, 1, \dots, 1, 2)$ of Γ for which $X_{\Gamma'}$ is not simply connected (by [4, Theorem 3]), a contradiction. It follows that if $r_i \neq w$, then Γ is a $(2,1)$-collapsible tree with root r_i.

For the second part of the proposition, suppose that G contains a $b111$-subtree with black vertex b such that none of its white vertices w_1, w_2, w_3 is a root of any of the Γ_j, and let Γ_i be the pruned subgraph of G containing w_i $(i = 1, 2, 3)$ and with roots $r_i \neq w_i$. Then the trivalent pruned subgraph Γ' of G, which is the union of the edges and vertices of the three simple paths in G from the roots r_i to b ($i = 1, 2, 3$), has all terminal edges of label 2. Then, by Lemma 2.6, $\pi_1(X_{\Gamma'}) \neq 1$, a contradiction. $\square$

The figure below shows that the converse of Lemma 3.2 is false. There are two $b111$-vertices, all labels are 1 except as indicated, and $H_1(X_G; \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Deleting the two $b12$-trees at the white center vertex (but not the center vertex) yields a horned tree H_T.

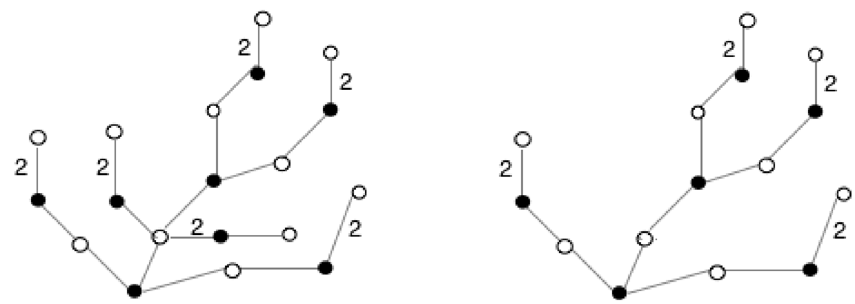

FIGURE 5. G and a horned tree H_T

Definition 3.3. A *horned tree* H_T is a trivalent finite connected bipartite labeled tree such that all white vertices have label 0, all terminal vertices are white, and

(1) every black vertex b which has distance 1 to a terminal white vertex has degree 2; otherwise, b has degree 3;
(2) every nonterminal white vertex has degree 2;
(3) every terminal edge has label 2 and every nonterminal edge has label 1;
(4) there is at least one black vertex of degree 3.

A horned tree H_T may be constructed from a tree T that has at least two edges and all of whose nonterminal vertices have degree 3 as follows: Color a vertex of T white (black, respectively) if it has degree 1 (3, respectively). Trisect the terminal edges of T and bisect the nonterminal edges, obtaining the graph H_T. Color the additional vertices v so that H_T is bipartite; that is, v is colored black if v is a neighbor of a terminal vertex of H_T and white otherwise. Then label the edges so that (3) holds.

A main property of H_T is that $\pi_1(X_{H_T}) = \mathbb{Z}_2$, which can be seen as follows: Let w be a nonterminal white vertex of H_T with black neighbors b_1 and b_2 and denote by $b_1{-}w{-}b_2$ the two-edge path from b_1 to b_2. Let Γ_{b_1} be the graph obtained from H_T by collapsing $b_1{-}w{-}b_2$ to b_1. Then $\pi_1(X_\Gamma) \cong \pi_1(X_{H_T})$ since the edge labels on $b_1{-}w{-}b_2$ are 1. Collapsing successively all such two-edge paths yields a connected (not necessarily trivalent) graph Γ_b with a single black vertex b and all terminal edges of label 2. Then $\pi_1(X_{H_T}) \cong \pi_1(X_{\Gamma_b}) \cong \mathbb{Z}_2$.

Finally, we consider a "reduced graph" of G which encodes information on how the $(2,1)$-collapsible trees of Lemma 3.2 are attached to the stars of the black vertices of degree 3.

Denote by B the union of all the black vertices of degree 3 of G, let $St(B)$ be the (closed) star of B in G, and let $st(B)$ be the open star of B. Note that $G - st(B)$ consists of the components $\Gamma_1, \dots, \Gamma_n$ as in Lemma 3.2.

Definition 3.4. Let G be a bipartite labeled tree such that the components of $G - st(B)$ are $(2,1)$-collapsible trees. The *reduced subgraph* $R(G)$ of G is the graph obtained from $St(B)$ by attaching to each white vertex w of $St(B)$ that is not a root, a $b12$-graph such that the terminal edge has label 2.

As an example, the reduced graph $R(G)$ for the graph in Figure 4 is in Figure 6.

Lemma 3.5. $H_1(X_G, \mathbb{Z}_2) \approx H_1(X_{R(G)}, \mathbb{Z}_2)$.

Proof. $H_1(X_G, \mathbb{Z}_2)$ is generated by the black vertices (more precisely by the components of $X_G^{(1)}$ corresponding to the black vertices). Let w be a white vertex of G of degree $n \geq 2$ with incident edges $e_1, \dots, e_n$. Suppose the label on e_i is 2 for $i = 1, \dots, k$ $(k \leq n)$. Split w into $k+1$ (disjoint) vertices $w_1, \dots, w_k, w'$ so that each w_j has degree 1 with adjacent edge e_j and w' has degree $n-k$ with adjacent edges $e_{k+1}, \dots, e_n$. This change of the graph does not change $H_1(X_G, \mathbb{Z}_2)$.

Now let Γ be a $(2,1)$-collapsible component of $G - st(B)$. If w is a terminal (white) vertex of Γ that is also a terminal vertex of G, delete the $b12$-subgraph of Γ that contains w, if there is one (Γ might consist of a single vertex). Continue doing this operation until all terminal vertices of Γ belong to $St(B)$. This does not change $\pi_1(X_G)$. If w is a non-terminal white vertex of (the new Γ), then, if w is the root, all edges incident to w have label 2; if w is not the root, all edges but one incident to w have label 2. Do the above construction on each such w to change Γ to Γ'. Then each component of Γ' that does not contain a terminal white vertex of Γ is a linear graph of type $w_1 - b_1 - \cdots - w_k - b_k$ with successive edge labels

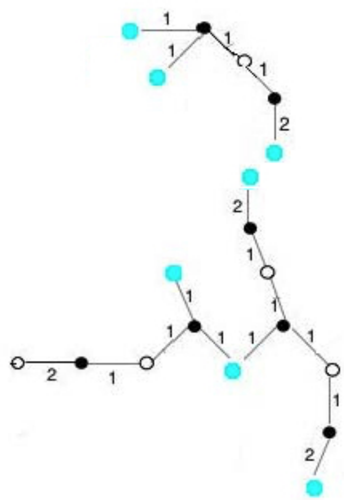

FIGURE 6

$2 - 1 - \cdots - 2 - 1$ (or consists of a single white vertex). Deleting these homologically trivial components from Γ' does not change $H_1(X_G, \mathbb{Z}_2)$.

Doing this for all $(2,1)$-collapsible components Γ of $G - st(B)$ results in $R(G)$. □

We now state the Classification Theorem.

Theorem 3.6. *Let X_G be a trivalent connected 2-stratifold. The following are equivalent:*

(1) *X_G is simply connected;*
(2) *G_X is a tree with all white vertices of genus 0 and all terminal vertices white such that the components of $G - st(B)$ are $(2,1)$-collapsible trees and the reduced graph $R(G)$ contains no horned tree.*

Proof. If X_G is simply connected, then by Proposition 2.2 and Lemma 3.2, the components of $G - st(B)$ are $(2,1)$-collapsible trees. If the reduced graph $R(G)$ contains a horned tree H, let Γ be the component of $R(G)$ containing H. Note that H is a pruned subgraph of Γ and since $\pi_1(X_H) = \mathbb{Z}_2$, it follows from Remark 2.4 that $H_1(\Gamma; \mathbb{Z}_2) \neq 0$. Lemma 3.5 then shows that $H_1(X_G; \mathbb{Z}_2) \cong H_1(X_{R(G)}; \mathbb{Z}_2) \neq 0$. Hence, $R(G)$ does not contain a horned tree.

Conversely, suppose the components of $G - st(B)$ are $(2,1)$-collapsible trees and $R(G)$ contains no horned trees. Let Γ be a component of $R(G)$.

First, we show by induction on $n :=$ number of black vertices of degree 3 in Γ, that $H_1(X_\Gamma; \mathbb{Z}_2) = 0$. If $n = 1$, then Γ is a $b111$-tree with at most two $b21$-trees attached to its terminal vertices, and so $H_1(X_\Gamma; \mathbb{Z}_2) = 0$.

Let $n > 1$. We claim that at least one terminal label of Γ is 1. If not, then Γ satisfies conditions (1) and (3) of Definition 3.3. We can find a sequence $\Gamma = \Gamma_0, \Gamma_1, \dots, \Gamma_m$, where Γ_m is a horned tree and Γ_{i+1} is obtained from Γ_i by deleting all but two components of $\Gamma_i - \{w\}$ for some nonterminal white vertex w of Γ_i, contradicting the assumption that $R(G)$ contains no horned trees. This proves the claim.

Now let $\Gamma' = \Gamma - st(b)$ where $b - w$ is a terminal edge of Γ with label 1. Then $H_1(\Gamma; \mathbb{Z}_2) = H_1(X_{\Gamma'}; \mathbb{Z}_2)$ which by induction is 0. Therefore, $H_1(X_G, \mathbb{Z}_2) = H_1(X_{R(G)}; \mathbb{Z}_2) = 0$ and it follows from Theorem 2.7 that X_G is 1-connected. □

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