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Alireza Zamani Bahabadi

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Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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μ-EXPANSIVE MEASURE FOR FLOWS

ALIREZA ZAMANI BAHABADI

Abstract. In this paper, we show that if X is in the C^1-interior of the set of μ-measure expansive divergence free vector fields, then X admits a dominated splitting. We also show that in dimension 3, the above result would be Anosov by considering δ-μ-measure expansiveness.

1. Introduction

Let M be a closed, connected, and smooth Riemannian manifold endowed with a volume form, which has a measure μ, called the Lebesgue measure. We denote by $\mathcal{X}^1_\mu(M)$, the set of divergence-free vector fields endowed with the C^1 Whitney topology.

Let $X \in \mathcal{X}^1_\mu(M)$ and let $x \in M$ be a regular point. Let $N_x = X(x)^\perp \subset T_xM$ denote the normal bundle of X at x.

We define the *linear Poincaré flow*

$$P^t_X(x) := \sqcap_{X^t(x)} \circ D_xX^t,$$

where $\sqcap_{X^t(x)} : T_{X^t(x)}M \longrightarrow N_{X^t(x)}$ is the canonical orthogonal projection.

A vector field X has an associated flow, denoted by X^t, $t \in \mathbb{R}$. Denote by Sing(X) the union of the singularities of X and by Crit(X) the set of the closed orbits and the singularities of X. For $x \in X$, the set $O_X(x) = \{X^t(x) : t \in \mathbb{R}\}$ is said to be the orbit of X^t through the point x.

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Let Λ be a compact, X^t-invariant, and regular set. If N_Λ admits a P_X^t invariant splitting $N_\Lambda = N_\Lambda^s \oplus N_\Lambda^u$, such that there is $l > 0$ satisfying

$$||P_X^l|_{N_x^s}|| \leq \frac{1}{2} \quad \text{and} \quad ||P_X^{-l}\left(X^l(x)\right)|_{N^u_{X^l(x)}}|| \leq \frac{1}{2}$$

for any $x \in \Lambda$, then we say that Λ is hyperbolic. A vector field X is said to be Anosov if the whole manifold M is hyperbolic.

We say that a vector field X without singularity admits an l-dominated splitting if there exists a continuous P_X^t invariant splitting $N = N^1 \oplus N^2$ and an integer $l > 0$ such that

$$||P_X^l|_{N_x^1}||.||P_X^{-l}\left(X^l(x)\right)|_{N^2_{X^l(x)}}|| \leq \frac{1}{2}.$$

The notion of measure expansivity for flows is introduced in [9] by D. Carrasco-Olivera and C. A. Morales and they extend some properties of measure expansive homeomorphisms to flows. Jumi Oh [10] shows that C^1-stably measure expansive flows satisfy quasi-Anosov. Let $X \in \chi_\mu^1(M)$ and $\Lambda \subset M$ be a compact invariant set. A flow X^t on Λ is called expansive if for any $\varepsilon > 0$ there is $\delta > 0$ such that if $x, y \in X$ satisfy $d\big(X^t(x),\ X^{h(t)}(y)\big) \leq \delta$ for every $t \in \mathbb{R}$ and for some $h \in B$, then $y \in X^{[-\varepsilon,\varepsilon]}(x)$ where $B = \{h : \mathbb{R} \to \mathbb{R}$ is an increasing continuous map, $h(0) = 0\}$. For $\delta > 0$, put

$$\Gamma_\delta(x) = \bigcup_{h\in B} \bigcap_{t\in\mathbb{R}} X^{-h(t)}(N[X^t(x), \delta])$$

where $N[x, \delta] = \{y \in M, d(x, y) \leq \delta\}$.

Denote by $\mathcal{M}(M)$ the set of Borel's probability measures on M endowed with weak* topology.

Definition 1.1. Let $\mathcal{M}_{X^*}(M) = \{\mu \in \mathcal{M}(M)$ and $\mu(O_X(x)) = 0$ for any $x \in M\}$. Let $\mu \in \mathcal{M}(M)$. We say that X is *δ-μ-measure expansive* if $\mu\big(\Gamma_\delta(x)\big) = 0$ for every $x \in M$ and X is *μ-measure expansive* if X is δ-μ-measure expansive for some $\delta > 0$. If X is μ-measure expansive for all $\mu \in \mathcal{M}_{X^*}(M)$, then $X \in \chi_\mu^1(M)$ is said to be *measure expansive.*

Jiweon Ahn, Manseob Lee, and Oh [1] show that the C^1-interior of measure expansive divergence free vector field is Anosov.

In this paper we work with the following definitions.

Definition 1.2. Let $\mu \in \mathcal{M}(M)$ and μ be positive on every open subset of M, and let $\delta > 0$ be given. We say that $X \in \chi_\mu^1(M)$ belongs to the C^1-interior of the set of μ-measure (δ-μ-measure) expansive if there exists a C^1-neighborhood $\mathcal{W}$ of X in $\chi_\mu^1(M)$ such that every $Y \in \mathcal{W}$ is μ-measure (δ-μ-measure) expansive.

Denote by $\mathrm{Int}(\mu)$ $\big(\mathrm{Int}(\mu,\delta)\big)$ the set of all vector fields belonging to the C^1-interior of the set of μ-measure (δ-μ-measure) expansive.

The following theorems are the main results of this paper.

Theorem 1.3. *If $X \in \mathrm{Int}(\mu)$, then X admits a dominated splitting.*

Theorem 1.4. *If $\dim(M) = 3$ and $X \in \mathrm{Int}(\mu,\delta)$, then X is Anosov.*

Remark that μ-measure expansivity is strictly weaker than measure expansivity.

2. Proofs of Theorem 1.3 and Theorem 1.4

The following proposition is proved in [4] and is the volume-preserving version of [8].

Proposition 2.1. *Let $X \in \chi^1_\mu(M)$ and $\varepsilon_0 > 0$ be given. There exist $\pi_0, l \in \mathbb{N}$ such that, for any closed orbit x with period $\pi(x) > \pi_0$, we have either*

(i) *that p^t_X has an l-dominated splitting along the orbit of x or else,*
(ii) *for any neighborhood U of $\bigcup_t X^t(x)$, there exists an ε-C^1-perturbation Y of X, coinciding with X outside U and on $\bigcup_t X^t(x)$ and such that $p^{\pi(x)}_Y(x)$ has all eigenvalues equal to 1 and –1.*

Let $X \in \chi^1_\mu(M)$ and let $p \in M$ be a regular point of X. Consider a linear differential system [5] as follows:

$$s^t : \mathbb{R}^d_p \to \mathbb{R}^d_{X_t(p)},$$

such that

- $s^t \in sl(d,\mathbb{R})$ for every t;
- $s^o = id$ and $s^{t+r} = s^t \circ s^r$ for every s and t
- s^t is differentiable in t.

For the proof of Lemma 2.2 we use "good coordinates" obtained in [5, subsection 2.2] to get the following local linear representation of the flow Y, ε-C^1-perturbation of X:

$$\hat{X}^t(q) = tv + s^t(q), \tag{2.1}$$

where $q \in N_p = X(p)^\perp$, s^t represents the action of the linear Poincaré flow p^t_Y.

Lemma 2.2. *If $X \in \mathrm{Int}(\mu)$, then any $Y \in \chi^1_\mu(M)$ sufficiently C^1-close to X does not contain closed orbits with trivial real spectrum.*

Proof. By contradiction, we assume that any Y which is C^1-close to X has a non-hyperbolic closed orbit q of period π and with trivial real spectrum. We consider $\hat{X}^t$ the linear coordinate given in (2.1). Thus, there exists an eigenvalue λ with $|\lambda| = 1$ for $S^{\pi}(p)$. So $\hat{X}^{2\pi}(q) = 2\pi\nu + S^{q\pi}(q) = id$ holds in a neighborhood v_p of p in N_p. We consider an open neighborhood w of p such that $\operatorname{diam}(w \cap v_p) < \frac{\operatorname{diam}(v_p)}{2}$ and

$$d\big(\hat{X}^t(q), \hat{X}^t(p)\big) < \frac{\delta}{2}$$

for every $q \in w$ and $0 \leq t \leq 2\pi$ where δ is the expansivity constant of Y. One can see $w \subset \Gamma_\delta(p)$; therefore, $\mu\big(\Gamma_\delta(p)\big) > \mu(w) > 0$, which is a contradiction with $X \in \operatorname{Int}(\mu)$. $\square$

Proof of Theorem 1.3. Let $X \in \operatorname{Int}(\mu)$. Since topologically mixing vector fields are residual in $\chi^1_\mu(M)$ (see [3]), there exists $X_n \in \chi^1_\mu(M)$ which is C^1-close to X which is topologically mixing. We can find $Z_n \in \chi^1_\mu(M)$ C^1-close to X_n having a closed orbit $O(p_n)$ such that $d_H\Big(\bigcup_t Z^t_n\big(O(p_n)\big), M\Big) < \frac{1}{n}$ where d_H is the Hausdorff distance. By Lemma 2.2 and Proposition 2.1, $O(p_n)$ admits an l-dominated splitting. Since $Z_n \xrightarrow{C^1} X$ and, in the Hausdorff metric, $\bigcup_t Z^t_n\big(O(p_n)\big)$ converges to M, we obtain that $M \setminus \operatorname{Sing}(X)$ has an l-dominated splitting. By [9], $\operatorname{Sing}(X) = \phi$, so the proof is complete. $\square$

A C^1-divergence-free vector field X is said to be a *divergence-free star vector field* if there exists a C^1-neighborhood U of X in $\mathcal{X}^1_\mu(M)$ such that if $Y \in U$, then every point in Crit(Y) is hyperbolic. The set of divergence-free star vector fields is denoted by $\mathcal{G}^1_\mu(M)$.

The following theorem is proved in [6].

Theorem 2.3. *Let M be a three-dimensional closed manifold. If $X \in \mathcal{G}^1_\mu(M)$, then X is Anosov.*

In [7], Mário Bessa and Jorge Rocha obtain an upgrade of the C^1-pasting lemma for vector fields [2] as follows.

Lemma 2.4. *Let M be a compact Riemannian manifold without boundary with dimension $n \geq 2$.*

Given $\varepsilon > 0$, $X \in \mathcal{X}^1_\mu(M)$, K a compact subset of M, and an open neighborhood U of K, there are $\delta > 0$ and an open set $K \subset V \subset U$ such that if $Y \in \mathcal{X}^2_\mu(M)$ is δ-C^1-close to X in U, then there exists $Z \in \mathcal{X}^1_\mu(M)$ satisfying

(a) *$Z = Y$ in V,*
(b) *Z is ϵ-C^1-close to X, and*
(c) *$Z = X$ outside U.*

Proof of Theorem 1.4. Let $X \in \operatorname{Int}(\mu, \delta)$ and let $p \in \operatorname{Crit}(X)$ be an elliptic closed orbit of period π. Consider a small open neighborhood

v_p of p such that $d\big(X^t(p), X^t(q)\big) < \frac{\delta}{2}$ for every $q \in v_p$ and $0 \leq t \leq \pi + 1$. Remark that we can take v_p sufficiently small such that for every $q \in v_p$, $\tau(q) \leq \pi + 1$, where $\tau(q)$ is the first time that $X^t(q)$ meets N_p. We set k a small compact neighborhood of orbit p (a tabular neighborhood) such that $\mathrm{diam}(k \cap v_p) < \frac{\mathrm{diam}(v_p)}{8}$ and, also, the linear flow induced by the periodic orbit restricted to k is C^1-close to X. Use Lemma 2.4 to obtain a vector field Y C^1-close to X which is $DX(p)$ inside an invariant neighborhood w where $\mathrm{diam}(w \cap v_p) < \frac{\mathrm{diam}(v_p)}{2}$. By choosing v_p and since w is invariant, one can see that $w \cap v_p \subset \Gamma_\delta(p)$. Therefore, $\mu\big(\Gamma_\delta(p)\big) \geq \mu(w \cap v_p) > 0$, which is a contradiction. Hence, since X has no singularities, we have $X \in \mathcal{G}^1_\mu(M)$. By Theorem 2.3, X is Anosov. □

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Department of Pure Mathematics; Ferdowsi University of Mashhad; P. O. Box 1159; Mashhad 91775, Iran

E-mail address: `zamany@um.ac.ir`