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A CO-LIFTING THEOREM FOR IMAGES OF ARCS

B. D. DANIEL AND MURAT TUNCALI

Abstract. Let X be a continuum and let G be an upper semi-continuous decomposition of X such that each element of G is the continuous image of an arc. If the quotient space X/G is the continuous image of an arc, under what conditions is X also the continuous image of an arc? Examples demonstrate that one must place severe conditions on G if one wishes to obtain positive results. Such results that characterize those locally connected continua that admit a lifting of a map of an arc onto a decomposition space back to the base space are often called lifting theorems. Such lifting theorems have taken several forms and their proofs have utilized a variety of methods. In this paper, we prove what we call a "co-lifting" result, since we are not directly lifting a map of an arc to a decomposition space to the base space. We then note that a number of typical lifting theorems from the literature follow somewhat easily from this co-lifting theorem.

1. Introduction

The classical Hahn–Mazurkiewicz theorem asserts that a metric continuum is the continuous image of the closed unit interval $[0, 1]$ if and only if it is locally connected. In the non-metric case, the situation turns out to be quite complicated. The case of continuous images of non-metric arcs did not begin to be systematically studied until 1960. Characterizations of continuous images of non-metric arcs are given by Witold Bula and Marian Turzański [1], by Jacek Nikiel [6], and by L. B. Treybig [8]. For

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a survey of the problem, the reader is referred to the early survey paper of Treybig and L. E. Ward, Jr. [9] or to the more recent survey of Sibe Mardešić [5].

One natural question is if there exists an upper semi-continuous decomposition G of a space X so that X/G is the image of an arc by continuous surjection f and each $g \in G$ is the image of some ordered compactum by continuous surjection f_g, then may the map f and the collection $\{f_g : g \in G\}$ of maps be utilized so as to construct a continuous surjection of an arc onto X? In general, the answer is no. Two instructive counter-examples are given in [7]. What are sufficient conditions on the map f, the collection $\{f_g : g \in G\}$, and the elements of G to ensure that X is the image of some ordered compactum? The examples noted above provide evidence that quite strong conditions are required in order to obtain positive results on the central question. Earlier work in this and related areas includes [3] and [4].

Such positive results that characterize those locally connected continua that admit a lifting of a map of an arc onto a decomposition space to the base space are often called lifting theorems. Such lifting theorems have taken several forms and their proofs have utilized a variety of methods. Herein, we prove a "co-lifting" theorem, which differs somewhat from those lifting theorems that characterize locally connected continua that admit a lifting of a map of an arc onto a decomposition space to the base space. We also use our theorem to provide alternate proofs of some lifting theorems in the literature.

2. Preliminaries

A *compactum* is a compact Hausdorff space. A *continuum* is a connected compactum. An *ordered compactum* is a compact space X which admits a linear ordering such that the order topology coincides with the given topology. An *arc* is a connected ordered compactum. Each metric arc is homeomorphic to the closed unit interval $[0, 1]$ in the real line. It is well known and easy to prove that each ordered compactum is contained in an arc.

A Hausdorff space X is said to be an IOK if it is the continuous image of some ordered compactum. If $x \in X$ and x is in the open set O', and there exists an open set O such that $x \in O \subseteq \mathrm{Cl}(O) \subseteq O'$ and $\mathrm{Cl}(O)$ is an IOK, then X said to be *locally an IOK* at x. If K is an ordered space and $x, y \in K$ with $x \neq y$, then the set $\{x, y\}$ is referred to as a *gap* of K if and only if $\{k \in K : x < k < y\}$ is empty. A Hausdorff space X is said to be an *IOC* if it is the continuous image of some arc.

3. Main Results

We first give the aforementioned co-lifting result.

Theorem 3.1. *The locally connected continuum X is an IOC if and only if there exists an upper semi-continuous decomposition G of X into continua and a continuous map $f : [a,b] \to X/G$ of an arc $[a,b]$ (with order $<_{[a,b]}$) onto X/G such that*

(1) *$f^{-1}(g)$ is totally disconnected for each $g \in G$,*
(2) *each $g \in G$ is an IOK,*
(3a) *for each $x \in [a,b]$, there exist points l_x and r_x of $g_x = f(x)$ so that $l_x \in \lim_{t\to x^-} f(t)$ and $r_x \in \lim_{t\to x^+} f(t)$, where $f(t) \neq g_x$ (at $x = a$, only r_a exists and at $x = b$, only l_b exists), such that*
(3b) *if K is the ordered compactum $K = \{(x, r_x) : x \in [a,b]\} \cup \{(x, l_x) : x \in [a,b]\}$ with the order $<$ defined by $(x, l_x) < (x, r_x)$ and $(x, r_x) < (y, l_y)$ if and only if $x <_{[a,b]} y$, then the (into) map $h : K \to X$ defined by $h((x, l_x)) = l_x$ and $h((x, r_x)) = r_x$ is continuous.*

Proof. ($\Rightarrow$) Since X is IOC, there exists an arc $[c,d]$ and a continuous onto map $F : [c,d] \to X$. For each $x \in X$, let $h'_x = F^{-1}(x)$. Let $H' = \{h'_x : x \in X\}$ and $H = \{h_x : h_x$ is a component of h'_x for some $h'_x \in H'\}$. It is well known that H is then an upper semi-continuous decomposition of X into continua.

We now show that $[c,d]/H$ is an arc. Let $\psi : [c,d] \to [c,d]/H$ denote the natural map. Let $x, y \in [c,d]$. Then $\psi(x) < \psi(y)$ if and only $\psi(x) \neq \psi(y)$ and $x < y$ in $[c,d]$. Since this gives a linear order on $[c,d]/H$, it is an arc.

Let $[c,d]/H$ be denoted by $[a,b]$. We define G to be the trivial upper semi-continuous decomposition of X into points, and define $f = F \circ \psi$.

We now demonstrate that each of the properties holds.

(1) Let $g \in X = X/G$ and let C be a component of $f^{-1}(g)$. Then, by definition of H, $C = h_x$ for some $h_x \in H$. The components of $f^{-1}(g)$ are, therefore, points of $[a,b]$, and $f^{-1}(g)$ is totally disconnected.

(2) As a closed subset of X, each point in X is trivially an IOK.

(3a) By the continuity of f and by definition of G, $l_x \in \lim_{t\to x^-} f(t)$ and $r_x \in \lim_{t\to x^+} f(t)$, where $f(t) \neq g_x$ and $x \neq a$ and $x \neq b$. Set $l_x = r_x = f(x)$. The cases for $x = a$ and $x = b$ are similar.

(3b) With $h : K \to X$ as in the theorem, $h((x, l_x)) = h((x, r_x)) = f(x)$. So, if $M \subseteq K$, it follows from (3a),

$$\begin{aligned} h(Cl(M)) &\subseteq \{f(x) : (x, l_x) \text{ or } (x, r_x) \text{ is in } Cl(M)\} \\ &\subseteq Cl(\{f(x) : (x, l_x) \text{ or } (x, r_x) \text{ is in } Cl(M)\}) \\ &\subseteq Cl(h(M)). \end{aligned}$$

Therefore, h is continuous.

($\Leftarrow$) Suppose that there exists an upper semi-continuous decomposition G of X into continua and a continuous map $f : [a,b] \to X/G$ of an arc $[a,b]$ onto X/G such that properties (1)–(3b) hold. For each $x \in [a,b]$, select l_x and r_x in $g_x = f(x)$ satisfying (3a). Form K and define $h : K \to X$ as in (3b). For each $g_x \in G$, select an ordered compactum K_x and a continuous onto map $f_x : K_x \to g_x$. Suppose that the least element of K_x is $k_1(x)$ and the greatest element of K_x is $k_2(x)$. Then we may assume that $f(k_1(x)) = l_x$ and $f(k_2(x)) = r_x$. For each $g_x \in G$, select $y \in f^{-1}(g_x)$, and let Y denote the collection of y's so selected. For each $y \in Y$ and $y \in f^{-1}(g_x)$, select a copy $K_x(y)$ of K_x so that if $w, z \in Y$ and $w \neq z$, then $K_x(w) \cap K_x(z) = \emptyset$ and $K \cap K_x(z) = \emptyset$ for all $z \in Y$.

We now form an ordered compactum L. $L = K \cup \{K_x(y) : y \in Y\}$, and if $r, s \in L$, then $r < s$ in L if and only if

(i) $r, s \in K$ and $r < s$ in K, or
(ii) $r, s \in K_x(y)$ and $r < s$ in $K_x(y)$, or
(iii) $r \in K$, $s \in K_x(y)$, and $r \leq (x, l_x)$ in K.

Define $g : L \to X$ by

$$g(r) = \begin{cases} h(r), & r \in K \\ f_x(r), & r \in K_x(y) \text{ for some } y \in Y. \end{cases}$$

We now show that g is continuous; we consider two cases.

Case 1. Suppose $r \in K_x(y)$, and $g(r) \in V$ open. Select U open in $K_x(y)$ so that $f_x(U) \subseteq V$ and $r \in U$. Since U is open in L, g is continuous at r.

Case 2. Suppose $r \in K$, and $g(r) \in V$ open. Select U open in $K_x(y)$ so that $h(U) \subseteq V$ and $r \in U$. Select $t = (w, r_w) \in K$ such that $t \in U$ and $t < r$; if no such t exists, set $t = r$. Similarly, select $q = (z, l_z) \in K$ such that $q \in U$ and $q > r$; if no such q exists, set $q = r$. Let $O_1 = \{s \in L : t \leq s \leq q\}$. Then O_1 is open in L. Select O_2 open in $K_x(y)$, where $h(r) \in g_x$, so that $f_x(O_2) \subseteq V$. Then $(O_1 \cup O_2)$ is open in L, $r \in (O_1 \cup O_2)$, and $g(U) \subset V$.

X is then an IOK. Since X is connected and locally connected, it is then an IOC by [8, Theorem 3]. □

Note that the assumption of property (1) that $f^{-1}(g)$ is totally disconnected for each $g \in G$ is not, in fact, needed for the sufficiency of the previous results. If f' is a continuous map $f' : [a,b] \to X/G$ of an arc $[a,b]$ onto X/G, then the decomposition of $[a,b]$ into components of the point inverses of f' clearly yields a continuous map of an arc onto X/G with totally disconnected point inverses. We also note that, as indicated in the proof of the necessity, property (3b), in fact, follows from property (3a).

Although our Theorem 3.1 is a stand-alone result, its real utility is as a co-lifting theorem in that it provides a natural result by which to approach (some) lifting theorems. As an initial example, we note that the following corollary, which is [3, Theorem 3.1], follows relatively easily from the previous theorem.

Corollary 3.2. *Let X be a locally connected continuum and let G be an upper semi-continuous decomposition of X such that*

(i) *each $g \in G$ is connected and has zero-dimensional boundary, and*
(ii) *each $g \in G$ is a continuous image of an ordered compactum.*

If the quotient space X/G is the continuous image of an arc then so too is X.

Proof. We need only show that property (3a) holds.

Suppose $x \in [a, b]$ and consider $f(x) = g_x \in G$. Then $\lim_{t\to x^-} f(t)$ and $\lim_{t\to x^+} f(t)$ both exist and must be singletons, else the zero dimensionality of the boundary of g_x is contradicted. □

As a second example, we address a setting in which the zero-dimensionality of the boundary of the elements of the decomposition is not directly hypothesized.

Corollary 3.3. *Let X denote a locally connected continuum and let G be an upper semi-continuous decomposition of X into continua such that each $g \in G$ is an IOK. If there exists a continuous onto map $f : [a, b] \to X/G$ so that $[a, b]$ is an arc and for each $g \in G$ either $f^{-1}(g)$ is finite or g is a point, then X is an IOC.*

Proof. Suppose that each $g \in G$ is an IOK. For $x \in [a, b]$, let $g_x = f(x) \in G$. If $f^{-1}(g_x)$ is finite, let (u_x, v_x), ($[a, v_x)$, or $(u_x, b]$) denote an open (half-open) interval containing x but which intersects $f^{-1}(f(x))$ only in x. One can construct sequences (u_{α_n}) and (v_{β_n}), indexed by well-ordered sets, such that $u_x < u_{\alpha_1} < u_{\alpha_2} < \cdots < x < \cdots < v_{\beta_2} < v_{\beta_1} < v_x$, and $u_{\alpha_n} \to x$ and $v_{\beta_n} \to x$.

We first show that there is a point $l_x \in g_x$ ($r_x \in g_x$, respectively) so that if U (V, respectively) is an open set containing l_x (r_x, respectively), then for each α_i (β_j, respectively), we have $[\cup f((u_{\alpha_n}, x))] \cap U \neq \emptyset$ ($[\cup f((x, v_{\beta_n}))] \cap V \neq \emptyset$, respectively). Assume that for every $y \in g_x$, there exists U open in X such that $y \in U$ and an α_i so that $\cup f((u_{\alpha_i}, x)) \cap U = \emptyset$. By the compactness of g_x, there exists O open in X and an α_m so that $g_x \subset O$ and $\cup f((u_{\alpha_m}, x)) \cap O = \emptyset$. Define $R = \{r : r \in G, r \subset O\}$. Then R is open in X/G and $\cup R$ is open in X. Then $\cup f((u_{\alpha_m}, x)) \cap (\cup R) = \emptyset$

and therefore $\cup\text{Cl}(f((u_{\alpha_m}, x))) \cap g_x = \emptyset$. Then since $g_x \in f([u_{\alpha_m}, x])$, we have that $f(\text{Cl}(u_{\alpha_m}, x)) = f([u_{\alpha_m}, x]) \not\subset \text{Cl} f((u_{\alpha_m}, x))$ which contradicts the continuity of f. The case for r_x is similar.

We now show that l_x (r_x, respectively) is unique. Suppose $l'_x \in g_x$, $l_x \neq l'_x$, and l'_x has the property above. Let O and O' and M and M' be pairs of open connected sets such that
(1) $Cl_X(O) \cap Cl_X(O') = \emptyset$,
(2) $Cl_X(M) \cap Cl_X(M') = \emptyset$,
(3) $l_x \in O \subset Cl_X(O) \subset M$,
(4) $l'_x \in O' \subset Cl_X(O') \subset M'$.

Let $\mathcal{O}$ denote a finite open cover of $\text{Bd}_X(g_x)$ such that each of O and O' is an element of $\mathcal{O}$ and the only element of $\mathcal{O}$ that contains l_x is O and the only element of $\mathcal{O}$ that contains l'_x is O'. Then $(\cup\mathcal{O} \cup g_x)$ is open in X and $S = \{g \in G : g \subset (\cup\mathcal{O} \cup g_x)\}$ is open in X/G. Select a subsequence (u'_{α_n}) of (u_{α_n}) so that

(1) $[u'_{\alpha_i}, u'_{\alpha_{i+1}}] \cap [u'_{\alpha_j}, u'_{\alpha_{j+1}}] = \emptyset$ for all naturals i and j such that $(i+1) < j$,
(2) $f([u'_{\alpha_i}, u'_{\alpha_{i+1}}]) \subset S$ for each natural i, and
(3) for each $n = 1, 3, 5, \ldots$, $C_n = \cup(f([u'_{\alpha_n}, u'_{\alpha_{n+1}}])$ is a continuum in X intersecting both of O and O'.

The limiting set of the sequence (C_n) contains a continuum C in X which intersects both O and O'. Therefore, $g_x - (O \cup O')$ intersects C in a point $p \in Bd_X(g_x)$, a contradiction. The case for r_x is similar, and property (3a) then follows. □

As seen in the two previous results, it is relatively straightforward to select the points l_x and r_x, form the ordered compactum K, and construct the mapping h as in Theorem 3.1 in the setting that the elements g of G have totally disconnected boundary. If this is not the case, Theorem 3.1 may still be applied but requires more effort. The following is proved in both [2, Corollary 6] and [3, Theorem 4.1].

Corollary 3.4. *Let X be a locally connected continuum and let Y be a closed G_δ metric subspace of X. If X is locally IOK at each point of $X - Y$, then X is IOC.*

The techniques of [2, Lemma 4 and Theorem 5] may be modified slightly to yield a mapping of an arc onto X/G where G is the decomposition of X in which Y is the only non-degenerate element. The resulting map then satisfies our Theorem 3.1 by construction. The details are left to the reader.

4. Remaining Questions

We close with some questions related to the results of this work. Some of these we have not studied in depth but they are clearly related and are of interest. Some are new and some are quite old.

Question 4.1. Let X be a locally connected continuum and let G be a null cover of X such that each g in G is an IOC. Suppose also that if $h, h' \in G$ are such that $h \cap h' = \emptyset$, then there exists an element g of G and a closed subset A of g that separates h and h' in X. Is X an IOC? ([4, Question 1])

Question 4.2. Let X be a locally connected continuum and let G be a null cover of X by IOCs. Suppose $g, h \in G$ with $g \cap h = \emptyset$ implies there exists an integer n and $g_1, g_2, \cdots, g_n \in G$ such that a closed subset of $g_1 \cup g_2 \cup \cdots \cup g_n$ separates g and h in X. If also for $g, g' \in G$, $g \setminus g'$ is contained in a component of $X \setminus g'$, is X an IOC? ([4, Question 2])

Question 4.3. Let X be a locally connected continuum and let $G = \{g_1, g_2, \dots\}$ be a sequence of closed pairwise disjoint G_δ subsets with metric boundaries. Let G be the upper semi-continuous decomposition of X whose elements are points and the elements $g_i \in G$. Suppose that G is a null family and X/G is IOC. Is X then an IOC? (See [3, Theorem 5.1].)

Question 4.4. Let X be a locally connected continuum which is not an IOK. Must there exist an upper semi-continuous decomposition G of X into continua so that $N = \{g \in G : X/G$ is not locally an IOK at $g\}$ has uncountably many components? (See [2, Theorem 8].)

Question 4.5. Let X be a (first countable) locally connected continuum which is not metrizable. Must there exist an upper semi-continuous decomposition G of X into continua so that $N = \{g \in G : X/G$ is not locally metrizable at $g\}$ has uncountably many components?

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