http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 53, 2019

Pages 15–26

http://topology.nipissingu.ca/tp/

A Banach-Stone type theorem for C^1-function spaces over the circle

by

Kazuhiro Kawamura

Electronically published on February 11, 2018

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 53 (2019)
Pages 15-26

E-Published on February 11, 2018

A BANACH-STONE TYPE THEOREM FOR C^1-FUNCTION SPACES OVER THE CIRCLE

KAZUHIRO KAWAMURA

Abstract. We prove and apply an elementary theorem in calculus to improve the Banach-Stone type theorem on C^1-function spaces over the unit circle on the complex plane proved in [8, Theorem 3.1].

1. Introduction

This is a continuation of the paper [8]. The Banach-Stone theorem states that every linear isometry on the space of continuous functions over a compact Hausdorff space (with the sup norm) is a weighted composition operator with a unimodular weight. Various extensions of the theorem have been studied by many authors (see monographs [3],[4]). Banach-Stone type theorems for the C^1-function spaces over $[0,1]$ have been proved in [1], [6], [7], [11], [12], [13], [14] etc. Recently the author obtained a similar theorem for $C^1(\mathbb{T})$, the space of C^1-function space over the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ on the complex plane [8]. In [8, Theorem 3.1] it was shown that every linear isometry $T : C^1(\mathbb{T}) \to C^1(\mathbb{T})$ with respect to a suitable norm is a weighted composition operator or its variant if the isometry T satisfies an additional hypothesis:

$$(*)\ T(\mathrm{id}_{\mathbb{T}}) \text{ and } T(\overline{\mathrm{id}_{\mathbb{T}}}) \text{ are } C^3\text{-functions}$$

where $\mathrm{id}_{\mathbb{T}}$ and $\overline{\mathrm{id}}_{\mathbb{T}}$ denote the identity function on $\mathbb{T}$ and its complex conjugate respectively. The above technical hypothesis was assumed only to prove the following lemma.

2010 *Mathematics Subject Classification.* Primary 46E15; Secondary 57N20.
Key words and phrases. Isometry group, weighted composition operator.
The author was supported in part by JSPS KAKENHI Grant Number 17K05241.

Lemma 1.1. [8, Lemma 3.12] *Let $\psi : \mathbb{T} \to \mathbb{T}$ be a C^2-diffeomorphism and let $\beta : \mathbb{T} \to \mathbb{T}$ be a C^1-map such that, for each C^1-function $f \in C^1(\mathbb{T})$, there exists a C^1-function $F \in C^1(\mathbb{T})$ such that*

$$F'(z) = \beta(z) f'(\psi(z)),\ \ z \in \mathbb{T}.$$

Then β is a constant map and ψ is an isometry.

In this note we prove the above lemma without the C^1- and C^2-hypotheses of the maps β and ψ. First we prove:

Theorem 1.2. *Let $\beta : \mathbb{R} \to \mathbb{C}$ be a continuous function of period 1 and let $\psi : \mathbb{R} \to \mathbb{R}$ be a homeomorphism with a constant $\gamma \in \{\pm 1\}$ such that*

$$\psi(t+1) = \psi(t) + \gamma,\ \ t \in \mathbb{R}.$$

Suppose that

$$\int_0^1 \beta(t) f'(\psi(t)) dt = 0 \tag{1.1}$$

for each periodic C^1-function $f : \mathbb{R} \to \mathbb{R}$ of period 1. Then

(1) *β is a constant function and*
(2) *there exists a constant c such that $\psi(t) = \gamma t + c,\ \ t \in \mathbb{R}$.*

Then we derive the same conclusion as that of Lemma 1.1 under a weaker hypothesis. More precisely we have:

Proposition 1.3. *Let $\psi : \mathbb{T} \to \mathbb{T}$ be a homeomorphism and let $\beta : \mathbb{T} \to \mathbb{T}$ be a continuous map such that, for each C^1-function $f \in C^1(\mathbb{T})$, there exists a C^1-function $F \in C^1(\mathbb{T})$ such that*

$$F'(z) = \beta(z) f'(\psi(z)),\ \ z \in \mathbb{T}.$$

Then β is a constant map and ψ is an isometry.

The above proposition enables us to generalize [8, Theorem 3.1]: replacing [8, Lemma 3.12] with Proposition 1.3, the argument in [8, Section 3, 3.1-3.2] works with no change to derive the same conclusion as that of [8, Theorem 3.1] without the hypothesis $(*)$. In order to state the generalization, we fix our notation and define some terminologies. Let $\pi : \mathbb{R} \to \mathbb{T}$ be the standard covering map defined by $\pi(t) = \exp(2\pi i t)$ ($t \in \mathbb{R}$). For a function $f : \mathbb{T} \to \mathbb{C}$, let

$$f'(e^{2\pi i\theta}) = \frac{1}{2\pi} \frac{d}{dt}\Big|_{t=\theta} f(\exp(2\pi i t)). \tag{1.2}$$

We say that the function f is a C^1-function if the above derivative exists and is continuous. The space of all complex-valued C^1-functions on $\mathbb{T}$ is denoted by $C^1(\mathbb{T})$. Let $p_i : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ be the projection to the i-th factor $i = 1, 2$. For a compact connected subset D of $\mathbb{T} \times \mathbb{T}$ with $p_2(D) = \mathbb{T}$, we define a norm $\|\cdot\|_{<D>}$ on $C^1(\mathbb{T})$ which induces the C^1-topology by:

$$\|f\|_{<D>} = \sup_{(x,y)\in D} (|f(x)| + |f'(y)|), \ f \in C^1(\mathbb{T}). \tag{1.3}$$

Theorem 1.4. *(cf.* [8, Theorem 3.1]*) Let $T : C^1(\mathbb{T}) \to C^1(\mathbb{T})$ be a surjective $\mathbb{C}$-linear $\|\cdot\|_{<D>}$-isometry and let $I = P,(D)$.*

(1) *Assume that $I = \{a\}$. Then there exist constants $\beta, \kappa \in \mathbb{T}$ and an isometry $\varphi : \mathbb{T} \to \mathbb{T}$ such that*

$$Tf(z) = \beta f(\varphi(z)) + (\kappa f(a) - \beta f(\varphi(a))), \ z \in \mathbb{T} \tag{1.4}$$

for each $f \in C^1(\mathbb{T})$.

(2) *Assume that I is not a singleton. Then there exist a constant $\kappa \in \mathbb{T}$ and an isometry $\varphi : \mathbb{T} \to \mathbb{T}$ such that $\varphi(I) = I$ and*

$$Tf(z) = \kappa f(\varphi(z)), \ z \in \mathbb{T} \tag{1.5}$$

for each $f \in C^1(\mathbb{T})$.

As a corollary we obtain structure theorems on isometry groups with respect to some typical norms on $C^1(\mathbb{T})$ defined below. For a C^1-function $f \in C^1(\mathbb{T})$ and a point $c \in \mathbb{T}$, let

$$\begin{aligned} \|f\|_\Sigma &= ||f||_\infty + ||f'||_\infty, \\ \|f\|_C &= \sup_{t\in\mathbb{T}}(|f(t)| + |f'(t)|), \\ \|f\|_{\sigma,c} &= |f(c)| + ||f'||_\infty. \end{aligned} \tag{1.6}$$

For a norm $\|\cdot\|$ on $C^1(\mathbb{T})$, $\mathcal{U}(\|\cdot\|)$ denotes the group of all surjective $\mathbb{C}$-linear $\|\cdot\|$-isometries. The groups of isometries on $\mathbb{T}$ is denoted by $\mathrm{Isom}(\mathbb{T})$ and, for a subset J of $\mathbb{T}$, let $\mathrm{Isom}(\mathbb{T}; J) = \{\varphi \in \mathrm{Isom}(\mathbb{T}) \mid \varphi(J) = J\}$. Notice that $\mathrm{Isom}(\mathbb{T}) \cong \mathbb{T} \times \mathbb{Z}_2$.

Corollary 1.5. (cf. [8, Corollary 3.3].)

(1) $\mathcal{U}(\|\cdot\|_\Sigma) = \mathcal{U}(\|\cdot\|_C) \cong \mathbb{T} \times \mathrm{Isom}(\mathbb{T})$.
(2) $\mathcal{U}(\|\cdot\|_{\sigma,c}) \cong \mathbb{T} \times \mathbb{T} \times \mathrm{Isom}(\mathbb{T})$ *for each* $c \in \mathbb{T}$.
(3) *For an interval I of $\mathbb{T}$ which is not a singleton,* $\mathcal{U}(\|\cdot\|_{<I\times\mathbb{T}>}) \cong \mathbb{T} \times \mathrm{Isom}(\mathbb{T}; I)$.

Recently Hatori and Oi [5] studied the space $C^1(\mathbb{T})$ with the norm $\|\cdot\|_\Sigma$ in a much broader context. In particular they studied the function space $C^1(\mathbb{T}, C(Y))$ of the $C(Y)$-valued C^1-functions on $\mathbb{T}$, where $C(Y)$ denotes the Banach space of continuous functions on a compact Hausdorff space Y. In particular Corollary 1.5 (1) for the norm $\|\cdot\|_\Sigma$ follows from their result.

The present note is devoted to prove Theorem 1.2 and Proposition 1.3. Although the proof of Theorem 1.2 belongs to elementary calculus, the author could not find it in the literature and a proof is supplied in detail.

2. Proofs

For a map $\varphi : \mathbb{T} \to \mathbb{T}$, let $\bar{\varphi} : \mathbb{R} \to \mathbb{R}$ be a lift of φ, that is, a map satisfying $\pi \circ \bar{\varphi} = \varphi \circ \pi$. If φ is a homeomorphism, then $\bar{\varphi}$ is a homeomorphism and there exists a constant $d_\varphi \in \{\pm 1\}$ such that $\bar{\varphi}(t+1) = \bar{\varphi}(t) + d_\varphi,\ t \in \mathbb{R}$.

Proof of Theorem 1.2. We assume at the outset that $\gamma = 1$. The case $\gamma = -1$ can be proved similarly. Since $\psi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, we see that ψ is, under the assumption $\gamma = 1$, a monotone increasing function.

Step 1. First we show that β is a constant function. Applying (1.1) to

$$f_1(t) = \frac{1}{2\pi n}\sin(2\pi nt),\ f_2(t) = -\frac{1}{2\pi n}\cos(2\pi nt)$$

for $n \in \mathbb{Z} \setminus \{0\}$, we have

$$\int_0^1 \beta(t)\cos(2\pi n\psi(t))dt = \int_0^1 \beta(t)\sin(2\pi n\psi(t))dt = 0. \tag{2.1}$$

Suppose $\beta(a) \neq \beta(b)$ for some $a, b \in [0,1]$ and let $\epsilon = |\beta(a) - \beta(b)| > 0$. For simplicity let $p = \psi(a), q = \psi(b)$ and take a $\delta > 0$ so small that

(i) $[a-\delta, a+\delta] \cap [b-\delta, b+\delta] = \emptyset$ and $\psi([a-\delta, a+\delta]) \cap \psi([b-\delta, b+\delta]) = \emptyset$,

(ii) $\mathrm{diam}\beta([a-\delta, a+\delta]) < \epsilon/2$ and $\mathrm{diam}\beta([b-\delta, b+\delta]) < \epsilon/2$.

Let $I_p = \psi([a-\delta, a+\delta] \cap [0,1]), I_q = \psi([b-\delta, b+\delta] \cap [0,1])$. Take a real-valued smooth functions $g_a : I_p \to \mathbb{R}$ and $g_b : I_q \to \mathbb{R}$ such that $g_a, g_b \geq 0,\ g_a|\partial I_p = g_b|\partial I_q \equiv 0$ and

$$\int_{a-\delta}^{a+\delta} g_a(\psi(t)) = \int_{b-\delta}^{b+\delta} g_b(\psi(t)) = 1. \tag{2.2}$$

Let $\psi([0,1]) = [A, A+1]$. Then we have $I_p \cup I_q \subset [A.A+1]$. Define a C^1-function $g : [A, A+1] \to \mathbb{R}$ by

$$g(t) = \begin{cases} g_a(t) & t \in I_p \\ -g_b(t) & t \in I_q \\ 0 & t \notin I_p \cup I_q \end{cases} \tag{2.3}$$

The function g naturally extends to a periodic C^1-function, denoted by $g : \mathbb{R} \to \mathbb{R}$, of period 1 which satisfies

$$\int_0^1 g(t)dt = \int_A^{A+1} g(t)dt = 0. \tag{2.4}$$

The function

$$f(t) = \int_0^t g(s)ds,\ t \in \mathbb{R}$$

is a C^1-function and has period 1 due to (2.4). Thus its Fourier series converges uniformly on $[0, 1]$:

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(2\pi nt) + b_n \sin(2\pi nt)$$

with a uniform convergent series of the derivative:

$$\begin{aligned} g(t) = f'(t) &= \sum_{n=0}^{\infty} (-2\pi n a_n \sin(2\pi nt) + 2\pi n b_n \cos(2\pi nt)) \\ &= \lim_{n\to\infty} S_n(t), \end{aligned}$$

where $S_n(t) = \sum_{k=0}^{n} (-2\pi k a_k \sin(2\pi kt) + 2\pi k b_k \cos(2\pi kt))$. Applying (2.1) we have $\int_0^1 \beta(t) S_n(\psi(t))dt = 0$ and by taking the limit we obtain

$$\int_0^1 \beta(t) g(\psi(t))dt = 0. \tag{2.5}$$

On the other hand we see from (2.2) and (ii),

$$\begin{aligned} &\left| \int_{a-\delta}^{a+\delta} \beta(t) g_a(\psi(t))dt - \beta(a) \right| \qquad (2.6) \\ = &\left| \int_{a-\delta}^{a+\delta} \beta(t) g_a(\psi(t))dt - \int_{a-\delta}^{a+\delta} \beta(a) g_a(\psi(t))dt \right| \\ \le &\int_{a-\delta}^{a+\delta} |\beta(t) - \beta(a)| g_a(\psi(t))dt \\ < &\frac{\epsilon}{2} \int_{a-\delta}^{a+\delta} g_a(\psi(t))dt = \epsilon/2. \end{aligned}$$

Similarly we have

$$\left| \int_{b-\delta}^{b+\delta} \beta(t) g_b(\psi(t))dt - \beta(b) \right| < \epsilon/2. \tag{2.7}$$

We also have by (2.5) and (2.3)

$$\int_{a-\delta}^{a+\delta} \beta(t) g_a(\psi(t))dt - \int_{b-\delta}^{b+\delta} \beta(t) g_b(\psi(t))dt = \int_0^1 \beta(t) g(\psi(t))dt = 0.$$

However since $|\beta(a) - \beta(b)| = \epsilon$, the last equality contradicts (2.6) and (2.7), which proves that β is a constant function.

Step 2. We prove that ψ takes the form as in the conclusion. Since β is a constant map by Step 1, (1.1) is reduced to

$$\int_0^1 f'(\psi(t))dt = 0 \tag{2.8}$$

for each C^1-function $f : \mathbb{R} \to \mathbb{R}$ of period 1. Recall $\psi([0,1]) = [A, A+1]$. We prove:

Claim. Let p, r be two points of $[A, A+1]$ with $p < r$ and let $q = \frac{1}{2}(p+r)$. Then we have the equality

$$\psi^{-1}(q) = \frac{1}{2}(\psi^{-1}(p) + \psi^{-1}(r)). \tag{2.9}$$

First we show that Claim implies the desired conclusion. Let $\theta = \psi^{-1}$ and note that θ is a monotone increasing function such that $\theta([A, A+1]) = [0,1]$. Fix $p, q \in [A, A+1]$ with $p \leq q$ arbitrarily. Claim implies

$$\theta(\frac{p+q}{2}) = \frac{\theta(p) + \theta(q)}{2}.$$

By a straightforward induction, we can prove, for each $n \geq 1$, the following equality

$$\theta\left(\frac{k}{2^n}p + (1 - \frac{k}{2^n})q\right) = \frac{k}{2^n}\theta(p) + (1 - \frac{k}{2^n})\theta(q), \ 0 \leq k \leq 2^n.$$

The continuity of θ implies

$$\theta(tp + (1-t)q) = t\theta(p) + (1-t)\theta(q), \ 0 \leq t \leq 1.$$

Since p and q are arbitrary points in $[A, A+1]$, we see that θ is an affine function. Noticing $\theta(A) = 0$ we see that $\theta(t) = t - A$ $(A \leq t \leq A+1)$ and hence $\psi(t) = t + A$ $(0 \leq t \leq 1)$, as desired.

For the proof of Claim we make a preliminary construction. For $u < v$ and $\delta \in (0, \frac{u+v}{2})$ we define a function $h_\delta^{u,v} : [u,v] \to [0,1]$ by

$$h_\delta^{u,v}(t) = \begin{cases} \frac{1}{\delta}(t-u) & u \leq t \leq u+\delta, \\ 1 & u+\delta \leq t \leq v-\delta, \\ \frac{1}{\delta}(v-t) & v-\delta \leq t \leq v. \end{cases} \tag{2.10}$$

In what follows we assume that $\delta = \frac{v-u}{K}$ for a large integer $K > 0$. For a positive integer $m > 0$, we define a partition $\Delta_m(u,v)$ of $[u,v]$ into subintervals of equal length δ/m:

(2.11)

$$u = u_0 < u_1 < \cdots < u_m = u + \delta, \qquad u_j = u + \frac{\delta}{m} j,\ 0 \le j \le m,$$
$$u + \delta = w_0 < w_1 < \cdots < w_{m(K-2)} = v - \delta,$$
$$w_j = u + \delta + \frac{\delta}{m} j,\ 0 \le j \le m(K-2),$$
$$v - \delta = v_0 < v_1 < \cdots < u_m = v, \qquad v_j = v - \delta + \frac{\delta}{m} j,\ 0 \le j \le m.$$

We use the partition $\Delta_m(u, v)$ for an approximation of the integral

$$\int_{\psi^{-1}(u)}^{\psi^{-1}(v)} h_\delta^{u,v}(\psi(t))dt$$

by a Riemannian sum $S_m(h_\delta^{u,v})$ as follows: let

$$\begin{array}{ll} a_j = \psi^{-1}(u_j),\ b_j = \psi^{-1}(v_j) & (0 \le j \le m) \\ c_j = \psi^{-1}(w_j) & (0 \le j \le m(K-2)) \end{array}$$

and notice

$$\begin{array}{l} a_0 = \psi^{-1}(u),\ a_m = c_0 = \psi^{-1}(u + \delta), \\ c_{m(K-2)} = b_0 = \psi^{-1}(v - \delta),\ b_m = \psi^{-1}(v). \end{array}$$

Let

$$\begin{aligned} S_m(h_\delta^{u,v}) &= \sum_{j=1}^{m} h_\delta^{u,v}(\psi(a_j))(a_j - a_{j-1}) \\ &+ \sum_{j=1}^{m(K-2)} h_\delta^{u,v}(\psi(c_j))(c_j - c_{j-1}) \\ &+ \sum_{j=1}^{m} h_\delta^{u,v}(\psi(b_j))(b_j - b_{j-1}). \end{aligned} \tag{2.12}$$

Using (2.10) and (2.11), we obtain

$$\begin{aligned} \sum_{j=1}^{m} h_\delta^{u,v}(\psi(a_j))(a_j - a_{j-1}) &= \sum_{j=1}^{m} \frac{1}{\delta}(u_j - u)(a_j - a_{j-1}) \\ &= \sum_{j=1}^{m} \frac{1}{m} j(a_j - a_{j-1}) \\ &= a_m - \frac{1}{m} \sum_{j=0}^{m-1} a_j. \end{aligned} \tag{2.13}$$

Similarly we have

$$\sum_{j=1}^{m} h_\delta^{u,v}(\psi(b_j))(b_j - b_{j-1}) = \frac{1}{m}\sum_{j=0}^{m-1} b_j - b_0, \tag{2.14}$$

and also

$$\begin{aligned}\sum_{j=1}^{m} h_\delta^{u,v}(\psi(c_j))(c_j - c_{j-1}) &= \sum_{j=1}^{m}(c_j - c_{j-1}) \\ &= c_{m(K-2)} - c_0 = b_0 - a_m.\end{aligned} \tag{2.15}$$

Using (2.12), (2.13), (2.14) and (2.15) we obtain

$$S_m(h_\delta^{u,v}) = \frac{1}{m}\sum_{j=0}^{m-1} b_j - \frac{1}{m}\sum_{j=0}^{m-1} a_j. \tag{2.16}$$

Proof of Claim. Fix $p, q, r \in [A, A+1]$ with

$$A \le p < r \le A+1,\ q = \frac{1}{2}(p+r)$$

and let $d = \frac{1}{2}(r-p)$. Take a large positive integer K and let $\delta = d/K$. Use (2.10) to define a continuous function $g_\delta : [A, A+1] \to [-1, 1]$ by

$$g_\delta(t) = \begin{cases} h_\delta^{p,q}(t) & \text{if } p \le t \le q, \\ -h_\delta^{q,r}(t) & \text{if } q \le t \le r, \\ 0 & \text{otherwise.} \end{cases}$$

Since $g_\delta(A) = g_\delta(A+1) = 0$, g_δ naturally extends to a periodic function, denoted by $g_\delta : \mathbb{R} \to [-1, 1]$, of period 1 such that $\int_0^1 g_\delta(t)dt = 0$. Define a C^1-function $f_\delta : [A, A+1] \to [0, 1]$ by

$$f_\delta(t) = \int_A^t g_\delta(s)ds,\ t \in [A, A+1]$$

and again take its natural extension $f_\delta : \mathbb{R} \to [0, 1]$ as a C^1-function of period 1 so that $f_\delta' = g_\delta$. By the hypothesis (2.8) we have

$$\begin{aligned} 0 = \int_0^1 f_\delta'(\psi(t))dt &= \int_0^1 g_\delta(\psi(t))dt \\ &= \int_0^{\psi^{-1}(p)} + \int_{\psi^{-1}(p)}^{\psi^{-1}(q)} + \int_{\psi^{-1}(q)}^{\psi^{-1}(r)} + \int_{\psi^{-1}(r)}^{1} \\ &= \int_{\psi^{-1}(p)}^{\psi^{-1}(q)} h_\delta^{p,q}(\psi(t))dt - \int_{\psi^{-1}(q)}^{\psi^{-1}(r)} h_\delta^{q,r}(\psi(t))dt. \end{aligned}$$

Defining $a = \psi^{-1}(p), b = \psi^{-1}(q), c = \psi^{-1}(r)$, we have

$$\int_a^b h_\delta^{p,q}(\psi(t))dt = \int_b^c h_\delta^{q,r}(\psi(t))dt. \tag{2.17}$$

Fix an $\epsilon > 0$ arbitrarily and take a positive integer m so large that

$$\begin{aligned} &|\textstyle\int_a^b h_\delta^{p,q}(\psi(t))dt - S_m(h_\delta^{p,q})| < \epsilon/2, \\ &|\textstyle\int_b^c h_\delta^{q,r}(\psi(t))dt - S_m(h_\delta^{q,r})| < \epsilon/2, \end{aligned}$$

and hence we have

$$|S_m(h_\delta^{p,q}) - S_m(h_\delta^{q,r})| < \epsilon. \tag{2.18}$$

Let

$$\begin{aligned} &p = p_0 < \cdots < p_j < \cdots < p_m = x_0 < \cdots \\ &\cdots < x_j < \cdots < x_{K(m-2)} = q_0^- < \cdots < q_j^- < \cdots < q_m^- = q, \\ &q = q_0^+ < \cdots < q_j^+ < \cdots q_m^+ = y_0 < \cdots \\ &\cdots < y_j < \cdots < y_{K(m-2)} = r_0 < \cdots < r_j \cdots < r_m = r \end{aligned}$$

be the partitions $\Delta_m(p,q)$ and $\Delta_m(q,r)$ of $[p,q]$ and $[q,r]$ respectively that appear in the definition of $S_m(h_\delta^{p,q})$ and $S_m(h_\delta^{q,r})$ (see (2.11)). By (2.16) we have

$$S_m(h_\delta^{p,q}) = \frac{1}{m}\sum_{j=0}^{m-1} \psi^{-1}(q_j^-) - \frac{1}{m}\sum_{j=0}^{m-1} \psi^{-1}(p_j), \tag{2.19}$$

$$S_m(h_\delta^{q,r}) = \frac{1}{m}\sum_{j=0}^{m-1} \psi^{-1}(r_j) - \frac{1}{m}\sum_{j=0}^{m-1} \psi^{-1}(q_j^+). \tag{2.20}$$

Since ψ^{-1} is monotone increasing, we have $\psi^{-1}(p) \le \psi^{-1}(p_j) \le \psi^{-1}(p+\delta)$ and $\psi^{-1}(q-\delta) \le \psi^{-1}(q_j^-) \le \psi^{-1}(q)$, which imply

$$\psi^{-1}(p) \le \frac{1}{m}\sum_{j=1}^{m} \psi^{-1}(p_j) \le \psi^{-1}(p+\delta), \tag{2.21}$$

$$\psi^{-1}(q-\delta) \le \frac{1}{m}\sum_{j=1}^{m} \psi^{-1}(q_j^-) \le \psi^{-1}(q). \tag{2.22}$$

Similarly we have

$$\psi^{-1}(q) \le \frac{1}{m}\sum_{j=1}^{m} \psi^{-1}(q_j^+) \le \psi^{-1}(q+\delta), \tag{2.23}$$

$$\psi^{-1}(r-\delta) \le \frac{1}{m}\sum_{j=1}^{m} \psi^{-1}(r_j) \le \psi^{-1}(r). \tag{2.24}$$

Using (2.19) - (2.24) and (2.18), we obtain

$$\begin{aligned}
&\psi^{-1}(q-\delta)+\psi^{-1}(q)-\psi^{-1}(p+\delta)-\psi^{-1}(r)\\
&\leq \frac{1}{m}\sum_{j=1}^{m}\psi^{-1}(q_j^-)+\frac{1}{m}\sum_{j=1}^{m}\psi^{-1}(q_j^+)-\\
&\qquad -\frac{1}{m}\sum_{j=1}^{m}\psi^{-1}(p_j)-\frac{1}{m}\sum_{j=1}^{m}\psi^{-1}(r_j)\\
&\leq S_m(h_\delta^{p,q})-S_m(h_\delta^{q,r})\leq \epsilon.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\psi^{-1}(q)+\psi^{-1}(q+\delta)-\psi^{-1}(p)-\psi^{-1}(r-\delta)\\
&\geq S_m(h_\delta^{p,q})-S_m(h_\delta^{q,r})\\
&\geq -\epsilon,
\end{aligned}$$

and hence

$$\begin{aligned}
\psi^{-1}(q-\delta)+\psi^{-1}(q) &\leq \psi^{-1}(p+\delta)+\psi^{-1}(r)+\epsilon,\\
\psi^{-1}(q)+\psi^{-1}(q+\delta) &\geq \psi^{-1}(p)+\psi^{-1}(r-\delta)-\epsilon.
\end{aligned}$$

Taking the limit $K\to\infty$, we have $\delta=d/K\to 0$. Taking into account of the continuity of ψ^{-1} and taking the limit $\epsilon\to 0$ we obtain

$$\psi^{-1}(p)+\psi^{-1}(r)\leq 2\psi^{-1}(q)\leq \psi^{-1}(p)+\psi^{-1}(r),$$

and therefore $\psi^{-1}(p)+\psi^{-1}(r)=2\psi^{-1}(q)$, as desired. This proves Claim and hence completes the proof of the theorem. □

Proof of Proposition 1.3. Let ψ,β be functions as in the proposition and take a lift $\bar\psi:\mathbb{R}\to\mathbb{R}$. Since ψ is a homeomorphism, $\bar\psi$ is a homeomorphism and there exists a constant $\gamma\in\{\pm 1\}$ such that $\bar\psi(t+1)=\bar\psi(t)+\gamma$ $(t\in\mathbb{R})$. Let $\bar\beta=\beta\circ\pi:\mathbb{R}\to\mathbb{C}$. For each C^1-function $f:\mathbb{T}\to\mathbb{C}$, take a function $F:\mathbb{T}\to\mathbb{C}$ as in the hypothesis of the proposition and let $\bar f=f\circ\pi:\mathbb{R}\to\mathbb{C}$ and $\bar F=F\circ\pi:\mathbb{R}\to\mathbb{C}$. Then $\bar f$ is a periodic C^1-function of period 1 and every C^1-function $\mathbb{R}\to\mathbb{C}$ of period 1 is obtained in this way. By the choice of F we have

$$\frac{d}{dt}\bar F(t)=\bar\beta(t)\frac{d\bar f}{dt}(\bar\psi(t))$$

and hence

$$\int_0^1\bar\beta(t)\frac{d\bar f}{dt}(\bar\psi(t))dt=\bar F(1)-\bar F(0)=0.$$

Applying Theorem 1.2 we conclude that $\bar\beta$ and hence β is a constant function and $\bar\psi(t)=\gamma t+c$ for some constant c. Recalling that $\gamma=\pm 1$, we see that ψ is an isometry on $\mathbb{T}$. This proves the proposition. □

The above proof uses the metric feature of the unit circle $\mathbb{T}$ which is locally isometric to the 1-dimensional Euclidean space $\mathbb{R}$. It is not known to the author whether a similar statement holds when the circle is equipped with other metrics. The author recently proved a Banach-Stone type theorem for C^1-function spaces over compact Riemannian manifolds in [9] under a "regularity-preserving-hypothesis" similar to $(*)$. It is not known whether the same result holds without the hypothesis. See also [10] for isometries with respect to the $\|\cdot\|_\Sigma$-type norm.

Acknowledgement. The author is grateful to the referee for helpful comments.

References

[1] F. Botelho and J. Jamison, *Surjective isometries on spaces of differentiable vector-valued functions,*, Studia Math. **192** (2009), 39–50.

[2] M. Cambern, *Isometries of certain Banach algebras,* Studia Math. **25** (1964-1965), 217–225.

[3] R. Fleming and J. Jamison, Isometries on Banach spaces; function spaces, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Math. 129, Boca Raton 2003.

[4] ———, Isometries on Banach spaces vol.2; Vector-valued function spaces, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Math. 138, Boca Raton 2008.

[5] O. Hatori and Sh. Oi, *Isometries on Banach algebras of vector-valued maps,* preprint.

[6] K. Jarosz and V.D. Pathak, *Isometries between function spaces,* Trans. Amer. Math. Soc. **305** (1988), 193–206.

[7] K. Kawamura, *Isometries of vector-valued function spaces preserving the kernel of a linear operator*, Publ. Math. Debrecen **92** (2018), 17–57.

[8] ———, *Perturbations of norms on C^1-function spaces and associated isometry groups*, Top. Proc. **51** (2018), 169–196.

[9] ———, *Banach-Stone type theorem for C^1-function spaces over Riemannian manifolds,* Acta Sci. Math. (Szeged) **83** (2017), 551–591.

[10] ———, *Isometries of function spaces over Riemannian manifolds*, Acta. Sci.Math. (Szeged) **83** (2017), 593–617.

[11] K. Kawamura, H. Koshimizu and T. Miura, *Norms on $C^1([0,1])$ and their isometries,* preprint.

[12] H. Koshimizu, *Linear isometries on spaces of continuously differentiable and Lipschitz continuous functions,* Nihonkai Math. J. **22** (2011), 73–90.

[13] V.D. Pathak, *Isometries of $C^{(n)}[0,1]$,* Pacific J. Math. **94** (1981), 211–222.

[14] N.V. Rao and A.K. Roy, *Linear isometries of some function spaces,* Pacific J. Math. **38** (1971), 177–192.

Institute of Mathematics, University of Tsukuba; Tsukuba, Ibaraki 305-8571 Japan
E-mail address: kawamura@math.tsukuba.ac.jp