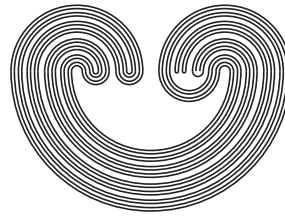


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SPACES WITH NO INFINITE DISCRETE SUBSPACE

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SPACES WITH NO INFINITE DISCRETE SUBSPACE

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ABSTRACT. We show that the spaces with no infinite discrete subspace are exactly those in which every closed set is a finite union of irreducibles. Call them FAC spaces: this generalizes a theorem by Erdős and Tarski (1943), according to which a preordered set has no infinite antichain—the finite antichain, or FAC, property—if and only if all its downwards-closed subsets are finite unions of ideals. All Noetherian spaces are FAC spaces, and we show that sober FAC spaces have a simple order-theoretic description.

1. INTRODUCTION

A preorder is *FAC* (for: has the *finite antichain* property) if and only if all its antichains are finite. An antichain is a subset of pairwise incomparable elements. A well-known result in the theory of preorders states that a preorder is FAC if and only if all its downwards-closed subsets are finite unions of ideals—an ideal is a downwards-closed, directed subset. This was discovered many times [1, 9, 10, 3, 7], and is credited to Erdős and Tarski [2].

The purpose of this paper is to generalize that result to the case of topological spaces. An antichain will simply be a discrete subspace, namely a subspace whose topology is discrete. Downwards-closed subsets will be replaced by closed subsets, and ideals by irreducible closed subsets. We shall retrieve the above preorder-theoretic result by looking at spaces with the Alexandroff topology of the preorder. This is in line with related results, such as the fact that Noetherian spaces are a topological generalization of well-quasi-orders [4].

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2. MAIN RESULTS

The main result of this paper is the following, and will be proved in Section 3.

Theorem 2.1. *For a topological space X , the following are equivalent:*

- (1) *X has no infinite discrete subspace;*
- (2) *every closed subset of X is a finite union of irreducible closed subsets.*

We recall that a closed subset C of X is *irreducible* if and only if it is non-empty, and for all closed subsets C_1, C_2 of X , if C is included in $C_1 \cup C_2$ then C is included in C_1 or in C_2 . Equivalently, C is irreducible (closed) if and only if it is non-empty and, for any two open subsets U and V of X , if C intersects both U and V then it intersects $U \cap V$.

Perhaps the closest result in the literature, except for the preorder-theoretic results mentioned in the introduction, is due to A. H. Stone [11, Theorem 2]: a space X has no infinite discrete subspace if and only if every open cover of every subspace A of X has a finite subfamily whose union is dense in A , if and only if every continuous real-valued function on every subspace of X is bounded. Another one is due to Milner and Pouzet [8], and we shall return to it in Remark 3.4.

Definition 2.2. A space satisfying any of the equivalent properties of Theorem 2.1 is called a *FAC space*.

The closures of points are always irreducible. A space is *sober* if and only if every irreducible closed subset is the closure of a unique point. The *sobrification* $\mathcal{S}(X)$ of X can be defined as the set of all irreducible closed subsets of X , with the topology whose open subsets are $\diamond U = \{C \in \mathcal{S}(X) \mid C \cap U \neq \emptyset\}$, where U ranges over the open subsets of X (see [5, Section 8.2.3] for example).

The map $U \mapsto \diamond U$ defines an isomorphism between the lattice of open subsets of X and of $\mathcal{S}(X)$. It follows that X and $\mathcal{S}(X)$ have isomorphic lattices of closed subsets. Since Item 2 of Theorem 2.1 only depends on the lattice of closed subsets, we immediately obtain:

Fact 2.3. *A space X is FAC if and only if its sobrification $\mathcal{S}(X)$ is.*

Hence it is legitimate to study *sober* FAC spaces. This is best done by comparing the situation with Noetherian spaces (see Section 9.7 of [5] for an introduction to Noetherian spaces).

A Noetherian space is a space where every open subset is compact. (A subset is *compact* if and only if all its open covers have a finite subcover; we do not require Hausdorffness.) Equivalently, a space is Noetherian if and only if every monotone chain of open subsets stabilizes, if and only if the inclusion ordering on the lattice of closed subsets is well-founded.

It is well-known that, in a Noetherian space, all closed subsets are finite unions of irreducibles. A simple proof goes as follows. Imagine there were a closed subset C that one cannot write as a finite union of irreducibles. Since inclusion is well-founded on closed sets, we can require C to be minimal. C is not a finite union of irreducibles, hence is not empty and not irreducible itself. Hence there are two closed sets C_1 and C_2 such that $C \subseteq C_1 \cup C_2$ but C is included neither in C_1 nor in C_2 . Then $C \cap C_1$ and $C \cap C_2$, being strictly smaller than C , are finite unions of irreducibles. It follows that $C = (C \cap C_1) \cup (C \cap C_2)$ is also a finite union of irreducibles—contradiction. That means that Item 2 of Theorem 2.1 is true of all Noetherian spaces, whence:

Fact 2.4. *Every Noetherian space is FAC.*

That can also be obtained by using Theorem 1 of [11], which says that a space is Noetherian if and only if it has no weakly discrete infinite subspace, and using Item 1 of Theorem 2.1. (A space is *weakly discrete* if and only if every point has a finite neighborhood. Clearly, every discrete space is weakly discrete.)

The converse of Fact 2.4 fails. For example, consider the real line \mathbb{R} , with the Scott topology, whose non-trivial opens are the half-open intervals $]r, +\infty[$, $r \in \mathbb{R}$. That is FAC, because all non-empty closed sets are of the form $] - \infty, r]$ with $r \in \mathbb{R}$, or the whole of \mathbb{R} , and all of them are irreducible. However, \mathbb{R} with the Scott topology is not Noetherian, since no open set $]r, +\infty[$ is compact.

Another example comes from the theory of preorders. Every topological space X comes with a so-called *specialization* preorder \leq , defined by $x \leq y$ if and only if every open neighborhood of x contains y , if and only if x is in the closure of y . X is *Alexandroff* if and only if the open subsets of X are the upwards-closed subsets with respect to some preordering (which must be \leq). The full subcategory of the category of topological spaces consisting of Alexandroff spaces is equivalent to the category of preorders. Hence, equating the two categories, a Noetherian Alexandroff space is the same thing as a well-quasi-order [5, Proposition 9.7.17]. The discrete subspaces of an Alexandroff space are exactly its antichains, hence:

Fact 2.5. *The FAC Alexandroff spaces are exactly the preorders with no infinite antichain.*

Fact 2.4, once specialized to Alexandroff spaces, yields the following trivial fact: in a well-quasi-ordered set, every antichain is finite. The converse fails, take \mathbb{Z} with its usual ordering for example.

In an Alexandroff space, a closed subspace is the same as a downwards-closed subset, and an irreducible closed subset is the same thing as an

ideal, namely a downwards-closed directed subset (see Fact 8.2.49 in [5], or [6].) Hence, specialized to Alexandroff spaces, Theorem 2.1 is a restatement of the Erdős-Tarski result mentioned in the introduction.

There is a purely order-theoretic characterization of sober Noetherian spaces (see Theorem 9.7.12 of [5]): they are exactly the posets X whose ordering \leq is well-founded and satisfies properties T and W, in the upper topology. Property T states that X is *finitary*, where a finitary subset is defined as the downwards-closure $\downarrow E$ of a finite set E . Property W states that, for any two points x and y , $\downarrow x \cap \downarrow y$ is finitary. (We write $\downarrow x$ for $\downarrow\{x\}$. This coincides with the closure of x .) The upper topology is the coarsest topology that has \leq as specialization ordering. Alternatively, this is the topology whose closed sets are the intersections of finitary subsets.

Theorem 2.6. *The sober FAC spaces are exactly the posets X in which every intersection of finitary subsets is finitary, in the upper topology. Every closed set is finitary.*

Properties T and W are equivalent to the statement that *finite* intersections of finitary subsets are finitary. Together with well-foundedness, they imply Noetherianness, hence FACness, but as we have seen, the fact that intersections of finitary subsets are finitary is strictly weaker than requiring that \leq is well-founded.

Proof. Let X be a sober FAC space. Since X is sober, its only irreducible closed subsets are the closures of points, and those are exactly the sets of the form $\downarrow x$, $x \in X$, where downward closure is taken with respect to the specialization ordering \leq of X . Item 2 of Theorem 2.1 then implies that every closed set is finitary. Since intersections of closed sets are closed, we conclude.

Conversely, let X be a set with an ordering \leq in which intersections of finitary subsets are finitary. Since finite unions of finitary subsets are finitary, the finitary subsets form the closed sets of a topology, and that must be the upper topology. By Item 2 of Theorem 2.1, and since every set $\downarrow x$ is irreducible closed, X is FAC in that topology.

Finally, we show that X is sober. Let $C = \downarrow E$ be closed, where E is finite. We may assume that E is an antichain, by removing elements below another one, one by one. If C is irreducible, then it is non-empty, so E contains some element x . Then C is included in the union of the closed sets $\downarrow x$ and $\downarrow(E \setminus \{x\})$, hence must be contained in one of them. It cannot be contained in the second one, since $x \in C$ and E is an antichain. It follows that C is included in $\downarrow x$, hence is equal to it. Recall that $\downarrow x$ is the closure of x . We have shown that every irreducible closed set is the closure of some point. This point must be unique because \leq is an ordering, not just a preordering. \square

We finish with the following observation. That generalizes the fact that the Hausdorff Noetherian spaces are the finite discrete spaces.

Proposition 2.7. *Given a FAC space X , the following are equivalent:*

- (1) X is sober and T_1 ;
- (2) X is Hausdorff;
- (3) X is a KC-space, namely one where each compact subset is closed;
- (4) X is finite and discrete.

Proof. (1) \Rightarrow (4). In a sober space, every irreducible closed set is the closure of a point, and in a T_1 space, all points are closed. Hence the finite unions of irreducible closed subsets are the finite subsets. That applies to every closed subset, in particular to X itself. Finally, every finite T_1 space is discrete.

(4) \Rightarrow (2) \Rightarrow (1), (4) \Rightarrow (3): obvious.

(3) \Rightarrow (4). Every KC-space is T_1 , since one-element sets are compact, hence closed. Then (3) \Rightarrow (4) follows from the well-known fact that every infinite KC-space contains an infinite discrete subspace. We give a proof below, for completeness. Hence, assuming that X is a FAC KC-space, X must be finite, and since every KC-space is T_1 , its topology is discrete.

As promised, let us show that every infinite KC-space X contains an infinite discrete subspace. We first observe that X cannot be Noetherian. Otherwise all its subsets would be compact [5, Proposition 9.7.7], hence closed since X is KC. Then the topology of X would be discrete. The finite subsets of X would then form an open cover of X . Since X is Noetherian, hence compact, that would have a finite subcover, contradicting the fact that X is infinite.

Since X is not Noetherian, the inclusion ordering is not well-founded on the lattice of closed subsets of X . We can therefore find a strictly decreasing sequence $C_0 \supset C_1 \supset \cdots \supset C_n \supset \cdots$ of closed subsets of X . For each $n \in \mathbb{N}$, pick a point x_n in $C_n \setminus C_{n+1}$, and let $A = \{x_n \mid n \in \mathbb{N}\}$. Clearly, A is infinite. Since C_{n+1} is closed in X , and X is T_1 , $C'_n = \{x_1, \dots, x_{n-1}\} \cup C_{n+1}$ is closed, and its complement U_n is open. However, $U_n \cap A$ contains just x_n . This shows that the subspace topology on A is discrete. \square

3. PROOF OF THE MAIN THEOREM

The proof of Theorem 2.1 is inspired from Lawson, Mislove and Priestley's proof of the Erdős-Tarski theorem [7, Proposition 4]. In fact, this *is* their proof, with all preorder-theoretic notions replaced by suitable topological notions.

We first need the following lemma. We write $cl(A)$ for the closure of A in X .

Lemma 3.1. *Let A be a subset of a topological space X . The subspace topology on A is discrete if and only if for every $x \in A$, x is not in the closure $cl(A \setminus \{x\})$.*

Proof. If A is discrete, then for every $x \in A$, $\{x\}$ is open in A , so $A \setminus \{x\}$ is closed in A . That means that $A \setminus \{x\}$ occurs as the intersection of some closed subset of X with A , and the smallest one is $cl(A \setminus \{x\})$. That cannot contain x since its intersection with A is $A \setminus \{x\}$.

Conversely, for every $x \in A$, if x is not in the closure $cl(A \setminus \{x\})$, then it lies in some open subset U of X that does not intersect $A \setminus \{x\}$. Then $U \cap A$ is open in A , but only contains the point x from A . It follows that every one-element subset of A is open in A , whence A is discrete. \square

Given a closure operator φ on a set X (namely, φ is monotonic, $A \subseteq \varphi(A)$ and $\varphi(\varphi(A)) \subseteq \varphi(A)$ for every $A \subseteq X$), a subset A is called φ -independent by Milner and Pouzet [8] if and only if, for every $x \in A$, x is not in $\varphi(A \setminus \{x\})$. One may therefore restate Lemma 3.1 by saying that A is discrete if and only if it is cl -independent. Note that cl is not just a closure operator, but a *topological* closure operator, namely one which commutes with finite unions. This will be important in two places in the proof below.

Next, we prove a useful lemma, generalizing Lemma 3 of [7] to the topological case. For a family of sets M , we write $\bigcup M$ for the union of all elements of M .

Lemma 3.2. *Let X be a topological space, and M be an infinite family of irreducible closed subsets of X , with the property that for every infinite subfamily N of M , $cl(\bigcup N) = cl(\bigcup M)$. Then $cl(\bigcup M)$ is irreducible closed.*

Proof. We shall repeatedly use the fact that if an open set intersects the closure of a set E , then it intersects E . Let U and V be two open subsets of X that intersect $cl(\bigcup M)$. It suffices to show that $cl(\bigcup M)$ intersects $U \cap V$. Let M' be the subfamily of those $I \in M$ such that U intersects I . M' is non-empty: since U intersects $cl(\bigcup M)$, it intersects $\bigcup M$, hence some $I \in M$.

If M' were finite, then $M \setminus M'$ would be infinite, so by assumption $cl(\bigcup(M \setminus M'))$ would be equal to $cl(\bigcup M)$. However, U intersects the latter, so it would intersect the former, hence also $\bigcup(M \setminus M')$, hence also some $I \in M \setminus M'$. That would contradict the definition of M' . Therefore M' is infinite.

Since M' is infinite, we use the assumption again, and so $cl(\bigcup M') = cl(\bigcup M)$. We now use the fact that V , not just U , intersects $cl(\bigcup M)$.

Then V must also intersect $cl(\bigcup M')$, hence $\bigcup M'$, and therefore it must intersect some $I \in M'$. By definition of M' , U also intersects I . Since I is irreducible closed, it must therefore intersect $U \cap V$. Since I is included in $cl(\bigcup M)$, the latter must also intersect $U \cap V$. \square

3.1. Proving the hard implication (2) \Rightarrow (1). We prove the contrapositive, namely: we assume that there is a closed subset C of X that cannot be written as a finite union of irreducible closed subsets, and we shall build an infinite discrete subspace of X .

Lemma 3.3. *(Assuming that C is closed in X .) Every irreducible closed subset of X included in C is contained in some maximal irreducible closed subset of X included in C .*

Proof. Recall that $\mathcal{S}(X)$ is sober, and that every sober space is a dcpo in its specialization ordering [5, Proposition 8.2.34]. A dcpo is a poset in which every directed subset has a supremum. The supremum of a directed family $(I_k)_{k \in K}$ of irreducible closed subsets is the closure of their union. To show that, it suffices to show that $cl(\bigcup_{k \in K} I_k)$ is irreducible: if it intersects two open subsets U and V , then U intersects some I_k , V intersects some $I_{k'}$, and by directedness we may assume $k = k'$; then I_k intersects both U and V , hence $U \cap V$ since I_k is irreducible; this implies that the larger set $cl(\bigcup_{k \in K} I_k)$ also intersects $U \cap V$.

In particular, the supremum of a directed family of irreducible closed subsets included in C is again included in C . It follows that the poset of all irreducible closed subsets of X included in C is a dcpo, in particular an inductive poset. We apply Zorn's Lemma and conclude. \square

Alternatively, one may show that the irreducible closed subsets of X included in C are exactly the irreducible closed subsets of the subspace C . (For that, the fact that C is closed is important.) We obtain a slightly different proof by reasoning in $\mathcal{S}(C)$, also a dcpo, hence also an inductive poset, instead of $\mathcal{S}(X)$.

We now have everything to produce the argument, imitating the proof of [7, Proposition 4].

Let M_0 be the collection of all maximal irreducible closed subsets of X included in C . By Lemma 3.3, since $\downarrow x$ is irreducible closed in X for every $x \in C$, and also included in C , we obtain that x is a member of some maximal irreducible closed subset of X included in C . Therefore $C = \bigcup M_0$, in particular, $C = cl(\bigcup M_0)$.

Since C is not a finite union of irreducible closed subsets, it is in particular not irreducible. The contrapositive of Lemma 3.2 then implies the existence of an infinite subset M_1 of M_0 such that $cl(\bigcup M_1)$ is strictly included in $cl(\bigcup M_0)$.

If $cl(\bigcup M_1)$ were an irreducible closed subset, and recalling that it is included in C , it would be included in some maximal irreducible closed subset of X included in C by Lemma 3.3, namely in some member of M_0 . In particular, $\bigcup M_1$ would be included in some member of M_0 . Since all elements of M_0 are maximal, hence pairwise incomparable, that would imply that M_1 contains only one element. That is impossible, since M_1 is infinite.

Hence we can reapply Lemma 3.2: there is an infinite subset M_2 of M_1 such that $cl(\bigcup M_2)$ is strictly included in $cl(\bigcup M_1)$.

Again $cl(\bigcup M_2)$ cannot be irreducible, hence we can find an infinite subset M_3 of M_2 such that $cl(\bigcup M_3)$ is strictly included in $cl(\bigcup M_2)$. Let us proceed infinitely that way. To make that clear, we have an infinite, antitone sequence of infinite subsets $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ such that $cl(\bigcup M_0) \supset cl(\bigcup M_1) \supset cl(\bigcup M_2) \supset \dots \supset cl(\bigcup M_n) \supset \dots$ (all inclusions here are strict; from which we can conclude that the inclusions $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$ are strict as well).

We can therefore pick an irreducible closed subset I_n in M_n that is not contained in $cl(\bigcup M_{n+1})$, for each natural number n . Indeed, if every element I of M_n were in $cl(\bigcup M_{n+1})$, then $\bigcup M_n$ would be included in $cl(\bigcup M_{n+1})$, so $cl(\bigcup M_n)$ would be included in $cl(\bigcup M_{n+1})$ as well, contradicting $cl(\bigcup M_n) \supset cl(\bigcup M_{n+1})$.

We now claim that I_n cannot be included in $cl(\bigcup \{I_m \mid m \neq n\})$. Assume the contrary. Since for every $m > n$, I_m is in $M_m \subseteq M_{n+1}$, I_n would be included in $cl(I_0 \cup I_1 \cup I_2 \cup \dots \cup I_{n-1} \cup (\bigcup M_{n+1}))$. However, closure commutes with finite unions—this is where we require a *topological* closure operator, and where a mere closure operator φ as in [8] will not suffice. Therefore I_n would be included in $cl(I_0) \cup cl(I_1) \cup cl(I_2) \cup \dots \cup cl(I_{n-1}) \cup cl(\bigcup M_{n+1})$, that is, in $I_0 \cup I_1 \cup I_2 \cup \dots \cup I_{n-1} \cup cl(\bigcup M_{n+1})$. Since I_n is irreducible, it must be included in one term of the union. It cannot be included in $cl(\bigcup M_{n+1})$, by definition. And it cannot be included in any I_k with $k < n$, since all those irreducible closed subsets, being maximal, are pairwise incomparable.

Finally, since I_n is not included in $cl(\bigcup \{I_m \mid m \neq n\})$, we can pick a point x_n of I_n that is not in $cl(\bigcup \{I_m \mid m \neq n\})$, for each $n \in \mathbb{N}$. Since x_m is in I_m , for every $m \in \mathbb{N}$, $cl(\bigcup \{I_m \mid m \neq n\})$ contains $cl\{x_m \mid m \neq n\}$. It follows that x_n is not in $cl\{x_m \mid m \neq n\}$. Lemma 3.1 states that the family $\{x_n \mid n \in \mathbb{N}\}$ is a (necessarily infinite) discrete subspace.

Remark 3.4. The implication (2) \Rightarrow (1) can also be proved by using a result of Milner and Pouzet [8]. Stating it bluntly would not be helpful. Instead, we draw a few consequences of (2), walking quietly towards their result. Let C be a closed subset of X . By (2), C is a union of finitely many irreducible closed subsets I_1, \dots, I_n , and choosing them maximal in C ,

one may assume that they are pairwise incomparable. Let $A_i = I_i \setminus \bigcup_{j \neq i} I_j$ (a non-empty set), and N_i be the family of all subsets E of A_i whose closure is different from I_i . N_i contains the empty set, hence is non-empty, and is clearly downwards-closed. For every pair $E, E' \in N_i$, $E \cup E'$ is also in N_i : $cl(E \cup E') = cl(E) \cup cl(E')$ cannot be equal to I_i , since I_i is irreducible. Finally, N_i is proper, meaning that $A_i \notin N_i$. Hence N_i is a proper ideal of $\mathbb{P}(A_i)$.

For every subset E of C , one can see that $cl(E) = C$ if and only if no $E \cap A_i$ belongs to N_i . In one direction, if no $E \cap A_i$ is in N_i , then $cl(E) = cl(\bigcup_{i=1}^n E \cap A_i) = \bigcup_{i=1}^n cl(E \cap A_i) = \bigcup_{i=1}^n I_i = C$. In the other direction, assume $cl(E) = C$ and $E \cap A_i \in N_i$ for some i . Then $cl(E \cap A_i) \neq I_i$, hence $cl(E \cap A_i)$ is a proper subset of I_i . Since $I_i \subseteq C = cl(E) = cl(E \cap A_i) \cup cl(E \cap \bigcup_{j \neq i} A_j) \subseteq cl(E \cap A_i) \cup \bigcup_{j \neq i} I_j$, by irreducibility we would obtain that either $I_i \subseteq cl(E \cap A_i)$, or $I_i \subseteq \bigcup_{j \neq i} I_j$. Both cases are impossible.

Hence, for φ equal to the closure operator cl , we have obtained that (2) implies: for each closed subset C of X , $(*)$ there are finitely many, pairwise disjoint subsets A_1, \dots, A_n and proper ideals N_i of $\mathbb{P}(A_i)$, $1 \leq i \leq n$, such that for every subset E of $\bigcup_{i=1}^n A_i$, $\varphi(E) = C$ if and only if no $E \cap A_i$ belongs to N_i . Theorem 1.2 of [8] states that, for a closure operator φ (not necessarily topological), $(*)$ is equivalent to the fact that C contains no infinite φ -independent set. \square

3.2. Proving the easier implication $(1) \Rightarrow (2)$. We assume an infinite discrete subspace A of X . We shall simply show that $cl(A)$ cannot be written as a finite union of irreducible closed subsets of X . Assume one could write $cl(A)$ as the finite union $I_0 \cup I_1 \cup I_2 \cup \dots \cup I_{n-1}$, where each I_k is irreducible closed in X .

For each k , if I_k contains some point x of A , then, writing $cl(A)$ as $cl(\{x\} \cup (A \setminus \{x\})) = \downarrow x \cup cl(A \setminus \{x\})$ (recall—for the second and final time—that closures commute with finite unions), and remembering that I_k is irreducible and included in $cl(A)$, we obtain that I_k is included in $\downarrow x$ or in $cl(A \setminus \{x\})$. The latter is impossible if I_k contains x , since A is discrete: indeed, Lemma 3.1 states that x is not in $cl(A \setminus \{x\})$. Therefore $I_k = \downarrow x$.

That implies that each I_k can contain at most one point x from A . Explicitly, if it contained two distinct points x and y of A , then I_k would be equal to $\downarrow x$, and also to $\downarrow y$. If we had assumed X to be T_0 , we could conclude $x = y$, but we haven't made that assumption, and we must therefore work slightly harder. Since A is discrete, x is not in $cl(A \setminus \{x\})$, hence neither in the smaller subset $cl(\{y\}) = \downarrow y$. That shows that x is not less than or equal to y . That contradicts $\downarrow x = \downarrow y$.

We now have infinitely many points in A , hence in $cl(A) = I_0 \cup I_1 \cup I_2 \cup \dots \cup I_{n-1}$, but each I_k can contain at most one point from A . That is impossible, by the pigeonhole principle.

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