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by

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SPACES WITH NO INFINITE DISCRETE SUBSPACE

JEAN GOUBAULT-LARRECQ

Abstract. We show that the spaces with no infinite discrete subspace are exactly those in which every closed set is a finite union of irreducibles. Call them FAC spaces: this generalizes a theorem by Erdős and Tarski (1943), according to which a preordered set has no infinite antichain—the finite antichain, or FAC, property—if and only if all its downwards-closed subsets are finite unions of ideals. All Noetherian spaces are FAC spaces, and we show that sober FAC spaces have a simple order-theoretic description.

1. Introduction

A preorder is *FAC* (for: has the *finite antichain* property) if and only if all its antichains are finite. An antichain is a subset of pairwise incomparable elements. A well-known result in the theory of preorders states that a preorder is FAC if and only if all its downwards-closed subsets are finite unions of ideals—an ideal is a downwards-closed, directed subset. This was discovered many times [1, 9, 10, 3, 7], and is credited to Erdős and Tarski [2].

The purpose of this paper is to generalize that result to the case of topological spaces. An antichain will simply be a discrete subspace, namely a subspace whose topology is discrete. Downwards-closed subsets will be replaced by closed subsets, and ideals by irreducible closed subsets. We shall retrieve the above preorder-theoretic result by looking at spaces with the Alexandroff topology of the preorder. This is in line with related results, such as the fact that Noetherian spaces are a topological generalization of well-quasi-orders [4].

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Key words and phrases. Discrete subspace, irreducible closed subset, finite antichain, ideal, sober space, Noetherian space.

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