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OPEN INDUCED MAPPINGS, AN EXAMPLE

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Abstract. For a Hausdorff space X, let $S_c(X)$ denote the hyperspace of all nontrivial convergent sequences in X, endowed with the Vietoris topology. Given a mapping between Hausdorff spaces $f : X \to Y$, the induced mapping $S_c(f) : S_c(X) \to S_c(Y)$ is defined by $S_c(f)(A) = f(A)$ (the image of A under f). In this paper we show an example of a strong light open mapping between Hausdorff Fréchet-Urysohn spaces $f : X \to Y$ such that $S_c(f)$ is not open. This answers a question by David Maya, Patricia Pellicer-Covarrubias, and Roberto Pichardo-Mendoza.

1. Introduction

The symbol $\mathbb{N}$ denotes the set of positive integers. A *mapping* is a continuous function.

All spaces in this paper are Hausdorff spaces. Given a space X, let

$$CL(X) = \{A \subset X : A \text{ is closed and nonempty}\}.$$

Given $n \in \mathbb{N}$ and open subsets $U_1, \ldots, U_n$ in X, let

$$\langle U_1, \ldots, U_n \rangle = \{A \in CL(X) : A \subset U_1 \cup \ldots \cup U_n \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, n\}\}.$$

We consider $CL(X)$ with the Vietoris topology which has as a basis the family of all sets of the form $\langle U_1, \ldots, U_n \rangle$.

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A *nontrivial convergent sequence in* X is a countably infinite subset S of X for which there is $x \in S$ such that for each open subset of X with $x \in U$, we have $S \setminus U$ is finite. In this case we say that S *converges to* x and we write $\lim S = x$.

We consider the hyperspace of nontrivial convergent sequences in X defined by

$$S_c(X) = \{S \in CL(X) : S \text{ is a nontrivial convergent sequence in } X\}.$$

Recall that X is a *Fréchet-Urysohn* space if for each $A \subset X$ and each $p \in \mathrm{cl}_X(A)$, there exists a sequence $\{p_n\}_{n=1}^{\infty}$ in A such that $\lim_{n\to\infty} p_n = p$. The space X is *sequential* if for each $A \subset X$ with $\mathrm{cl}_X(A) \neq A$, there exists a sequence $\{p_n\}_{n=1}^{\infty}$ in A and a point $p \in X \setminus A$ such that $\lim_{n\to\infty} p_n = p$.

A mapping between Hausdorff spaces $f : X \to Y$ is a *strong light mapping* if for each $y \in Y$, $f^{-1}(y)$ is a discrete subspace of X.

Given a strong light mapping $f : X \to Y$ one can consider the *induced function* $S_c(f) : S_c(X) \to S_c(Y)$ given by $S_c(f)(S) = f(S)$ (the image of X under f). It is easy to show that $S_c(f)$ is continuous (see sections 2 and 3 in [5]).

The hyperspace of nontrivial sequences has been recently studied in [2], [5], [6], and [7]. In [5], the authors introduced the study of induced mappings $S_c(f)$.

Given a family $\mathcal{M}$ of mappings, the authors of [5] consider the possible implications between the conditions:

(a) $f \in \mathcal{M}$, and

(b) $S_c(f) \in \mathcal{M}$.

They studied implications for the following families of mappings: open, almost-open, quotient, monotone, surjective, finite-to-1, homeomorphism, closed, strong light, light, sequence-covering and 1-sequence covering.

In particular, they found [5, Theorem 4.20] equivalent conditions on the mapping f in order that the mapping $S_c(f)$ is open. As a consequence they showed [5, Corollary 4.21] that if X is sequential, the set of points in X that are limit of a sequence is dense in X, and the mapping $S_c(f) : S_c(X) \to S_c(Y)$ is open, then f is open. They gave conditions [5, Theorem 4.23 and Corollary 4.24] under which the openness of f implies the openness of $S_c(f)$. They also asked the following [5, Question 5.9]: Is there a strong light and open mapping f such that $S_c(f)$ is not open? (where X is a sequential Hausdorff space such that it contains at least one convergent sequence).

The aim of this paper is to answer this question by showing two Fréchet-Urysohn Hausdorff spaces X and Y and a strong light open mapping $f : X \to Y$ such that $S_c(f)$ is not open.

For the openness of induced mappings in other hyperspaces, the reader could be interested in the papers [1], [3], and [7].

2. An Auxiliary Space

Lemma 2.1. *There exists a family $\mathcal{F}$ of subsets of $\mathbb{N}$ such that*

(1) $\bigcup \mathcal{F} = \mathbb{N}$,
(2) $\mathbb{N}$ *is not a finite union of elements of* $\mathcal{F}$,
(3) *each infinite subset of* $\mathbb{N}$ *contains an element of* $\mathcal{F}$,
(4) *each element of* $\mathcal{F}$ *is infinite.*

Proof. Let $\mathbb{Q}$ be set of rational numbers in the real line. Since $\mathbb{Q} \times \mathbb{Q}$ is countable, taking all the sets of one of the forms $([z, z+1) \times \mathbb{Q}) \cap (\mathbb{Q} \times \mathbb{Q})$ and $(\mathbb{Q} \times [z, z+1)) \cap (\mathbb{Q} \times \mathbb{Q})$ (for integer numbers z) it is possible to find an infinite family $\mathcal{F}_0$ of subsets of $\mathbb{N}$, satisfying (1), (2), and (4) and the property:
(5) each element of $\mathbb{N}$ belongs to two distinct elements of $\mathcal{F}_0$.
Define

$$\mathcal{Z} = \{\mathcal{G} : \mathcal{G} \text{ is a family of subsets of } \mathbb{N} \text{ satisfying (4), } \mathcal{F}_0 \subset \mathcal{G} \text{ and no element } A \text{ of } \mathcal{G} \text{ is contained in a finite union of elements of } \mathcal{G} \setminus \{A\}\}.$$

We consider $\mathcal{Z}$ with the order given by the inclusion.
A simple application of Zorn's Lemma implies that $\mathcal{Z}$ contains a maximal element $\mathcal{M}$.
Define

$$\mathcal{F} = \{A \subset \mathbb{N} : A \text{ is infinite and } A \text{ is contained in an element of } \mathcal{M}\}.$$

Since $\mathcal{F}_0 \subset \mathcal{M} \subset \mathcal{F}$, $\mathcal{F}$ satisfies (1). By definition, $\mathcal{F}$ satisfies (4).
If $\mathcal{F}$ does not satisfy (2), by the definition of $\mathcal{F}$, there exist $n \in \mathbb{N}$ and $M_1, \ldots, M_n \in \mathcal{M}$ such that $\mathbb{N} = M_1 \cup \ldots \cup M_n$, since $\mathcal{F}_0 \subset \mathcal{M}$, $\mathcal{M}$ is infinite, so there exists $M \in \mathcal{M} \setminus \{M_1, \ldots, M_n\}$ and $M \subset M_1 \cup \ldots \cup M_n$. This is a contradiction since $\mathcal{M} \in \mathcal{Z}$. Hence, $\mathcal{F}$ satisfies (2).
In order to prove that $\mathcal{F}$ satisfies (3), take an infinite subset A of $\mathbb{N}$. In the case that $A \in \mathcal{M} \subset \mathcal{F}$, we are done. Suppose then that $A \notin \mathcal{M}$. By the maximality of $\mathcal{M}$, the family $\mathcal{M}_0 = \mathcal{M} \cup \{A\}$ does not belong to $\mathcal{Z}$, so there are $m \in \mathbb{N}$ and pairwise distinct elements $L_1, \ldots, L_{m+1} \in \mathcal{M}_0$ such that $L_{m+1} \subset L_1 \cup \ldots \cup L_m$.
Since $\mathcal{M} \in \mathcal{Z}$, there is $i \in \{1, \ldots, m+1\}$ such that $A = L_i$.
In the case that $i = m+1$, we have that $A \subset L_1 \cup \ldots \cup L_m$ and $\{L_1, \ldots, L_m\} \subset \mathcal{M}$. Since A is infinite, there exists $j \in \{1, \ldots, m\}$ such that $A \cap L_j$ is infinite. By definition, $A \cap L_j \in \mathcal{F}$ and we are done.

Now, we consider the case that $i \leq m$. Without loss of generality, we suppose that $i = 1$. Notice that the set $B = L_{m+1} \setminus (L_2 \cup \ldots \cup L_m)$ is contained in A. If B is finite, since $\mathcal{F}_0 \subset \mathcal{M}$, by (5), B can be covered by a finite number $C_1, \ldots, C_k$ of elements of $\mathcal{M} \setminus \{L_{m+1}\}$ and $L_{m+1} \subset C_1 \cup \ldots \cup C_k \cup L_2 \cup \ldots \cup L_m$, which contradicts the fact that $\mathcal{M} \in \mathcal{Z}$. Thus, B is infinite. Therefore, $B \in \mathcal{F}$ and $B \subset A$, and we are done.

This ends the proof that $\mathcal{F}$ has the required properties. □

Consider $\mathbb{N}$ with the discrete topology and let $\mathbb{N}_\infty = \mathbb{N} \cup \{p_\infty\}$ be the one-point compactification of $\mathbb{N}$ ($p_\infty \notin \mathbb{N}$).

Let $\mathcal{F}$ be a family satisfying properties (1)–(4) in Lemma 2.1.

For each $F \in \mathcal{F}$, let $T_F = F \times \{F\} \subset \mathbb{N} \times \mathcal{F}$.

Let $\pi_{\mathbb{N}} : \mathbb{N} \times \mathcal{F} \to \mathbb{N}$ be the projection.

Define $Z_0 = \bigcup\{T_F \subset \mathbb{N} \times \mathcal{F} : F \in \mathcal{F}\}$. Take a point $p_0 \notin Z_0$.

Set $Z = Z_0 \cup \{p_0\}$.

Let $\mathcal{G} = \{B \subset Z_0 : B \cap T_F$ is finite for each $F \in \mathcal{F}\}$, $\mathcal{B}_0 = \{Z \setminus B : B \in \mathcal{G}\}$, and $\mathcal{B} = \mathcal{B}_0 \cup \{\{p\} : p \in Z_0\} \cup \{\emptyset\}$. Notice that $p_0 \in U$ for each $U \in \mathcal{B}_0$ and $\mathcal{B}_0$ and $\mathcal{B}$ are closed under finite intersections.

We endow Z with the topology τ that has $\mathcal{B}$ as a basis.

Given $p \in Z_0$, $\{p\} \in \mathcal{G}$, $Z \setminus \{p\}$ is open and $\{p\} \in \mathcal{B} \subset \tau$. This implies that Z is a Hausdorff space.

Claim 2.2. *Z is Fréchet-Urysohn.*

Proof. Let $A \subset Z$ and $p \in \mathrm{cl}_X(A) \setminus A$. Since for each point $q \in Z_0$, $\{q\}$ is open in Z, we have $p \notin Z_0$. So, $p = p_0$.

If $A \cap T_F$ is finite for each $F \in \mathcal{F}$, then $A \in \mathcal{G}$, $Z \setminus A \in \mathcal{B}_0 \subset \tau$, and A is closed, which is absurd. Thus, there exists $F \in \mathcal{F}$ such that the set $B = A \cap T_F$ is infinite. Since $T_F = F \times \{F\}$, B is of the form $B = C \times \{F\}$, where C is infinite and $C \subset \mathbb{N}$. Then there exists a sequence $n_1 < n_2 < \ldots$ in $\mathbb{N}$ such that $C = \{n_1, n_2, \ldots\}$. Notice that $\{(n_k, F) : k \in \mathbb{N}\} \subset A$.

We claim that $\lim_{k\to\infty}(n_k, F) = p_0$. Let $U \in \mathcal{B}$ be such that $p_0 \in U$. Then $U = Z \setminus D$ for some $D \in \mathcal{G}$. So $D \cap T_F$ is finite. In particular, $D \cap \{(n_k, F) : k \in \mathbb{N}\}$ is finite. So there exists $K \in \mathbb{N}$ such that $(n_k, F) \notin D$ for each $k \geq K$. Hence, $(n_k, F) \in U$ for each $k \geq K$. Therefore, Z is Fréchet-Urysohn. □

With a similar argument as in the last paragraph it can be proved that if $F \in \mathcal{F}$, then $\lim T_F = p_0$.

Let $g : Z \to \mathbb{N}_\infty$ be the function defined as

$$g(p) = \begin{cases} \pi_{\mathbb{N}}(p), & \text{if } p \in Z_0, \\ p_\infty, & \text{if } p = p_0. \end{cases}$$

Claim 2.3. *g is continuous.*

Proof. Since for each $p \in Z_0$, $\{p\}$ is open, we have that g is continuous at p. To see that g is continuous at p_0, take $n \in \mathbb{N}$ and let $R_n = \{0, \dots, n-1\}$ and $W_n = \{n, n+1, \dots\} \cup \{p_\infty\}$. Given $F \in \mathcal{F}$, $g^{-1}(R_n) \cap (\mathbb{N} \times \{F\})$ is finite. This implies that $g^{-1}(R_n) \cap T_F$ is finite. Thus, $g^{-1}(R_n)$ is closed and $Z \setminus g^{-1}(R_n) = g^{-1}(W_n)$ is open in Z. Therefore, g is continuous. □

Claim 2.4. *g is strong light.*

Proof. Given $n \in \mathbb{N}$, $g^{-1}(n)$ is a subset of Z_0. Since τ induces the discrete topology on Z_0, we have that $g^{-1}(n)$ is discrete. On the other hand, $g^{-1}(p_\infty) = \{p_0\}$ which is also discrete. Hence, g is strong light. □

Claim 2.5. *g is open.*

Proof. Let U be an open subset of Z and let $q = g(p) \in g(U)$.

If $p \in Z_0$, since $g(p) \in \mathbb{N}$, $\{g(p)\}$ is open in $\mathbb{N}_\infty$ and q is an interior point of $g(U)$.

If $p = p_0$, then $q = p_\infty$. In order to see that q is an interior point of $g(U)$, we need to show that there exists $N \in \mathbb{N}$ such that $\{N, N+1, \dots\} \subset g(U)$. Suppose to the contrary that there is no such an N. Then there exists an infinite subset A of $\mathbb{N}$ such that $A \cap g(U) = \emptyset$. Since $\mathcal{F}$ satisfies (3) in Lemma 2.1, there exists $F \in \mathcal{F}$ such that $F \subset A$. As we mentioned before, $\lim T_F = p_0$. So, there exists $(n, F) \in T_F$, with $n \in F$, such that $(n, F) \in U$. Thus, $n = g((n, F)) \in F \cap g(U) \subset A \cap g(U)$. This contradiction proves that q is an interior point of $g(U)$. Therefore, g is open. □

Claim 2.6. *Let $\{p_n\}_{n=1}^\infty$ be a sequence in Z_0 such that $\lim_{n\to\infty} p_n = p_0$. Then $\mathbb{N} \setminus \{g(p_n) : n \in \mathbb{N}\}$ is infinite.*

Proof. Suppose to the contrary that the set $\mathbb{N} \setminus \{g(p_n) : n \in \mathbb{N}\}$ is finite. Let $A = \{p_n \in Z_0 : n \in \mathbb{N}\}$.

Let $\mathcal{F}_A = \{F \in \mathcal{F} : A \cap T_F \neq \emptyset\}$. Notice that $A \subset \bigcup\{T_F : F \in \mathcal{F}_A\}$ and $\{g(p_n) : n \in \mathbb{N}\} = g(A) \subset \bigcup\{F : F \in \mathcal{F}_A\}$.

In the case that $\mathcal{F}_A$ is infinite, since the elements of the family $\{T_F : F \in \mathcal{F}\}$ are pairwise disjoint, there exists a subsequence $\{p_{n_k}\}_{k=1}^\infty$ of $\{p_n\}_{n=1}^\infty$ and there exists a sequence $\{F_k\}_{k=1}^\infty$ in $\mathcal{F}$ such that for each $k \in \mathbb{N}$, $p_{n_k} \in T_{F_k}$ and the sets $F_1, F_2, \dots$ are pairwise distinct. Consider the set $B = \{p_{n_k} : k \in \mathbb{N}\}$. Notice that $B \cap T_F$ is finite for each $F \in \mathcal{F}$. Thus, $B \in \mathcal{G}$ and $Z \setminus B$ is an open subset of Z containing p_0. Then there exists (infinitely many) $k \in \mathbb{N}$ such that $p_{n_k} \in Z \setminus B$. This contradicts the fact that $p_{n_k} \in B$ and proves that $\mathcal{F}_A$ is finite.

Set $\mathcal{F}_A = \{F_1, \ldots, F_m\}$. Then $\{g(p_n) : n \in \mathbb{N}\} \subset F_1 \cup \ldots \cup F_m$. Since $\mathbb{N} \setminus \{g(p_n) : n \in \mathbb{N}\}$ is finite and $\bigcup \mathcal{F} = \mathbb{N}$, there exist $r \in \mathbb{N}$ and $G_1, \ldots, G_r \in \mathcal{F}$ such that $\mathbb{N} \setminus \{g(p_n) : n \in \mathbb{N}\} \subset G_1 \cup \ldots \cup G_r$. Thus, $\mathbb{N} = F_1 \cup \ldots \cup F_m \cup G_1 \cup \ldots \cup G_r$. This contradicts the fact that $\mathcal{F}$ satisfies (2) in Lemma 2.1 and ends the proof. □

3. The Example

Now, we construct the spaces X and Y and the mapping f.

Consider the space $Z \subset (\mathbb{N} \times \mathcal{F}) \cup \{p_0\}$ defined in the previous section and $W = Z \times \mathbb{N}$, where $\mathbb{N}$ is endowed with the discrete topology.

For each $n, m \in \mathbb{N}$, let $S(m) = \{m, m+1, \ldots\}$, $Z(n) = Z \cap (\{n\} \times \mathcal{F})$, $Z^+(n) = [Z \cap (S(n) \times \mathcal{F})] \cup \{p_0\}$, $W(n,m) = Z(n) \times S(m) \subset W$, and $W^+(n) = Z^+(n) \times S(n)$.

Notice that $Z(n)$ and $Z^+(n)$ are open in Z and $W(n,m)$ and $W^+(n)$ are open in W. Notice also that $Z(n_1) \cap Z^+(n_2) = \emptyset$ if $n_1 < n_2$ and $Z(n_1) \cap Z^+(n_2) = Z(n_1)$ if $n_1 \geq n_2$. Moreover, $W(n_1,m) \cap W^+(n_2) = \emptyset$ if $n_1 < n_2$ and $W(n_1,m) \cap W^+(n_2) = Z(n_1) \times S(\max\{m, n_2\}) = W(n_1, \max\{m, n_2\})$ if $n_1 \geq n_2$.

Consider the space $\mathbb{N}_\infty^{(\infty)} = \mathbb{N}_\infty \times \{p_\infty\}$.

Define

$$X = W \cup \mathbb{N}_\infty^{(\infty)}.$$

We will define a topology for X by giving a local basis at each point of X.

For a point p in W, the local basis is the family of open subsets of the product W containing p.

For a point $p = (n, p_\infty) \in \mathbb{N} \times \{p_\infty\}$, the local basis is the family

$$\{W(n,m) \cup \{p\} : m \in \mathbb{N}\}.$$

And for the point (p_∞, p_∞), the local basis is the family $\{T(n) : n \in \mathbb{N}\}$, where

$$T(n) = W^+(n) \cup ([S(n) \cup \{p_\infty\}] \times \{p_\infty\}).$$

Let $\mathcal{B}_X$ be the family containing the empty set and all the basic sets described above. It is easy to show that $\mathcal{B}_X$ is closed under finite intersections. Thus, $\mathcal{B}_X$ is a basis for a topology τ_X on X. Clearly, X is a Hausdorff space.

Claim 3.1. *X is Fréchet-Urysohn.*

Proof. Let $A \subset X$ and $p \in \mathrm{cl}_X(A) \setminus A$. We need to consider three cases.

If $p = (z,n) \in Z_0 \times \mathbb{N}$, since $\{z\}$ is open in Z, $\{p\} = \{(z,n)\}$ is open in W, so $\{p\}$ is also open in X. Thus, this case is impossible.

If $p = (p_0, n) \in \{p_0\} \times \mathbb{N}$, since $Z \times \{n\}$ is open in W, it is also open in X. Then p is in the closure (in X) of $A \cap (Z \times \{n\})$. Since $Z \times \{n\}$ as subspace of X is homeomorphic to $Z \times \{n\}$ as subspace of $Z \times \mathbb{N}$, we have that $Z \times \{n\}$ is homeomorphic to Z. By Claim 2.2, there exists a sequence in $A \cap (Z \times \{n\})$ converging to p.

If $p \in \mathbb{N}_\infty^{(\infty)}$, by definition, X has a countable local basis at p. This implies that there exists a sequence of points in A converging to p. This completes the proof. □

Define the space Y as

$$Y = \mathbb{N}_\infty \times \mathbb{N}_\infty.$$

We consider Y with the product topology.

Consider the mapping $g : Z \to \mathbb{N}_\infty$ defined in the previous section.

Define $f : X \to Y$ by

$$f(p) = \begin{cases} (g(z), n), & \text{if } p = (z, n) \in W, \\ p, & \text{if } p \in \mathbb{N}_\infty^{(\infty)}. \end{cases}$$

Claim 3.2. *f is continuous.*

Proof. To prove that f is continuous we take appropriate basic open subsets U of Y.

If $U = (S(n) \cup \{p_\infty\}) \times (S(n) \cup \{p_\infty\})$, for some $n \in \mathbb{N}$, then $f^{-1}(U) = T(n)$, which is open in X.

If $U = (S(n) \cup \{p_\infty\}) \times \{m\}$, for some $n, m \in \mathbb{N}$, then $f^{-1}(U) = Z^+(n) \times \{m\} = ([Z \cap (S(n) \times \mathcal{F})] \cup \{p_0\}) \times \{m\} = g^{-1}(S(n) \cup \{p_\infty\}) \times \{m\}$. By Claim 2.3, $f^{-1}(U)$ is open in W and then $f^{-1}(U)$ is open in X.

If $U = \{n\} \times (S(m) \cup \{p_\infty\})$, for some $n, m \in \mathbb{N}$, then $f^{-1}(U) = W(n, m) \cup \{(n, p_\infty)\}$, which is open in X.

Finally, if $U = \{(n, m)\}$ for some $n, m \in \mathbb{N}$, then $f^{-1}(U) = Z(n) \times \{m\}$. Since $Z(n) = Z \cap (\{n\} \times \mathcal{F}) \subset Z_0$, and every subset of Z_0 is open in Z, $f^{-1}(U)$ is open in W and then it is open in X. □

Claim 3.3. *f is strong light.*

Proof. Take $p = (u, v) \in Y$.

If $u, v \in \mathbb{N}$, then $f^{-1}(p) = Z(u) \times \{v\} = g^{-1}(u) \times \{v\}$. By Claim 2.4, $g^{-1}(u)$ is discrete in Z, so $f^{-1}(p)$ is discrete in W, and then it is discrete in X.

If $u = p_\infty$ and $v \in \mathbb{N}$, then $f^{-1}(p) = \{(p_0, v)\}$, which is discrete.

If $v = p_\infty$, then $f^{-1}(p) = \{(u, p_\infty)\}$, which is discrete. □

Claim 3.4. *f is open.*

Proof. Let U be an open subset of X and let $q = f(p) \in f(U)$.

If $q \in \mathbb{N} \times \mathbb{N}$, then $\{q\}$ is open in Y and q is an interior point of $f(U)$.

If $q = (p_\infty, n)$ for some $n \in \mathbb{N}$, then $p = (p_0, n)$. Let $V = U \cap (Z \times \{n\})$. Then V is an open subset of X containing p. Since V is open in $Z \times \{n\}$, there exists an open subset V_0 of Z such that $p \in V_0 \times \{n\} \subset V$. By Claim 2.5, $g(V_0)$ is open in $\mathbb{N}_\infty$ and then $g(V_0) \times \{n\}$ is open in Y. Since $q \in f(V_0 \times \{n\}) = g(V_0) \times \{n\} \subset f(U)$, we conclude that q is an interior point of $f(U)$.

If $q = (n, p_\infty)$ for some $n \in \mathbb{N}$, then $p = (n, p_\infty)$. Thus, there exists $m \in \mathbb{N}$ such that $W(n, m) \cup \{p\} \subset U$. Then $q \in (\{n\} \times (S(m) \cup \{p_\infty\})) = f(W(n, m) \cup \{p\}) \subset f(U)$. Since $\{n\} \times (S(m) \cup \{p_\infty\})$ is open in Y, we conclude that q is an interior point of $f(U)$.

If $q = (p_\infty, p_\infty)$, then $p = (p_\infty, p_\infty)$. So, there exists $n \in \mathbb{N}$ such that $p \in W^+(n) \cup ([S(n) \cup \{p_\infty\}] \times \{p_\infty\}) \subset U$. Then $q \in (S(n) \cup \{p_\infty\}) \times (S(n) \cup \{p_\infty\}) = f(W^+(n) \cup ([S(n) \cup \{p_\infty\}] \times \{p_\infty\})) \subset f(U)$. Therefore, q is an interior point of $f(U)$. □

Claim 3.5. *$S_c(f)$ is not open.*

Proof. Suppose to the contrary that $S_c(f)$ is open. Then $S_c(f)(S_c(X))$ is open in $S_c(Y)$. Let $S = \{(n, p_\infty) : n \in \mathbb{N}\} \cup \{(p_\infty, p_\infty)\}$ be the sequence in X which converges to the point (p_∞, p_∞). Notice that $S_c(f)(S)$ is the sequence $T = \{(n, p_\infty) : n \in \mathbb{N}\} \cup \{(p_\infty, p_\infty)\}$ in Y which converges to (p_∞, p_∞). For each $m \in \mathbb{N}$, let T_m be the sequence $\{(n, m) : n \in \mathbb{N}\} \cup \{(p_\infty, m)\}$ in Y which converges to (p_∞, m). Since the sequence $\{T_m\}_{m=1}^\infty$ in $S_c(Y)$ converges to T, there exists $m \in \mathbb{N}$ such that $T_m \in S_c(f)(S_c(X))$. Then there exists a sequence S_m in $S_c(X)$ such that $S_c(f)(S_m) = T_m$.

Since T_m is a sequence in $\mathbb{N}_\infty \times \{m\}$ and $f^{-1}(\mathbb{N}_\infty \times \{m\}) = Z \times \{m\}$, we have that S_m is a sequence in $Z \times \{m\}$. Since $f(S_m) = T_m$, for each $n \in \mathbb{N}$, there exists a point $(s_n, m) \in S_m$ such that $f((s_n, m)) = (n, m)$. This implies that $s_n \in Z_0$ and $f((s_n, m)) = (g(s_n), m)$. Since $f(S_m) = T_m$ and $f^{-1}((p_\infty, m)) = (p_0, m)$, we have $(p_0, m) \in S_m$. Since the only non-isolated point in $Z \times \{m\}$ is (p_0, m), we have that $\lim S_m = (p_0, m)$. Since $\{(s_n, m) : n \in \mathbb{N}\}$ is an infinite subset of S_m, this set can be ordered in a subsequence $\{(z_k, m)\}_{k=1}^\infty$ of S_m converging to the point (p_0, m).

Then the sequence $\{z_k\}_{k=1}^\infty$ is a sequence in Z_0 such that $\lim_{k\to\infty} z_k = p_0$ and with the property that $\{g(z_k) : k \in \mathbb{N}\} \times \{m\} = \{f((z_k, m)) : k \in \mathbb{N}\} = \mathbb{N} \times \{m\}$. Thus, $\{g(z_k) : k \in \mathbb{N}\} = \mathbb{N}$. This contradicts Claim 2.6 and ends the proof. □

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