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THE UNIVERSALITY OF THREE-DIMENSIONAL SUBDIVISION RULES

BRIAN RUSHTON

Abstract. We characterize the history graph of a finite subdivision rule in terms of its combinatorics. We use this to show that each finite subdivision rule is combinatorially equivalent to a three-dimensional finite subdivision rule. This shows that high-dimensional recursive sequences of cell complexes (such as those used to construct higher-dimensional analogues of the Sierpinski cube) have the same adjacency patterns as 3-dimensional sequences, which are easier to visualize.

1. Introduction

Finite subdivision rules are a very general construct for creating recursively defined structures in all dimensions (such as the Sierpinski triangle, carpet, and cube). They consist of a set of topological spaces and maps used recursively to create more and more refined cell structures. In this paper, we show that a finite subdivision rule (in any dimension) is combinatorially equivalent to a three-dimensional finite subdivision rule (in the sense that there is map between the two recursive sequences of cell structures that preserves adjacency of cells).

The main difficulty in the paper is not in finding a sequence of 3-dimensional cell structures with the same adjacencies, but in finding such a sequence that is defined recursively, i.e. another finite subdivision rule.

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2. Acknowledgments

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3. Background

Finite subdivision rules are a construction in geometric group theory originally described by Cannon, Floyd, and Parry in relation to Cannon's Conjecture [4]. In their work, finite subdivision rules are a collection of 2-complexes and maps that can be used to repeatedly refine a tiling of the plane, sphere, or disk. For instance: 2-dimensional barycentric subdivision, which replaces every triangle in a two-dimensional simplicial complex with six smaller triangles, can be represented as a finite subdivision rule in the sense of Cannon et. al. These 2-dimensional finite subdivision rules are strongly connected to rational maps on the sphere [1, 3].

More importantly, Cannon and Swenson have shown that every three-dimensional hyperbolic manifold group can be associated to a two-dimensional finite subdivision rule that subdivides the two-sphere, and that this finite subdivision rule on the sphere contains enough information to reconstruct the group itself [2, 5]. A finite subdivision rule together with a 2-complex that it subdivides is called a **subdivision pair**.

This was later generalized to show that many groups can be associated to a finite subdivision pair of dimension higher than 2, and that:

1. the subdivision pair associated to a group is sufficient for reconstructing the quasi-isometry class of the group through a graph called the **history graph** [9, 10], and
2. there is a dictionary between combinatorial properties of the subdivision pair and quasi-isometry properties of the group [12].

In these last three cited papers, the primary features of interest in these subdivision pairs of higher dimension are their combinatorics. In this paper, we define a **combinatorial subdivision graph**, which is a combinatorial analogue to a subdivision pair. We prove the following:

Theorem 2. *Let (R, X) be a subdivision pair. Then the history graph $\Gamma(R, X)$ is a combinatorial subdivision graph.*

Theorem 3. *Let Ξ be a combinatorial subdivision graph. Then there is a 3-dimensional subdivision pair (R, X) such that the history graph $\Gamma(R, X)$ is graph isomorphic to Ξ.*

These two theorems show that we can replace a subdivision pair of any dimension with a 3-dimensional subdivision pair without changing its combinatorial properties. It also shows that if a group is quasi-isometric to a combinatorial subdivision graph, then it can be associated to a subdivision pair, allowing us to use the dictionary between combinatorics and quasi-isometry properties to study the group.

4. Definitions

4.1. Finite subdivision rules.

Definition. A **(colored) finite subdivision rule** R **of dimension** n consists of:

1. A finite n-dimensional CW complex S_R, called the subdivision complex. We assume that for every closed cell $\tilde{s}$ of S_R there is a CW structure s on a closed disk of the same dimension such that the subcells of s are contained in ∂s and the characteristic map $\psi_s : s \to S_R$ which maps onto $\tilde{s}$ restricts to a homeomorphism onto each open cell (this makes the complex essentially a union of polytopes, but allows for degenerate cases like bigons),
2. a finite n-dimensional complex $R(S_R)$ that is a subdivision of S_R,
3. a **subdivision map** $\phi_R : R(S_R) \to S_R$,which is a cellular map that restricts to a homeomorphism on each open cell, and
4. A **coloring** of the cells of S_R, which is a partition of the set of cells of S_R into an **ideal set** I and a **limit set** N, such that ϕ_R maps I into itself and the union of the cells in I is open in S_R.

Each cell s in the definition above (with its appropriate characteristic map) is called a **tile type** of S_R.

Given a finite subdivision rule R of dimension n, an R-**complex** consists of:

1. a CW complex X, and
2. a continuous cellular map $f : X \to S_R$ called the **structure map of** X whose restriction to each open cell is a homeomorphism.

Definition. Cells of R-complexes are referred to as **tiles**. Given a tile A of an R complex and a tile type B of S_R, we say that A **has tile type** B if A maps onto B under the structure map f. If B is ideal, we say that A **has an ideal tile type**, and the same holds for limit tile types.

Given an R-complex X with map $f : X \to S_R$ and subdivision $R(S_R)$ we define the subdivision of X under R as the complex $R(X)$ which is obtained from X by pulling back the cell structure of $R(S_R)$ under the map f.

This gives an induced structure map $f : R(X) \to R(S_R)$ that restricts to a homeomorphism on each open cell. This means that $R(X)$ is an R-complex with structure map $\phi_R \circ f : R(X) \to S_R$.

The n**-th subdivision of** X **under** R (written $R^n(X)$) is given by setting $R^0(X) = X$ (with structure map $f : X \to S_R$) and $R^n(X) = R(R^{n-1}(X))$ (with structure map $\phi_R^n \circ f : R^n(X) \to S_R$) if $n \geq 1$.

A finite subdivision rule R with a given R-complex X is called a **subdivision pair** (R, X).

The **dimension** of a subdivision pair (R, X) is the dimension of X. The dimension of X may be less than the dimension of R; for instance, a point is an R-complex for every finite subdivision rule R.

The history graph is one of the most useful constructions involving finite subdivision rules. It is a metric space whose quasi-isometry properties are directly determined by the combinatorial properties of a given subdivision pair (R, X) .

Definition. Let R be a finite subdivision rule, and let X be an R-complex of dimension m.

A **face** is an m-dimensional tile of X or one of its subdivisions. A **limit face** is a face with a limit tile type, and an **ideal face** is a face with an ideal tile type.

A facet is an $(m-1)$-dimensional tile of X or its subdivisions. Limit facets and ideal facets are defined as for faces.

Let Λ_n denote the set of all limit faces in the nth level of subdivision $R^n(X)$. We call Λ_n the n**th limit set** of X.

Let U_n equal the subspace consisting of the union of all tiles in Λ_n. Note that U_n is a closed subset of X, due to item 4 in the definition of subdivision rules and the fact that the subdivision map is continuous. We define $\Lambda = \Lambda(R, X)$ to be

$$\Lambda = \bigcap_{n=1}^{\infty} U_n$$

The complement of Λ is called the **ideal set of** X and is denoted $\Omega = \Omega(R, X)$.

Given (R, X) as above, we define Γ_n as the graph containing:

(1) a vertex for each face in Λ_n, and
(2) an edge for each pair of faces of Λ_n which share a facet.

The **history graph** $\Gamma = \Gamma(R, X)$ consists of:

(1) a single vertex O called the **origin**,
(2) the disjoint union of the Γ_n, whose edges are called **horizontal**, and
(3) a collection of **vertical** edges defined as follows: if a vertex v in Γ_n corresponds to a limit face T, we add an edge connecting v to each of the limit faces contained in $R(T)$. We also connect the origin O to every vertex of Γ_0.

4.2. **Combinatorial subdivision graphs.** In this section, we provide a characterization of history graphs of subdivision rules.

Definition. A **(finitely) labeled graph** is a graph together with a map from the edges of the graph to a finite set of **edge labels**, and a map from the vertices of the graph to a finite set of **vertex labels**. For purposes of this article, we include unions of open edges as labeled graphs.

For background on labeled graphs, see [8].

Definition. A **labeled graph morphism** is a graph morphism between labeled graphs that preserves labels.

Definition. The **open star** of a vertex is the union of a vertex with all the open edges that have that vertex as an endpoint.

Definition. A finitely-labeled graph Ξ is a **combinatorial subdivision graph** if it contains disjoint subgraphs Ξ_n such that the following are satisfied:

(1) Ξ_0 is a single vertex.
(2) Every vertex is contained in some Ξ_n.
(3) Every vertex v of Ξ_n for $n > 0$ is connected to a unique vertex of Ξ_{n-1} called the **predecessor** of v. We define the predecessor of the unique vertex in Ξ_0 to be itself.
(4) If two vertices of Ξ_n are connected by an edge for some $n > 0$, then their predecessors are connected by an edge.
(5) The open stars of any two vertices with the same label are labeled-graph isomorphic.
(6) Conditions 3 and 4 allow us to define a graph morphism $\pi : \Xi \to \Xi$ which sends each vertex to its predecessor. We call the map π the **predecessor map**. Then we require the preimages under π of two edges with the same label to be labeled-graph isomorphic. A representative graph in such an isomorphism class is called an **edge subdivision**. Similarly, we require the preimage of two open stars of vertices with the same label to be labeled-graph isomorphic, and a representative graph in this isomorphism class is called a **vertex subdivision**.

Lemma 1. *Each edge subdivision is a disjoint union of edges.*

Proof. A labeled graph morphism is a homeomorphism when restricted to an open edge, so each edge subdivision is a union of edges. Because no vertices are included, the edges are necessarily disjoint. □

5. Main Theorems

Theorem 2. *Let (R, X) be a finite subdivision pair. Then the history graph $\Gamma(R, X)$ is a combinatorial subdivision graph.*

Proof. Let Γ be the history graph of a subdivision pair of dimension n. Items 1-3 in the definition of a combinatorial subdivision graph are automatically satisfied.

Note that faces of $R^n(S_n)$ are subsets of faces of $R^{n-1}(S_n)$. Thus, if faces A, B of $R^n(S_n)$ share a facet, then the faces A', B' of $R^{n-1}(S_n)$ that contain A and B also share at least one facet. Thus, condition 4 is satisfied.

Vertex labels in Γ correspond to n-dimensional limit tile types, and edge labels correspond to $(n-1)$-dimensional limit tile types which are contained between two n-dimensional tiles in X. The tile type of an n-dimensional cell determines the tile type of its boundary, so the edge labels surrounding a given vertex label in Γ are unique, satisfying item 4.

Finally, item 5 is satisfied by the nature of a subdivision rule: a subdivision rule acts locally, and always replaces a tile with a given type the exact same way. □

Theorem 3. *Let Ξ be a combinatorial subdivision graph. Then there is a 3-dimensional subdivision pair (R, X) such that the history graph $\Gamma(R, X)$ is graph isomorphic to Ξ.*

Proof. It is easy, using general position arguments, to embed a graph in $\mathbb{R}^3$ and thicken it into a cell complex. By doing this for each subgraph Γ_n of a combinatorial subdivision graph Γ, we can get a sequence of cell complexes which are similar to those created by a subdivision rule. The difficulty, though, is ensuring that the sequence of cell complexes is defined recursively.

Therefore, we need to construct the subdivision complex S_R, the related complex $R(S_R)$, and the subdivision map ϕ_R explicitly, as well as the R-complex X.

For each vertex label v, let $B(v)$ be a closed ball in $\mathbb{R}^3$. We place a cell structure on $B(v)$ such that:

(1) there is one 3-cell in $B(v)$, and
(2) the boundary sphere of $B(v)$ contains disjoint disks, one for each edge in the open star of a vertex with the label v. We give each disk the standard cell structure with one vertex, one edge, and one 2-cell. We consider the remainder of the sphere 'ideal'.

Let Y be the quotient of the disjoint union of the $B(v)$ given by identifying boundary disks corresponding to edges with the same label. Thus, Y has one disk for each edge label, via an orientation-preserving map. For each disk $D(e)$, let $D'(e)$ be a new cell structure on $D(e)$ that contains sub-disks, one for each edge in the edge subdivision corresponding to $D(e)$. By Lemma 1, these sub-disks are disjoint.

The vertex subdivision corresponding to a vertex label is a union of open vertex stars. Given a label v, we construct a complex $N(v)$ taking the disjoint union of copies of the $B(w)$, one for each vertex w in the vertex subdivision of v, and identifying boundary disks that correspond to the same edge in the vertex subdivision, via an orientation-preserving map. Because each copy of a $B(w)$ deformation retracts onto compactification of the closed star of the corresponding vertex (here, the compactification merely adds endpoints onto the boundary edges), the whole complex $N(v)$ deformation retracts onto the compactification of the vertex subdivision. The unidentified boundary disks of $N(v)$ are called the **exterior disks** of $N(v)$. Each exterior disk of $N(v)$ corresponds to a boundary edge of the vertex subdivision, which in turn corresponds to an edge in the edge subdivision of one of the edges of the original vertex star. Thus, each exterior disk corresponds to a subdisk in some $D'(e)$.

We now embed each $N(v)$ into $B(v) \subseteq Y$ so that:

(1) the intersection of $N(v)$ with the boundary of $B(v) \subseteq Y$ is the union of the exterior disks of $N(v)$,
(2) each exterior disk of $N(v)$ matches up with the appropriate subdisk of the appropriate $D'(e)$. This is possible because the subdisks are disjoint, and
(3) the closed complement $\overline{B(v) \setminus N(v)}$ is divided into 3-cells that are almost polyhedral (which we can do by triangulating and using barycentric subdivision twice, if necessary). We label this complement as ideal.

This gives us a new cell structure on each $B(v)$, which we can call $B'(v)$, and thus a new cell structure on Y, which we can call Y'; the two complexes Y and Y' have the same underlying topological space. We now let $I(v)$ be a cell complex isomorphic to $B(v)\setminus N(v)$. Note that the boundary of $I(v)$ can be partitioned into its intersection with the boundary of $B(v)$ (the **outer** portion of $\partial I(v)$) and its intersection with the boundary of $N(v)$ (the **inner** portion of $\partial I(v)$). Attach $I(v)$ to Y by using the identity map on the outer portion and, on the inner boundary, mapping each part of $\partial N(v)$ to the $\partial B(v)$ it is a copy of.

Call this new complex S_R. If we replace the part corresponding to Y with the Y' structure, we get a new complex which we call $R(S_R)$.

There is a natural map from $R(S_R)$ to S_R, which is obtained by:

(1) mapping each boundary sub-disk of the $N(v)$'s corresponding to an edge label e to the disk $D(e) \subseteq S_R$ via an orientation-preserving map,
(2) mapping each $I(v)$ to itself via the identity,
(3) sending each closed 3-cell in $N(v)$ to the $B(v)$ it is a copy of, and
(4) sending each complex $\overline{B(v) \setminus N(v)}$ to the $I(v)$ that is a copy of it.

We call this map ϕ_R, and, together with S_R and $R(S_R)$, this forms a finite subdivision rule R of dimension 3.

To create the cell complex X, recall that Ξ_1 is full subgraph supported on the set of all vertices at distance 1 from the origin in Ξ. Let X consist of a copy of $B(v)$ for each open vertex star in Ξ_1 with label v, where we identify boundary disks of two $B(v)$'s that correspond to the same edge of Ξ_1.

Then the dual graph of X is graph isomorphic to Ξ_1.

Let the structure map $f : X \to S_R$ be given by mapping each copy of a $B(v)$ to the corresponding $B(v)$ in S_R. Recall that the subdivision $R(X)$ is obtained by 'pulling back' the cell structure of $R(S_R)$ via f. In this case, it replaces each copy of a $B(v)$ with $B(v)'$. The limit cells of $B(v)'$ are the interior cells of $N(v)$ and its boundary disk; thus, the dual graph of $R(X)$ is obtained by replacing each open vertex star of the dual graph of X with its vertex subdivision, and each edge with its edge subdivision. Thus, the dual graph of $R(X)$ must be isomorphic to Ξ_2. By continuing this process, we see that $R^n(x)$ is dual to Ξ_{n+1}, and that the vertical edges of $\Gamma(R, X)$ connect a vertex to its predecessor under the subdivision.

This concluded the proof. □

Note that 3 is the smallest dimension possible in the theorem, as some history graphs are not planar (such as the history graph of the subdivision rules in [10]).

References

[1] J. Cannon, W. Floyd, W. Parry, and K. Pilgrim, *Nearly euclidean thurston maps.* Conformal Geometry and Dynamics of the American Mathematical Society, **16(12)** (2012), 209–255.
[2] J. W. Cannon, *The combinatorial riemann mapping theorem.* Acta Mathematica, **173(2)** (1994), 155–234.

[3] J. W. Cannon, W. J. Floyd, R. Kenyon, and W. R. Parry, *Constructing rational maps from subdivision rules.* Conformal Geometry and Dynamics, **7** (2003), 76–102. (electronic).
[4] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Finite subdivision rules.* Conformal Geometry and Dynamics, **5** (2001), 153–196.
[5] J. W. Cannon and E. L. Swenson, *Recognizing constant curvature discrete groups in dimension 3.* Transactions of the American Mathematical Society, **350(2)** (1998), 809–849.
[6] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Conformal modulus: the graph paper invariant or the conformal shape of an algorithm.* Geometric group theory down under (Canberra, 1996), pages 71–102, 1999.
[7] David Epstein, Mike S. Paterson, James W. Cannon, Derek F. Holt, Silvio V. Levy, and William P. Thurston, *Word processing in groups* AK Peters, Ltd. 1992
[8] Joseph A. Gallian, *A dynamic survey of graph labeling.* The electronic journal of combinatorics, **16(6)** (2009), 1–219.
[9] Brian Rushton, *Constructing subdivision rules from polyhedra with identifications.* Alg. and Geom. Top., **12** (2012), 1961–1992.
[10] ———. *A finite subdivision rule for the n-dimensional torus.* Geometriae Dedicata, (2012) pages 1–12.
[11] ———. *Subdivision rules for special cubulated groups.* arXiv preprint arXiv:1307.1788, 2013.
[12] ———. *Classification of subdivision rules for geometric groups of low dimension.* Conformal Geometry and Dynamics of the American Mathematical Society, **18(10)** (2014), 171–191.
[13] Stephen Willard, *General topology.* Courier Corporation, 2004.

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