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Joshua J. Clemons and Lorelei Koss

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Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
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HIGHER ORDER ELLIPTIC FUNCTIONS WITH CONNECTED JULIA SETS

JOSHUA J. CLEMONS AND LORELEI KOSS

Abstract. Connectivity properties of elliptic functions are completely understood for only four families of order two elliptic functions. In this paper, we find families of elliptic functions of arbitrarily high order on square and triangular lattices for which the Julia set is connected.

There are four families of elliptic functions for which the connectivity of the Julia set is completely understood. Let $\wp_\Lambda$ denote the Weierstrass elliptic function on the lattice Λ. The Julia set of $\wp_\Lambda$ is always connected if Λ is a triangular [8] or square [3, 5] lattice. The Julia set of $1/\wp_\Lambda$ is either connected or Cantor if Λ is a triangular lattice [10] or a square lattice [13]. Connectivity properties of the Julia set of certain higher order elliptic functions on real rectangular lattices appear in [9, 11], but these results do not cover all of the functions in those families.

Both $\wp_\Lambda$ and $1/\wp_\Lambda$ have order two. The proofs of the connectivity results rely on using properties of the special triangular and square lattice shapes to investigate the locations of the critical values. Symmetry properties of $\wp_\Lambda$ on a triangular or square lattices force any Fatou component to contain at most one critical value. These families contain functions for which the Fatou set is nonempty as well as functions for which the Julia set is the entire sphere. On the other hand, the functions $1/\wp_\Lambda$ on a triangular or square lattice always have a super attracting Fatou component at the origin. Again, symmetry properties of the function imply that either this Fatou component contains all of the critical values, resulting in a Cantor Julia set, or every Fatou component contains at most one critical value, resulting in a connected Julia set.

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In this paper, we investigate a number of different families of elliptic functions of arbitrarily high order whose Julia sets are always connected. Our approach is to find families for which we can strictly control the locations and behavior of the critical values. We focus on triangular and square lattices because those lattices give rise to additional symmetry properties in comparison to general lattice shapes. Section 1 gives background information on elliptic functions, and Section 2 explains general results on the dynamics of elliptic functions. In Section 3, we find two different families of elliptic functions on triangular lattices which always have connected Julia sets. Finally, in Section 4 we prove that the Julia set is connected for three families of elliptic functions on square lattices.

1. Background on elliptic functions

We begin with some preliminaries about elliptic functions, the Weierstrass $\wp$-function and period lattices.

We start by picking a lattice $\Lambda = [\lambda_1, \lambda_2] = \{m\lambda_1 + n\lambda_2 \colon m, n \in \mathbb{Z}, \lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}, \lambda_2/\lambda_1 \notin \mathbb{R}\}$. If Λ is a lattice, and $k \neq 0$ is any complex number, then $k\Lambda$ is also a lattice defined by taking $k\lambda$ for each $\lambda \in \Lambda$; $k\Lambda$ is said to be *similar* to Λ. Similarity is an equivalence relation between lattices, and an equivalence class of lattices is called a *shape.* In this paper, we focus on two special lattice shapes: triangular and square lattices. Triangular lattices Λ have the property that $\varepsilon\Lambda = \Lambda$, where $\varepsilon = e^{2\pi i/3}$. The period parallelograms of a triangular lattice are formed by two equilateral triangles. Square lattices Λ have the property that $i\Lambda = \Lambda$. The period parallelograms of a square lattice form squares.

We define the Weierstrass elliptic function on $z \in \mathbb{C}$

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

The Weierstrass elliptic function is an even, meromorphic function that is periodic with respect to the lattice Λ. It has order two and has double poles at lattice points.

The Weierstrass elliptic function can also be defined by the differential equation

$$(\wp'_\Lambda(z))^2 = 4(\wp_\Lambda(z))^3 - g_2\wp_\Lambda(z) - g_3, \tag{1.1}$$

where $g_2(\Lambda) = 60 \sum_{w \in \Lambda \setminus \{0\}} w^{-4}$, and $g_3(\Lambda) = 140 \sum_{w \in \Lambda \setminus \{0\}} w^{-6}$. Each pair of complex numbers (g_2, g_3) with $g_2^3 - 27g_3^2 \neq 0$ determines a unique equivalence class of lattices and vice versa, where equivalence means that they generate the same subgroup [4].

It will be useful to have an expression for $\wp''_\Lambda$, the second derivative of the Weierstrass elliptic function for a given lattice Λ:

$$\wp''_\Lambda(z) = 6(\wp_\Lambda(z))^2 - \frac{g_2(\Lambda)}{2}. \tag{1.2}$$

In this paper, we focus on general elliptic functions, which can be characterized by the following theorem.

Theorem 1.1. [4] *Every elliptic function f_Λ with period lattice Λ can be written as $f_\Lambda(z) = R(\wp_\Lambda(z)) + \wp'_\Lambda(z)Q(\wp_\Lambda(z))$, where R and Q are rational functions with complex coefficients.*

The Weierstrass elliptic function and its derivative satisfy the following homogeneity properties.

Proposition 1.2. [4] *For any lattice Λ and for any $m \in \mathbb{C}\backslash\{0\}$,*

$$\wp_{m\Lambda}(mz) = m^{-2}\wp_\Lambda(z),$$
$$\wp'_{m\Lambda}(mz) = m^{-3}\wp'_\Lambda(z).$$

1.1. **Critical Points and Values.** Critical points and values play an important role in complex dynamics, so it is useful for us to be able to locate these points for $\wp_\Lambda$ and $\wp'_\Lambda$. From [4], we have that the critical points of $\wp_\Lambda$ lie exactly on the half lattice points of $\Lambda = [\lambda_1, \lambda_2]$; that is, the critical points are

$$z = \frac{\lambda_j}{2} + \Lambda, \tag{1.3}$$

for $j = 1, 2, 3$, where we define $\lambda_3 = \lambda_1 + \lambda_2$. We use the notation

$$e_1 = \wp_\Lambda\left(\frac{\lambda_1}{2}\right), \quad e_2 = \wp_\Lambda\left(\frac{\lambda_2}{2}\right), \quad e_3 = \wp_\Lambda\left(\frac{\lambda_3}{2}\right) \tag{1.4}$$

to denote the critical values of $\wp_\Lambda$.

For the special lattice shapes of concern in this paper, the invariants and critical values of $\wp_\Lambda$ take an especially nice form.

Proposition 1.3. [4, 6]

(1) *Let Λ be a triangular lattice. Then*
   (a) $g_2(\Lambda) = 0$.
   (b) *e_1, e_2, e_3 all have the same modulus and are cube roots of $g_3/4$, so $e_1 = e^{4\pi i/3}e_3$, and $e_2 = e^{2\pi i/3}e_3$.*
   (c) *$\wp_\Lambda(z) = 0$ if and only if*

$$z = \pm\frac{1}{3}\lambda_4 + \Lambda;$$

where $\lambda_4 = \lambda_1 - e^{2\pi i/3}\lambda_1$; that is, the roots of $\wp_\Lambda$ are located at the centers of the equilateral triangles forming the period parallelograms.

(2) *Let* Λ *be a square lattice. Then*
   (a) $g_3(\Lambda) = 0$.
   (b) $e_1 = \sqrt{g_2}/2, e_2 = -e_1$, *and* $e_3 = 0$.
   (c) *If* $\Lambda = [\lambda, \lambda i]$ *where* $\lambda \in \mathbb{R}$, *then* $\wp_\Lambda(\lambda/2) = e_1$ *is the minimum of* $\wp_\Lambda$ *on* $\mathbb{R}$.
   (d) $\wp_\Lambda(z) = 0$ *if and only if* $z = \dfrac{\lambda_3}{2} + \Lambda$.

To find the critical points of $\wp'_\Lambda$, we begin with $\wp''_\Lambda(z) = 6(\wp_\Lambda(z))^2 - g_2(\Lambda)/2$ from Equation 1.2. Solving $\wp''_\Lambda(z) = 0$ gives us that $\wp'_\Lambda$ has critical points in the four congruence classes where

$$(\wp_\Lambda(z))^2 = \frac{g_2}{12}. \tag{1.5}$$

The critical values of $\wp'_\Lambda$ are found by solving $4(\wp_\Lambda(z))^3 - g_2\wp_\Lambda(z) - ((\wp'_\Lambda(z))^2 + g_3) = 0$ for $\wp'_\Lambda(z)$ after substituting $\pm\sqrt{g_2/12}$ for $\wp_\Lambda(z)$. Thus

$$4\left(\sqrt{\frac{g_2}{12}}\right)^3 - g_2\sqrt{\frac{g_2}{12}} \pm ((\wp'_\Lambda(z))^2 + g_3) = 0,$$

which implies that

$$-\frac{1}{3}g_2^{\frac{3}{2}} = \pm\sqrt{3}((\wp'_\Lambda(z))^2 + g_3).$$

Squaring both sides and rearranging terms shows that

$${g_2}^3 - 27((\wp'_\Lambda(z))^2 + g_3)^2 = 0,$$

and by solving for $\wp'_\Lambda(z)$, we see that the critical values of $\wp'_\Lambda(z)$ are

$$\pm\left(-g_3 \pm \left(\frac{g_2}{3}\right)^{\frac{3}{2}}\right)^{\frac{1}{2}}. \tag{1.6}$$

We collect some information about the form of critical points and critical values for $\wp'_\Lambda$ on special lattice shapes in the following proposition.

Proposition 1.4. [4]

(1) *Let* Λ *be a triangular lattice. Then* $\wp'_\Lambda$ *has exactly two equivalence classes of critical points lying at*

$$z = \pm\frac{1}{3}\lambda_4 + \Lambda$$

(where $\lambda_4 = \lambda_1 - e^{2\pi i/3}\lambda_1$*; that is, the critical points are located at the centers of the equilateral triangles forming the period parallelograms), and two distinct critical values at* $v_1 = \sqrt{-g_3}$ *and* $v_2 = -v_1$.

(2) *Let Λ be a square lattice. Then the four critical values of $\wp'_\Lambda$ are*

$$v_1 = \sqrt{\left(\frac{g_2}{3}\right)^{\frac{3}{2}}}, v_2 = -v_1, v_3 = iv_1, v_4 = -iv_1.$$

1.2. **Background on the Dynamics of Meromorphic Functions.** We give a brief overview of the dynamics of meromorphic functions; more details can be found in [1, 2, 14]. Let $f\colon \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function. In this paper, we use the notation f^n or $(f(z))^n$ to denote exponentiation, and $f^{\circ n}$ to denote iteration. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^{\circ n}\colon n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty \backslash F(f)$. A point z_0 is *periodic* of period p if there exists a $p \geq 1$ such that $f^{\circ p}(z_0) = z_0$. We also call the set $\{z_0, f(z_0), \ldots, f^{\circ p-1}(z_0)\}$ a *p-cycle.* The *multiplier* of a point z_0 of period p is the derivative $(f^{\circ p})'(z_0)$. A periodic point z_0 is classified as *attracting, repelling,* or *neutral* if $|(f^{\circ p})'(z_0)|$ is less than, greater than, or equal to 1 respectively. If $|(f^{\circ p})'(z_0)| = 0$ then z_0 is called a *superattracting* periodic point.

Suppose U is a connected component of the Fatou set. We say that U is *preperiodic* if there exists $n > m \geq 0$ such that $f^{\circ n}(U) = f^{\circ m}(U)$, and the minimum of $n - m = p$ for all such n, m is the *period* of the cycle. Elliptic functions have a finite number of critical values, and thus it turns out that the classification of periodic components of the Fatou set is no more complicated than that of rational maps of the sphere. Periodic components of the Fatou set of elliptic functions may be attracting domains, parabolic domains, Siegel disks, or Herman rings. In particular, elliptic functions have no wandering domains or Baker domains [1, 6, 14].

Let $C = \{U_0, U_1, \ldots, U_{p-1}\}$ be a periodic cycle of components of $F(f)$. If C is a cycle of immediate attractive basins or parabolic domains, then $U_j \cap Crit(f) \neq \emptyset$ for some $0 \leq j \leq p-1$. If C is a cycle of Siegel Disks or Herman rings, then $\partial U_j \subset \overline{\bigcup_{n \geq 0} f^{\circ n}(Crit(f))}$ for all $0 \leq j \leq p-1$. In particular, any periodic component of an elliptic function has an associated critical point.

2. General Results on Fatou and Julia Sets

If f_Λ is an elliptic function, then the periodicity of f_Λ results in periodic Fatou and Julia sets.

Theorem 2.1. [6] *For any lattice Λ and any elliptic function f_Λ, $F(f_\Lambda) = F(f_\Lambda) + \Lambda$ and $J(f_\Lambda) = J(f_\Lambda) + \Lambda$.*

Even elliptic functions exhibit additional symmetry on their Fatou and Julia sets.

Theorem 2.2. [6] *For any lattice* Λ *and any even elliptic function* f_Λ, $F(f_\Lambda) = -F(f_\Lambda)$ *and* $J(f_\Lambda) = -J(f_\Lambda)$.

In [7], Hawkins and the second author showed that, on any lattice Λ, $\wp_\Lambda$ has no cycles of Herman rings. This result was extended to $1/\wp_\Lambda$ where Λ is a triangular lattice in [10], and to other even, order two elliptic functions in the same conformal class of $\wp_\Lambda$, where Λ is any lattice, in [13].

Many of the elliptic functions in this paper are rational functions of $\wp_\Lambda$ which are even. The proof given in [7] to prove that $\wp_\Lambda$ has no Herman rings can be immediately extended to prove Theorem 2.3. The key idea is that even elliptic functions give rise to Julia sets with even symmetry by Theorem 2.2, and this is an obstruction to having a Fatou component that is conjugate to an irrational rotation on an annulus.

Theorem 2.3. *For any lattice* Λ, *if* f_Λ *is an even elliptic function with poles at* Λ *then* f_Λ *has no cycle of Herman rings.*

We move to an investigation of the dynamics of elliptic functions of the form $f_\Lambda(z) = (\wp'_\Lambda(z))^n$, where $n > 0$. These functions will play an important role later in the paper. We begin with an examination of the symmetries that arise in the Julia and Fatou sets.

Theorem 2.4. *For any lattice* Λ, *let* $f_\Lambda(z) = (\wp'_\Lambda(z))^n$, *where* $n > 0$. *Then*

(1) $F(f_\Lambda) = -1F(f_\Lambda)$ *and* $J(f_\Lambda) = -1J(f_\Lambda)$.
(2) *If* Λ *is square, then* $F(f_\Lambda) = iF(f_\Lambda)$ *and* $J(f_\Lambda) = iJ(f_\Lambda)$.
(3) *If* Λ *is triangular, then* $\varepsilon F(f_\Lambda) = F(f_\Lambda)$ *and* $\varepsilon J(f_\Lambda) = J(f_\Lambda)$ *where* ε *is a cube root of unity.*

Proof. Part (1) follows since $\wp'_\Lambda$ is odd; if U is a set for which $\{f_\Lambda^{\circ k}(U)\}$ forms a normal family, then $\{f_\Lambda^{\circ k}(-U) = \pm f_\Lambda^{\circ k}(U)\}$ forms a normal family. For part (2), we know that square lattices satisfy $i\Lambda = \Lambda$. Using Proposition 1.2, we have $(\wp'_\Lambda)(-iz) = (\wp'_{i\Lambda})(-iz) = -i(\wp'_\Lambda)(z)$. Thus $\{f_\Lambda^{\circ k}(U)\}$ forms a normal family if and only if $\{f_\Lambda^{\circ k}(iU)\}$ does. For part (3), if ε is a cube root of one, then so is $\varepsilon^2 = 1/\varepsilon$; thus $\varepsilon^2\Lambda = \Lambda$. Then from Proposition 1.2, $\wp'_\Lambda(\varepsilon^2 z) = \wp'_{\varepsilon^2\Lambda}(\varepsilon^2 z) = \wp'_\Lambda(z)$, and thus $f_\Lambda^{\circ k}(\varepsilon^2 z) = f_\Lambda^{\circ k}(z)$. □

Next, we prove that the elliptic functions $f_\Lambda(z) = (\wp'_\Lambda(z))^n$, where Λ is a triangular lattice and $n > 0$, do not have Herman rings.

Theorem 2.5. *For any triangular lattice* Λ, $f_\Lambda(z) = (\wp'_\Lambda(z))^n$ *for* $n > 0$ *has no cycle of Herman rings.*

Proof. We begin with the case $n = 1$. Suppose that $\wp'_\Lambda$ has a cycle of Herman rings $\{U_0, U_1, \dots, U_{p-1}\}$ of period $p \geq 1$. Then for any $i = 0, 1, \dots p-1$, $(\wp'_\Lambda)^{\circ p}\colon U_i \to U_i$ is conjugate to an irrational rotation of the annulus and thus has degree one. The preimages under this conjugacy of the circles $|\eta| = r$, $1 < r < R$ foliate the disks with $(\wp'_\Lambda)^{\circ p}$ forward invariant leaves on which $(\wp'_\Lambda)^{\circ p}$ is injective. Let γ be a $(\wp'_\Lambda)^{\circ p}$ invariant leaf of U_i, and let B_γ denote the bounded component of the complement of γ. Since U_i is multiply connected, we know that B_γ contains a prepole. Thus, there is a smallest nonnegative number n such that $(\wp'_\Lambda)^{\circ n}(\gamma)$ contains a lattice point μ in $B_{(\wp'_\Lambda)^{\circ n}(\gamma)}$. Let U_j denote the Herman ring $(\wp'_\Lambda)^{\circ n}(U_i)$.

Since U_j is homeomorphic to an annulus with the lattice point μ in $B_{(\wp'_\Lambda)^{\circ n}(\gamma)}$, Theorem 2.4 implies that $e^{2\pi i/3}U_j + (\mu - e^{2\pi i/3}\mu)$ is a Fatou component such that μ is in $e^{2\pi i/3}B_{(\wp'_\Lambda)^{\circ n}(\gamma)} + (\mu - e^{2\pi i/3}\mu)$. Since the topology is identical in every fundamental region, we assume by translating the entire setup by $(\mu - e^{2\pi i/3}\mu)$, that $\mu = 0$. Therefore both U_j and $e^{2\pi i/3}U_j$ are annuli containing simple closed loops γ and $e^{2\pi i/3}\gamma$ respectively, and each has 0 in its bounded component. But since U_j is an annulus, then by symmetry of the Fatou set it follows that U_j is symmetric with respect to rotation by $2\pi/3$ and $U_j = e^{2\pi i/3}U_j$.

Then translating back to the original μ, we have that if $z \in U_j$, then $e^{2\pi i/3}z + (\mu - e^{2\pi i/3}\mu) \in U_j$. But since $\wp'_\Lambda(z) = \wp'_\Lambda(e^{2\pi i/3}z + (\mu - e^{2\pi i/3}\mu))$, $(\wp'_\Lambda)^{\circ p}$ cannot be degree one on U_j, which is a contradiction.

The proof for $n > 1$ follows in a similar fashion. □

The following proposition was proved initially for rational maps [12] and was extended to the Weierstrass elliptic function [8]. The basis of the proof is that a ramified covering of a simply connected region that has only one ramification point must be simply connected.

Proposition 2.6. *Suppose f_Λ is an elliptic function on a lattice Λ that has no Herman rings. If every Fatou component contains 0 or 1 critical values, then $J(f_\Lambda)$ is connected.*

3. Families of functions on triangular lattices

In this section, we examine a number of different families of elliptic functions on triangular lattices for which the Julia set is connected. Our approach is to find functions for which we can show that every Fatou component contains 0 or 1 critical values.

The functions described in Theorem 3.1 have at most one or two critical values that are not poles. In the case where there are two critical values lying in the Fatou set, we use symmetry properties to prove that the two critical values cannot lie in the same component.

Theorem 3.1. *Let Λ be a triangular lattice. Then the Julia set of*

(1) $f_\Lambda(z) = [4(\wp_\Lambda(z))^3 - g_3(\Lambda)]^{\frac{n}{2}}$ *for even* $n > 0$

(2) $f_\Lambda(z) = \wp'_\Lambda(z)[(4(\wp_\Lambda(z))^3 - g_3(\Lambda)]^{\frac{n-1}{2}}$ *for odd* $n > 0$

is connected.

Proof. If the Julia set is not the entire sphere, then there exists a cycle of Fatou components. Since Λ is triangular, we have that $g_2 = 0$ by Proposition 1.3 (1a). We can then use Equation 1.1 to rewrite the functions in parts (1) and (2) as $f_\Lambda(z) = (\wp'_\Lambda(z))^n$. By Theorem 2.5, f_Λ has no Herman rings.

The derivative is $f'_\Lambda(z) = n(\wp'_\Lambda(z))^{n-1}\wp''_\Lambda(z)$. Using Equation 1.3 and 1.4(1), we see that the critical points consist of the equivalence classes of the half lattice points $\lambda_j/2 + \Lambda$ for $j = 1, 2, 3$ and the centers of the equilateral triangles forming the period parallelograms $\pm\lambda_4/3 + \Lambda$, where $\lambda_4 = \lambda_1 - e^{2\pi i/3}\lambda_1$. The half lattice points $\lambda_j/2 + \Lambda$ for $j = 1, 2, 3$ are all prepoles of f_Λ and thus lie in the Julia set. Thus the only free critical points of $f_\Lambda(z)$ are the centers of the equilateral triangles forming the period parallelograms.

If n is even then, then

$$f_\Lambda\left(\frac{\lambda_4}{3}\right) = \left(\wp'_\Lambda\left(\frac{\lambda_4}{3}\right)\right)^n = \left(-\wp'_\Lambda\left(\frac{\lambda_4}{3}\right)\right)^n$$
$$= \left(\wp'_\Lambda\left(-\frac{\lambda_4}{3}\right)\right)^n = f_\Lambda\left(-\frac{\lambda_4}{3}\right),$$

so there is only one free critical value. By Proposition 2.6, $J(f_\Lambda)$ is connected.

If n is odd, let $v_1 = f_\Lambda(\lambda_4/3)$ denote one of the critical values. Then

$$v_2 = f_\Lambda\left(-\frac{\lambda_4}{3}\right) = \left(\wp'_\Lambda\left(-\frac{\lambda_4}{3}\right)\right)^n = \left(-\wp'_\Lambda\left(\frac{\lambda_4}{3}\right)\right)^n$$
$$= -\left(\wp'_\Lambda\left(\frac{\lambda_4}{3}\right)\right)^n = -f_\Lambda\left(\frac{\lambda_4}{3}\right) = -v_1,$$

and thus $v_2 = -v_1$.

We claim that v_1 and v_2 must lie in distinct Fatou components. We proceed with a proof by contradiction. Suppose v_1 and v_2 lie in the same component of the Fatou set U. Since the critical points lying at the half-lattice points are prepoles and have a finite forward orbit, there can be no Siegel disks. Thus the only periodic Fatou cycles are (super) attracting or parabolic. Let $\{U_1, U_2, \ldots, U_k\}$ denote a forward invariant cycle of components corresponding to such a cycle $\{p_1, p_2, \ldots, p_k\}$.

Some component U_j must contain one, and hence both, critical values. Then $\lim_{m\to\infty} f_\Lambda^{\circ mk}(v_1) = p_j$. Since n is an odd positive integer and $\wp'_\Lambda$ is an odd function, $\lim_{m\to\infty} f_\Lambda^{\circ mk}(v_2) = -\lim_{m\to\infty} f_\Lambda^{\circ mk}(v_1) = -p_j$. But then $p_j = -p_j = 0$, which is a pole, a contradiction. Thus no component of the Fatou set contains both critical values, and Proposition 2.6 implies that $J(f_\Lambda)$ is connected. □

Next, we investigate a family of functions for which the points z, $e^{2\pi i/3}z$, and $e^{4\pi i/3}z$ have orbits that are related. We begin with a lemma that describes the relationships between the orbits of these points.

Lemma 3.2. *Let Λ be a triangular lattice, $\varepsilon = e^{2\pi i/3}$, and define*

$$f_\Lambda(z) = \sum_{k=0}^{n} \frac{\binom{n}{k} 4^{n-k}(-g_3(\Lambda))^k(\wp_\Lambda(z))^{3n-3k-1}}{3n-3k-1}$$

for $n > 0$. Then, for any $p > 0$,

$$f_\Lambda^{\circ p}(\varepsilon z) = \begin{cases} \varepsilon^2 f_\Lambda^{\circ p}(z) & p \text{ is odd} \\ \varepsilon f_\Lambda^{\circ p}(z) & p \text{ is even} \end{cases}$$

and

$$f_\Lambda^{\circ p}(\varepsilon^2 z) = \begin{cases} \varepsilon f_\Lambda^{\circ p}(z) & p \text{ is odd} \\ \varepsilon^2 f_\Lambda^{\circ p}(z) & p \text{ is even} \end{cases}$$

Proof. We begin by proving the first statement using induction on p. Since $\varepsilon\Lambda = \Lambda$, applying Proposition 1.2, we have

$$\begin{aligned} f_\Lambda(\varepsilon z) &= \sum_{k=0}^{n} \frac{\binom{n}{k} 4^{n-k}(-g_3(\Lambda))^k(\wp_\Lambda(\varepsilon z))^{3n-3k-1}}{3n-3k-1} \\ &= \varepsilon^2 \sum_{k=0}^{n} \frac{\binom{n}{k} 4^{n-k}(-g_3(\Lambda))^k(\wp_\Lambda(z))^{3n-3k-1}}{3n-3k-1} \\ &= \varepsilon^2 f_\Lambda(z) \end{aligned}$$

and $f_\Lambda^{\circ 2}(\varepsilon z) = f_\Lambda(\varepsilon^2 f_\Lambda(z)) = \varepsilon^4 f_\Lambda^{\circ 2}(z) = \varepsilon f_\Lambda^{\circ 2}(z)$. Assume the statement is true for $q = 1, \ldots, p$. If $p+1$ is even, we have

$$f_\Lambda^{\circ p+1}(\varepsilon z) = f_\Lambda(f_\Lambda^{\circ p}(\varepsilon z)) = f_\Lambda(\varepsilon^2 f_\Lambda^{\circ p}(z)) = \varepsilon^4 f_\Lambda^{\circ p+1}(z) = \varepsilon f_\Lambda^{\circ p+1}(z).$$

If $p+1$ is odd, we have

$$f_\Lambda^{\circ p+1}(\varepsilon z) = f_\Lambda(f_\Lambda^{\circ p}(\varepsilon z)) = f_\Lambda(\varepsilon f_\Lambda^{\circ p}(z)) = \varepsilon^2 f_\Lambda^{\circ p+1}(z).$$

The second statement follows from a similar proof. □

Lemma 3.2 implies that the Julia and Fatou sets exhibit a rotational symmetry.

Theorem 3.3. *If Λ is a triangular lattice and $\varepsilon = e^{2\pi i/3}$, then $\varepsilon F(f_\Lambda) = F(f_\Lambda)$ and $\varepsilon J(f_\Lambda) = J(f_\Lambda)$.*

The following proposition explains the locations of the critical points and poles of the function f_Λ

Proposition 3.4. *Let* Λ *be a triangular lattice and*

$$f_\Lambda(z) = \sum_{k=0}^{n} \frac{\binom{n}{k} 4^{n-k}(-g_3(\Lambda))^k(\wp_\Lambda(z))^{3n-3k-1}}{3n-3k-1}.$$

Then f_Λ *is even and has critical points at half lattice points* $\lambda_j/2+\Lambda$ *for* $j=1,2,3$ *and at the centers of the equilateral triangles forming the period parallelograms* $\pm\lambda_4/3+\Lambda$. *Further, the poles of* f_Λ *are at* $\pm\lambda_4/3+\Lambda$.

Proof. Since Λ is triangular, $g_2=0$ by Proposition 1.3 (1a). We begin by noting that

$$f_\Lambda = P(\wp_\Lambda) \pm \frac{g_3^n}{\wp_\Lambda}, \tag{3.1}$$

where P is a polynomial with no constant term, so f_Λ is even. By Theorem 2.3, $f_\Lambda(z)$ has no Herman rings.

To find the critical points of $f_\Lambda(z)$, we use the Binomial Theorem to calculate

$$\begin{aligned} f'_\Lambda(z) &= \sum_{k=0}^{n} \binom{n}{k} 4^{n-k}(-g_3(\Lambda))^k(\wp_\Lambda(z))^{3n-3k-2}\wp'_\Lambda(z) \\ &= \frac{\sum_{k=0}^{n}\binom{n}{k}(4(\wp_\Lambda(z))^3)^{n-k}(-g_3(\Lambda))^k\wp'_\Lambda(z)}{(\wp_\Lambda(z))^2} \\ &= \frac{(4(\wp_\Lambda(z))^3 - g_3(\Lambda))^n\wp'_\Lambda(z)}{(\wp_\Lambda(z))^2} \\ &= \frac{(\wp'_\Lambda(z))^{2n+1}}{(\wp_\Lambda(z))^2}, \end{aligned}$$

where the last line follows from Equation 1.1 and $g_2=0$.

Using Equation 1.3 and Proposition 1.3 (1c), $\wp'_\Lambda(z)=0$ at any half lattice point $\lambda_j/2+\Lambda$ for $j=1,2,3$ and the centers of the equilateral triangles forming the period parallelograms $\pm\lambda_4/3+\Lambda$. By Proposition 1.3 (1c), $\wp_\Lambda$ is nonzero at any half lattice point, and so every half lattice point of Λ is a critical point of f_Λ. Both $\wp_\Lambda$ and $\wp'_\Lambda$ are zero at $\pm\lambda_4/3+\Lambda$, but

$$\lim_{z\to\pm\lambda_3/3} \frac{(\wp'_\Lambda(z))^{2n+1}}{(\wp_\Lambda(z))^2} = 0,$$

so $\pm\lambda_4/3+\Lambda$ are critical points of f_Λ. Thus the critical points of f_Λ are

$$Crit(f_\Lambda) = \left\{\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2}, \pm\frac{\lambda_4}{3}\right\} + \Lambda.$$

By Proposition 1.3 (1c) and Equation 3.1, we see that $\pm\lambda_4/3 + \Lambda$ are poles of f_Λ and thus lie in the Julia set. □

In addition to the rotational symmetry of the Julia and Fatou sets described in Theorem 3.3, we also see symmetry with respect to the centers of the equilateral triangles determined by the lattice Λ.

Corollary 3.5. *If μ is a center of an equilateral triangle determined by the lattice Λ, then the Julia and Fatou sets of f_Λ are symmetric with respect to rotation around μ by $2\pi/3$.*

Proof. Let $\Lambda = [\lambda_1, \varepsilon\lambda_1]$, where $\varepsilon = e^{2\pi i/3}$, and let $s = \frac{1}{3}(\lambda_1 - \varepsilon\lambda_1)$ be a pole. Let $z = s + b$. Then the rotation of z around s by $2\pi/3$ is $y = s + \varepsilon b = \varepsilon(z - \lambda_1)$. Using Theorems 2.1 and 3.3, $z \in F(g_\Omega)$ if and only if $\varepsilon(z - \lambda_1) = y \in F(g_\Omega)$. Using Theorem2.1, Proposition 3.4, and Theorem 2.2, this symmetry passes to all poles $\mu = \pm s + \Lambda$. □

We use Lemma 3.2 to prove that functions in this family have connected Julia sets. The symmetries in the orbits of any three points of the form z, $e^{2\pi i/3}z$, and $e^{4\pi i/3}z$ eliminates the possibility that two critical values can lie in the same Fatou component.

Theorem 3.6. *Let Λ be a triangular lattice. Then the Julia set of*

$$f_\Lambda(z) = \sum_{k=0}^{n} \frac{\binom{n}{k} 4^{n-k}(-g_3(\Lambda))^k(\wp_\Lambda(z))^{3n-3k-1}}{3n-3k-1}$$

for $n > 0$ is connected.

Proof. By Proposition 3.4, $\pm\lambda_4/3 + \Lambda$ are poles of f_Λ. First, we show that f_Λ has no Herman rings, using an argument similar to that of the proof of Theorem 2.5. Again, we let $\{U_0, U_1, \ldots, U_{p-1}\}$ denote a cycle of Herman rings of period $p \geq 1$, γ be a $(f_\Lambda)^{\circ p}$ invariant leaf of U_i, and B_γ denote the bounded component of the complement of γ. Since U_i is multiply connected, we know that B_γ contains a prepole. In this case, there is a smallest nonnegative number n such that $(f_\Lambda)^{\circ n}(\gamma)$ contains a pole $\mu = \frac{1}{3}(\lambda_1 - \varepsilon\lambda_1) + \Lambda$ in $B_{(f_\Lambda)^{\circ n}(\gamma)}$. Let U_j denote the Herman ring $(f_\Lambda)^{\circ n}(U_i)$. Using Theorem 2.1, Theorem 2.2, Lemma 3.2, Proposition 3.4, and Corollary 3.5, if $z \in U_j$, then $e^{2\pi i/3}z + (\mu - e^{2\pi i/3}\mu) \in U_j$. But since $f_\Lambda(z) = f_\Lambda(e^{2\pi i/3}z + (\mu - e^{2\pi i/3}\mu))$, $(f_\Lambda)^{\circ p}$ cannot be degree one on U_j, which is a contradiction.

Using Proposition 3.4, there are three remaining equivalence classes of critical points $\lambda_i/2 + \Lambda$ for $i = 1, 2, 3$. We claim that these critical points have related critical values $v_i = f(\lambda_i/2)$. Since Λ is triangular, $\varepsilon(\lambda_i/2) + \Lambda = \lambda_{(i+1 \mod 3)}/2 + \Lambda$ where $\varepsilon = e^{2\pi i/3}$. By Lemma 3.2 and

Proposition 1.3 (1b), the three critical values of f_Λ on a triangular lattice satisfy the relationship $v_2 = \varepsilon v_1 = f_\Lambda(\lambda_1/2)$ and $v_3 = \varepsilon v_2$.

We claim that no Fatou component can contain more than one critical value. Let $\{U_1, U_2, \ldots, U_s\}$ denote a forward invariant cycle of attracting or parabolic Fatou components corresponding to a cycle $\{p_1, p_2, \ldots, p_s\}$. First, suppose two non-zero critical values, say v_i and εv_i, lie in the same component U_j. Then $\lim_{k\to\infty} f_\Lambda^{\circ ks}(v_i) = p_j$ and $\lim_{k\to\infty} f_\Lambda^{\circ ks}(\varepsilon v_i) = p_j$. But by Lemma 3.2,

$$f_\Lambda^{\circ ks}(\varepsilon v_i) = \begin{cases} \varepsilon^2 f_\Lambda^{\circ ks}(v_i) & kn \text{ is odd} \\ \varepsilon f_\Lambda^{\circ ks}(v_i) & kn \text{ is even.} \end{cases}$$

Therefore if εv_i lies in U_j, then $p_j = \varepsilon p_j$ or $p_j = \varepsilon^2 p_j$, so $p_j = 0$. But 0 is a pole, contradicting our assumption.

Therefore, no Fatou component can contain more than one critical value, so the Julia set is connected by Proposition 2.6. □

4. Families of functions on square lattices

In this section, we find families of elliptic functions on square lattices with connected Julia sets. Our approach is similar to that of Section 3 in that we prove that no Fatou component can contain more than one critical value.

We begin with families of functions of the form $f_\Lambda(z) = [\wp_\Lambda(z)]^n$ for $n > 0$ on square lattices. When n is even we show that at most one critical value lies in the Fatou set. For the case when n is odd, we extend the proof used in [3] on $\wp_\Lambda$ to the functions $f_\Lambda(z) = [\wp_\Lambda(z)]^n$ and show that there are at most two non-pole critical values which must lie in separate Fatou components.

Theorem 4.1. *Let Λ be a square lattice. Then the Julia set of $f_\Lambda(z) = [\wp_\Lambda(z)]^n$ for $n > 0$ is connected.*

Proof. If the Julia set is not the entire sphere, then there exists a cycle of Fatou components. By Theorem 2.3, f_Λ has no Herman rings because f_Λ is even and has poles at lattice points. The critical points are found by solving $0 = f'_\Lambda(z) = n(\wp_\Lambda(z))^{n-1}\wp'_\Lambda(z)$. So the critical points are the roots of $\wp_\Lambda$ and $\wp'_\Lambda$, which are the half lattice points by Equation 1.3 and Proposition 1.3(2d).

By Proposition 1.3(2b), $f_\Lambda(\lambda_3/2) = 0$; this critical value is a pole for all $n > 0$ and thus lies in the Julia set. The other two critical values are

$$f_\Lambda\left(\frac{\lambda_1}{2}\right) = \left(\wp_\Lambda\left(\frac{\lambda_1}{2}\right)\right)^n = e_1^n$$

and

$$f_\Lambda\left(\frac{\lambda_2}{2}\right) = \left(\wp_\Lambda\left(\frac{\lambda_2}{2}\right)\right)^n = e_2^n = (-e_1)^n.$$

By Proposition 1.3(2b), for even n we have $e_2^n = (-e_1)^n = e_1^n$. Thus there is only one critical value in the Fatou set, so Proposition 2.6 implies that $J(f_\Lambda)$ is connected.

For odd n, we claim that both critical values cannot lie in the same component of the Fatou set. We prove this by contradiction, assuming that a Fatou component U contains both e_1^n and e_2^n. Clearly, U must be in the immediate basin of an attracting or parabolic cycle $\{p_0, p_1, \ldots, p_{l-1}\}$. By definition, U is path connected, so let C be a curve connecting e_1^n and $e_2^n = (-e_1)^n$ in U.

Since C is a compact subset of the immediate basin, $\{f_\Lambda^{\circ lk}\}$ converges uniformly on C to the constant function p_0 in the Euclidean metric. Thus for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $\sup_{z\in C} |f_\Lambda^{\circ lk}(z) - p_0| < \epsilon$ for $k > N$.

Next, we construct four sets L_1, L_2, S_1, and S_2 that we will use to provide the contradiction. Fixing the generator $\lambda \in \mathbb{C} \setminus \{0\}$ of the lattice $\Lambda = [\lambda, \lambda i]$, define $L_1 = \{t\lambda^{-2n} \colon t \in \mathbb{R}^+\}$ and $L_2 = -L_1$. Further, define $S_1 = \{t\lambda \colon t \in \mathbb{R}\setminus\mathbb{Z}\}$ and $S_2 = iS_1$. We note that any curve A connecting $z_1 \in L_1$ and $z_2 \in L_2$ must intersect every line that passes through the origin.

Fix Γ to be the lattice $\Gamma = [1, i]$. The critical value of the real critical point $1/2$ of $\wp_\Gamma$ on this lattice is $\wp_\Gamma(1/2) = \gamma^2 \approx (2.62206\ldots)^2$ (see [4]). Proposition 1.3 (2c) implies that $\gamma^2 > 0$ is the minimum of $\wp_\Gamma$ on $\mathbb{R}$.

If $z \in S_1$, then write $z = t\lambda$ for some $t \in \mathbb{R}\setminus\mathbb{Z}$. Then Proposition 1.2 implies

$$f_\Lambda(t\lambda) = (\wp_\Lambda(t\lambda))^n = (\lambda^{-2}\wp_\Gamma(t))^n = \lambda^{-2n}(\wp_\Gamma(t))^n = \lambda^{-2n} f_\Gamma(t).$$

But $f_\Gamma(t) = (\wp_\Gamma(t))^n \geq (\gamma^2)^n$, and thus $f_\Lambda(t\lambda) \in L_1$. Thus $f_\Lambda(S_1) \subset L_1$.

If $z \in S_2$, then write $z = it\lambda$ for some $t \in \mathbb{R}\setminus\mathbb{Z}$. We have $i\Gamma = \Gamma$, so Proposition 1.2 implies

$$f_\Lambda(it\lambda) = (\wp_\Lambda(it\lambda))^n = ((i\lambda)^{-2}\wp_\Gamma(t))^n = -\lambda^{-2n}(\wp_\Gamma(t))^n = -\lambda^{-2n} f_\Gamma(t).$$

Again, since $f_\Gamma(t) > 0$, we have $f_\Lambda(S_2) \subset L_2$.

We claim that for each $m \in \mathbb{N}$, $f_\Lambda(C)$ contains a point on each of L_1, L_2, S_1, and S_2 and is connected. We use induction to show this.

When $m = 0$, we note that C contains the points e_1^n on L_1 and e_2^n on L_2. Then $\overline{S_1}$ and $\overline{S_2}$ are lines passing through the origin. Since C is in the Fatou set, it contains no poles, and so $C \cap S_1 \neq \emptyset$ and $C \cap S_2 \neq \emptyset$.

For the induction hypothesis, assume $f_\Lambda^{\circ m}(C)$ contains a point on each of L_1, L_2, S_1, and S_2. Since all iterates are defined on the Fatou set, $f_\Lambda^{\circ m+1}(C)$ is connected. Since $f_\Lambda^{\circ m}(C)$ contains points on S_1 and S_2, and $f_\Lambda(S_1) \subset L_1$ and $f_\Lambda(S_2) \subset L_2$, $f_\Lambda^{\circ m+1}(C)$ contains points on L_1 and L_2. Since $\overline{S_1}$ and $\overline{S_2}$ are lines passing through the origin, $f_\Lambda^{\circ m+1}(C)$ contains points on S_1 and S_2.

If $p_0 \notin L_2$, choose $a_k \in f_\Lambda^{\circ lk}(C) \cap L_2$ for each $k \in \mathbb{N}$ (if $p_0 \in L_2$, then choose $a_k \in L_1$). We have that

$$\sup_{z\in C} |f_\Lambda^{\circ lk}(z) - p_0| > |a_k - p_0| > \max\{\inf_{z\in L_1} d(z,p_0), \inf_{z\in L_2} d(z,p_0)\} > 0,$$

for all $k \in \mathbb{C}$, where d is the Euclidean metric on $\mathbb{C}$. This contradicts the uniform convergence of $\{f_\Lambda^{\circ lk}\}_{k\in\mathbb{N}}$ on C to the constant function p_0.

Thus every Fatou component contains at most one critical value, and Proposition 2.6 implies that $J(f_\Lambda)$ is connected. □

To prove the following theorem, we show that the functions have only one critical value that could belong to the Fatou set, and thus the Julia set is connected.

Theorem 4.2. *Let Λ be a square lattice. Then the Julia set of*

$$f_\Lambda(z) = [4(\wp_\Lambda(z))^3 - g_2(\Lambda)\wp_\Lambda(z)]^{2n}$$

for $n > 0$ is connected.

Proof. If the Julia set is not the entire sphere, then there exists a cycle of Fatou components. Since Λ is square, we have that $g_3 = 0$ by Proposition 1.3 (2a). We can then use Equation 1.1 to rewrite the functions as

$$f_\Lambda(z) = (4(\wp_\Lambda(z))^3 - g_2(\Lambda)\wp_\Lambda(z))^{2n} = (\wp_\Lambda'(z))^{4n}$$

for $n > 0$. Then f_Λ is even and has poles at lattices points, and thus has no Herman rings by Theorem 2.3.

The derivative is $f_\Lambda'(z) = 4n(\wp_\Lambda'(z))^{4n-1}\wp_\Lambda''(z)$. So the critical points of f_Λ are either zeros of $\wp_\Lambda'$ or $\wp_\Lambda''$. The zeros of $\wp_\Lambda'$ are the half lattice points $\lambda_j/2$ for $j = 1,2,3$ by Equation 1.3, so the half lattice points are critical points of f_Λ. As these half lattice points are all zeros of $\wp_\Lambda'$, they are prepoles of f_Λ and thus lie in the Julia set. The zeros of $\wp_\Lambda''$ are the critical points c of $\wp_\Lambda'$, and we know by Proposition 1.4(2) that any critical point c of $\wp_\Lambda'$ lands on one the four critical values v_1, v_2, v_3, v_4 where

$$v_1 = \sqrt{\left(\frac{g_2}{3}\right)^{\frac{3}{2}}}, v_2 = -v_1, v_3 = iv_1, v_4 = -iv_1.$$

However, $f_\Lambda(c) = (\wp_\Lambda'(c))^{4n} = (v_j)^{4n} = (v_1)^{4n}$ for $j = 2,3,4$, so f_Λ has only one critical value arising from any such a critical point c. Since there can be at most one critical value in the Fatou set, Proposition 2.6 implies that $J(f_\Lambda)$ is connected. □

For the next family of functions under investigation, we again show that f_Λ can have at most one critical value lying in the Fatou set.

Theorem 4.3. *Let Λ be a square lattice. The Julia set of*

$$f_\Lambda(z) = \sum_{k=0}^{m} \frac{\binom{m}{k} 4^{m-k}(-g_2(\Lambda))^k (\wp_\Lambda(z))^{3m-2k+n+1}}{3m-2k+n+1}$$

for $m, n > 0$, where m and n have opposite parity, is connected.

Proof. Since m and n have opposite parity, we have that every term in $f_\Lambda(z)$ is a positive, even power of $\wp_\Lambda$. Thus $f_\Lambda(z)$ is even and has poles at lattice points. Therefore, $f_\Lambda(z)$ has no Herman rings by Theorem 2.3.

We claim that the critical points of $f_\Lambda(z)$ are the half lattice points $\lambda_j/2$ for $j = 1, 2, 3$. We have

$$\begin{aligned} f'_\Lambda(z) &= \left(\sum_{k=0}^{m} \binom{m}{k} 4^{m-k}(-g_2(\Lambda))^k (\wp_\Lambda(z))^{3m-2k} \right) (\wp_\Lambda)^n \wp'_\Lambda \\ &= \left(\sum_{k=0}^{m} \binom{m}{k} (4(\wp_\Lambda)^3)^{m-k} (-g_2(\Lambda)\wp_\Lambda)^k \right) (\wp_\Lambda)^n \wp'_\Lambda \\ &= (4(\wp_\Lambda)^3 - g_2(\Lambda)\wp_\Lambda)^m (\wp_\Lambda)^n \wp'_\Lambda, \end{aligned}$$

by the Binomial Theorem. Using Equation 1.1 and Proposition 1.3 (2a), we can write $f'_\Lambda(z) = (\wp_\Lambda)^n(\wp'_\Lambda)^{2m+1}$. Since Λ is square, the critical points are the half lattice points by Equation 1.3 and Proposition 1.3(2b).

Since every term in $f_\Lambda(z)$ is a positive, even power of $\wp_\Lambda$, $f_\Lambda(\lambda_3/2) = 0$ by Proposition 1.3 (2b), so $\lambda_3/2$ is a prepole and lies in the Julia set. Using Proposition 1.3 (2b), $f_\Lambda(\lambda_1/2) = f_\Lambda(\lambda_2/2)$, so there is only one critical value of $f_\Lambda(z)$ that could possibly lie in the Fatou set. By Proposition 2.6, the Julia set is connected. □

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(Clemons) Department of Mathematics; 460 McBryde Hall; Virginia Tech; 225 Stanger Street; Blacksburg, VA 24061-0123

E-mail address: jclemons@vt.edu

(Koss) Department of Mathematics and Computer Science; Dickinson College; P.O. Box 1773; Carlisle, Pennsylvania 17013

E-mail address: koss@dickinson.edu