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Mikhail Patrakeev

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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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WHEN THE PROPERTY OF HAVING A π-TREE IS PRESERVED BY PRODUCTS

MIKHAIL PATRAKEEV

Abstract. We find sufficient conditions under which the product of spaces that have a π-tree also has a π-tree. These conditions give new examples of spaces with a π-tree: every at most countable power of the Sorgenfrey line and every at most countable power of the irrational Sorgenfrey line has a π-tree. Also we show that if a space has a π-tree, then its product with the Baire space, with the Sorgenfrey line, and with the countable power of the Sorgenfrey line also has a π-tree.

1. Introduction

We study topological spaces that have a π-tree, see Terminology 2.5 in §2. The notion of a π-tree was introduced in [10] and is equivalent [10, Remark 11] to the notion of a Lusin π-base, which was introduced in [8]. The Sorgenfrey line $\mathcal{R}_S$ and the Baire space $\mathcal{N}$ (that is, ${}^{\omega}\omega$ with the product topology) are examples of spaces with a π-tree [8]. Every space that has a π-tree shares many good properties with the Baire space. One reason for this is expressed in Lemma 2.6 and Lemma 3.2; another two are the following: If a space X has a π-tree, then X can be mapped onto $\mathcal{N}$ by a continuous one-to-one map [8] and also X can be mapped onto $\mathcal{N}$ by a continuous open map [8] (hence, X can be mapped by a continuous open map onto an arbitrary Polish space, see [1] or [6, Exercise 7.14]). Every space that has a π-tree also has a countable π-base, see Lemma 2.7.

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In this paper we study the following question: When does the product of spaces that have a π-tree also have a π-tree? We find several kinds of conditions (see Theorem 4.1, Theorem 5.1, and Corollary 4.2) under which an at most countable product of spaces that have a π-tree also has a π-tree. We consider only at most countable products because an uncountable product of spaces that have a π-tree has an uncountable pseudocharacter; therefore, it has no π-tree (see [5, 5.3.b)] and lemmas 2.6, 2.7, and 3.2).

The above results give new examples of spaces that have a π-tree, see Section 7. For instance, Corollary 7.2 asserts that if $1 \leqslant |A| \leqslant \omega$ and for each $\alpha \in A$,

$$\text{either} \quad X_\alpha = \mathcal{N} \quad \text{or} \quad X_\alpha \subseteq \mathcal{R_S} \text{ with } \mathcal{R_S} \setminus X_\alpha \text{ at most countable,}$$

then the product $\prod_{\alpha \in A} X_\alpha$ has a π-tree. In particular, the powers ${\mathcal{R_S}}^n$ and ${\mathcal{I_S}}^n$ ($\mathcal{I_S}$ denotes the irrational Sorgenfrey line $\mathcal{R_S} \setminus \mathbb{Q}$) have a π-tree for all natural $n \geqslant 1$, and the powers ${\mathcal{R_S}}^\omega$ and ${\mathcal{I_S}}^\omega$ also have a π-tree. (Note that no finite power of the irrational Sorgenfrey line is homeomorphic to finite power of the Sorgenfrey line [2].) Other examples of spaces with a π-tree can be obtained by using Corollary 7.4, which says that if a space X has a π-tree, then the products $X \times \mathcal{N}$, $X \times \mathcal{R_S}$, and $X \times {\mathcal{R_S}}^\omega$ also have a π-tree.

2. Notation and Terminology

We use standard set-theoretic notation from [4] and [7]. In particular, each ordinal is equal to the set of smaller ordinals, $\omega =$ the set of natural numbers = the set of finite ordinals = the first limit ordinal = the first infinite cardinal, and $n = \{0, \dots, n-1\}$ for all $n \in \omega$. A *space* is a topological space; we use terminology from [3] when we work with spaces. Also we use the following notation.

Terminology 2.1. The symbol $:=$ means "equals by definition"; the symbol $:\longleftrightarrow$ is used to show that an expression on the left side is an abbreviation for expression on the right side;

- ✎ $x \subset y \quad :\longleftrightarrow \quad x \subseteq y$ and $x \neq y$;
- ✎ $A \equiv \bigsqcup_{\lambda \in \Lambda} B_\lambda \quad :\longleftrightarrow$
  $A = \bigcup_{\lambda \in \Lambda} B_\lambda$ and $\forall \lambda, \lambda' \in \Lambda \, [\lambda \neq \lambda' \to B_\lambda \cap B_{\lambda'} = \varnothing]$;
- ✎ $[A]^\kappa := \{B \subseteq A : |B| = \kappa\}, \quad [A]^{<\kappa} := \{B \subseteq A : |B| < \kappa\}$
  (here κ is a cardinal);
- ✎ γ has the $\mathsf{FIP} \quad :\longleftrightarrow \quad \forall \delta \in [\gamma]^{<\omega} \setminus \{\varnothing\} \, \left[\bigcap \delta \neq \varnothing\right]$
  (FIP means the Finite Intersection Property);
- ✎ $\mathsf{cofin}\, A := \{A \setminus F : F \in [A]^{<\omega}\}$;

✎ $\mathsf{nbhds}(p, X)$:= the set of (not necessarily open) neighbourhoods of point p in space X;
✎ $f \restriction A$:= the restriction of function f to A;
✎ $\gamma \gg \delta \ :\longleftrightarrow\ \gamma$ *π-refines* $\delta \ :\longleftrightarrow\ \forall D \in \delta \setminus \{\varnothing\}\ \exists G \in \gamma \setminus \{\varnothing\}\ [G \subseteq D]$.

When we work with (transfinite) sequences, we use the following notation.

Terminology 2.2. Suppose $n \in \omega$ and s, t are sequences; that is, s and t are functions whose domain is an ordinal.

✎ $\mathsf{length}\, s$:= the domain of s;
✎ note that $s \subset t$ iff $\mathsf{length}\, s < \mathsf{length}\, t$ and $s = t \restriction \mathsf{length}\, s$;
✎ $\langle r_0, \ldots, r_{n-1} \rangle$:= the sequence r such that $\mathsf{length}\, r = n$ and $r(i) = r_i$ for all $i \in n$;
✎ $\langle\rangle$:= the sequence of length 0;
✎ $\langle r_0, \ldots, r_{n-1} \rangle ^\frown \langle s_0, \ldots, s_{m-1} \rangle := \langle r_0, \ldots, r_{n-1}, s_0, \ldots, s_{m-1} \rangle$;
✎ ${}^B A$:= the set of functions from B to A; in particular, ${}^0 A = \{\langle\rangle\}$;
✎ ${}^{<\alpha} A := \bigcup_{\beta \in \alpha} {}^\beta A$ (here α is an ordinal).

Also we work with partial orders and then we use the following terminology:

Terminology 2.3. Suppose $\mathcal{P} = (Q, \triangleleft)$ is a strict partial order; that is, $\triangleleft$ is irreflexive and transitive on Q. Let $x, y \in Q$ and $A \subseteq Q$.

✎ $\mathsf{nodes}\,\mathcal{P} = \mathsf{nodes}(Q, \triangleleft) := Q$;
✎ $x <_{\mathcal{P}} y \quad :\longleftrightarrow \quad x \triangleleft y$;
✎ $x \leqslant_{\mathcal{P}} y \quad :\longleftrightarrow \quad x <_{\mathcal{P}} y$ or $x = y$;
✎ $x{\uparrow^{\circ}}_{\mathcal{P}} := \{v \in \mathsf{nodes}\,\mathcal{P} : v <_{\mathcal{P}} x\}$, $\quad x{\downarrow_{\circ}}_{\mathcal{P}} := \{v \in \mathsf{nodes}\,\mathcal{P} : v >_{\mathcal{P}} x\}$;
✎ $x{\uparrow^{\bullet}}_{\mathcal{P}} := \{v \in \mathsf{nodes}\,\mathcal{P} : v \leqslant_{\mathcal{P}} x\}$, $\quad x{\downarrow_{\bullet}}_{\mathcal{P}} := \{v \in \mathsf{nodes}\,\mathcal{P} : v \geqslant_{\mathcal{P}} x\}$;
✎ $A{\overline{\uparrow}}_{\mathcal{P}} := \bigcup\{v{\uparrow^{\bullet}}_{\mathcal{P}} : v \in A\}$, $\quad A{\underline{\downarrow}}_{\mathcal{P}} := \bigcup\{v{\downarrow_{\bullet}}_{\mathcal{P}} : v \in A\}$;
✎ $\mathsf{sons}_{\mathcal{P}}(x) := \{s \in \mathsf{nodes}\,\mathcal{P} : x <_{\mathcal{P}} s$ and $x{\downarrow_{\circ}}_{\mathcal{P}} \cap s{\uparrow^{\circ}}_{\mathcal{P}} = \varnothing\}$;
✎ A is a *chain* in $\mathcal{P} \quad :\longleftrightarrow \quad \forall v, w \in A\, [v \leqslant_{\mathcal{P}} w$ or $v >_{\mathcal{P}} w]$;
✎ $\mathcal{P}$ has *bounded chains* $\quad :\longleftrightarrow \quad$ for each nonempty chain C in $\mathcal{P}$, there is $v \in \mathsf{nodes}\,\mathcal{P}$ such that $C \subseteq v{\uparrow^{\bullet}}_{\mathcal{P}}$;
✎ $\max \mathcal{P} := \{m \in \mathsf{nodes}\,\mathcal{P} : m{\downarrow_{\circ}}_{\mathcal{P}} = \varnothing\}$;
✎ $\min \mathcal{P} := \{m \in \mathsf{nodes}\,\mathcal{P} : m{\uparrow^{\circ}}_{\mathcal{P}} = \varnothing\}$;
✎ $0_{\mathcal{P}}$:= the node such that $(0_{\mathcal{P}}){\downarrow_{\bullet}}_{\mathcal{P}} = \mathsf{nodes}\,\mathcal{P}$
(here $\mathcal{P}$ is a partial order that has such node).

When a partial order is a (set-theoretic) tree, we use the following terminology:

Terminology 2.4. Suppose $\mathcal{T}$ is a tree; that is, $\mathcal{T}$ is a strict partial order such that for each $x \in \mathsf{nodes}\,\mathcal{T}$, the set $x{\uparrow^{\circ}}_{\mathcal{T}}$ is well-ordered by $<_{\mathcal{T}}$. Let $x \in \mathsf{nodes}\,\mathcal{T}$, let α be an ordinal, and let κ be a cardinal.

✎ $\mathsf{height}_{\mathcal{T}}(x) :=$ the ordinal isomorphic to $(x{\downarrow}_{\mathcal{T}}, <_{\mathcal{T}})$;
✎ $\mathsf{level}_{\mathcal{T}}(\alpha) := \big\{v \in \mathsf{nodes}\,\mathcal{T} : \mathsf{height}_{\mathcal{T}}(v) = \alpha\big\}$;
✎ $\mathsf{height}\,\mathcal{T} :=$ the minimal ordinal β such that $\mathsf{level}_{\mathcal{T}}(\beta) = \varnothing$;
✎ B is a *branch* in $\mathcal{T}$ $:\longleftrightarrow$ B is a $\subseteq$-maximal chain in $\mathcal{T}$;
✎ $\mathcal{T}$ is *κ-branching* $:\longleftrightarrow$ $\forall v \in \mathsf{nodes}\,\mathcal{T} \smallsetminus \max \mathcal{T}\ \big[|\mathsf{sons}_{\mathcal{T}}(v)| = \kappa\big]$.

Finally, we work with foliage trees, which where introduced in [10]. Recall that a *foliage tree* is a pair $\mathbf{F} = (\mathcal{T}, l)$ such that $\mathcal{T}$ is a tree and l is a function with $\mathsf{domain}\,l = \mathsf{nodes}\,\mathcal{T}$. For each $x \in \mathsf{nodes}\,\mathcal{T}$, the $l(x)$ is called the *leaf* of $\mathbf{F}$ at node X and is denoted by $\mathbf{F}_x$; the tree $\mathcal{T}$ is called the *skeleton* of $\mathbf{F}$ and is denoted by $\mathsf{skeleton}\,\mathbf{F}$. We adopt the following convention: If $\mathbf{F}$ is a foliage tree and $\bullet$ is a notation that can be applied to a tree, then $\bullet(\mathbf{F})$ is an abbreviation for $\bullet(\mathsf{skeleton}\,\mathbf{F})$; for example, $x <_{\mathbf{F}} y$ stands for $x <_{\mathsf{skeleton}\,\mathbf{F}} y$. Also we use the following terminology:

Terminology 2.5. Suppose $\mathbf{F}$ is a foliage tree, $v \in \mathsf{nodes}\,\mathbf{F}$, $A \subseteq \mathsf{nodes}\,\mathbf{F}$, X is a space, α is an ordinal, and κ is a cardinal.

✎ $\mathsf{flesh}\,\mathbf{F} := \bigcup\{\mathbf{F}_x : x \in \mathsf{nodes}\,\mathbf{F}\}$;
✎ $\mathsf{flesh}_{\mathbf{F}}(A) := \bigcup\{\mathbf{F}_x : x \in A\}$;
✎ $\mathsf{shoot}_{\mathbf{F}}(v) := \big\{\mathsf{flesh}_{\mathbf{F}}(C) : C$ is a cofinite subset of $\mathsf{sons}_{\mathbf{F}}(v)\big\}$;
✎ $\mathsf{scope}_{\mathbf{F}}(a) := \{x \in \mathsf{nodes}\,\mathbf{F} : \mathbf{F}_x \ni a\}$;
✎ $\mathbf{F}$ has *nonempty leaves* $:\longleftrightarrow$ $\forall x \in \mathsf{nodes}\,\mathbf{F}\ [\mathbf{F}_x \neq \varnothing]$;
✎ $\mathbf{F}$ is *nonincreasing* $:\longleftrightarrow$ $\forall x, y \in \mathsf{nodes}\,\mathbf{F}\ [y \geqslant_{\mathbf{F}} x \to \mathbf{F}_y \subseteq \mathbf{F}_x]$;
✎ $\mathbf{F}$ has *strict branches* $:\longleftrightarrow$ $\mathsf{nodes}\,\mathbf{F} \neq \varnothing$ and for each branch B in $\mathbf{F}$, the $\bigcap_{x\in B} \mathbf{F}_x$ is a singleton;
✎ $\mathbf{F}$ is *locally strict* $:\longleftrightarrow$ $\forall x \in \mathsf{nodes}\,\mathbf{F} \smallsetminus \max \mathbf{F}\ [\mathbf{F}_x \equiv \bigsqcup_{s\in\mathsf{sons}_{\mathbf{F}}(x)} \mathbf{F}_s]$;
✎ $\mathbf{F}$ is *open* in X $:\longleftrightarrow$ $\forall z \in \mathsf{nodes}\,\mathbf{F}\ [\mathbf{F}_z$ is an open subset of $X]$;
✎ $\mathbf{F}$ is a *foliage* α, κ*-tree* $:\longleftrightarrow$
$\mathsf{skeleton}\,\mathbf{F}$ is isomorphic to the tree $({}^{<\alpha}\kappa, \subset)$;
✎ $\mathbf{F}$ is a *Baire foliage tree* on X $:\longleftrightarrow$ $\mathbf{F}$ is an open in X locally strict foliage ω, ω-tree with strict branches such that $\mathbf{F}_{0_{\mathbf{F}}} = X$;
✎ $\mathbf{F}$ *grows into* X $:\longleftrightarrow$
$\forall p\in X\ \forall U\in \mathsf{nbhds}(p, X)\ \exists z\in \mathsf{scope}_{\mathbf{F}}(p)\ \big[\mathsf{shoot}_{\mathbf{F}}(z) \gg \{U\}\big]$;
✎ $\mathbf{F}$ is a *π-tree* on X $:\longleftrightarrow$
$\mathbf{F}$ is a Baire foliage tree on X and $\mathbf{F}$ grows into X;
✎ $\mathbf{S}$:= the *standard foliage tree* of ${}^{\omega}\omega$:= the foliage tree such that
  ➢ $\mathsf{skeleton}\,\mathbf{S} := ({}^{<\omega}\omega, \subset)$ and
  ➢ $\mathbf{S}_x := \{p \in {}^{\omega}\omega : x \subseteq p\}$ for every $x \in {}^{<\omega}\omega$;
✎ $\mathcal{N}$:= the *Baire space* :=
the space $({}^{\omega}\omega, \tau_{\mathcal{N}})$, where $\tau_{\mathcal{N}}$ is the Tychonoff product topology with ω carrying the discrete topology.

Lemma 2.6 ([10, Lemma 13])**.**
(a) $\{\mathbf{S}_x : x \in {}^{<\omega}\omega\}$ *is a base for* $\mathcal{N}$.
(b) $\mathbf{S}$ *is a* π*-tree on* $\mathcal{N}$.

(c) $\mathbf{S}$ *is a Baire foliage tree on a space* $({}^{\omega}\omega,\tau)$ iff $\tau \supseteq \tau_{\mathcal{N}}$.

Lemma 2.7. *If* $\mathbf{F}$ *is a* π*-tree on a space* X*, then*

- $\{\mathbf{F}_v : v \in \mathsf{nodes}\,\mathbf{F}\}$ *is a countable* π*-base for* X*,*
- *each* $\mathbf{F}_v$ *is closed-and-open in* X*, and*
- $\bigcap\{\mathbf{F}_v : \mathbf{F}_v \ni p\} = \{p\}$ *for all* $p \in X$.

3. New Notions: Isomorphism and Spectrum

The notion of isomorphism between foliage trees allows to simplify proofs (see the proof of Theorem 4.1) in the following way: When we have a π-tree $\mathbf{F}$ on a space X, we may (by using (c) of Lemma 3.2 and (c) of Lemma 2.6) assume "without loss of generality" that $\mathbf{F} = \mathbf{S}$ and $X = ({}^{\omega}\omega,\tau)$ with $\tau \supseteq \tau_{\mathcal{N}}$.

Definition 3.1. An **isomorphism** between foliage trees $\mathbf{F}$ and $\mathbf{G}$ is a pair (φ,ψ) such that

- φ is an order isomorphism from $\mathsf{skeleton}\,\mathbf{F}$ onto $\mathsf{skeleton}\,\mathbf{G}$,
- ψ is a bijection from $\mathsf{flesh}\,\mathbf{F}$ onto $\mathsf{flesh}\,\mathbf{G}$, and
- $\psi[\mathbf{F}_x] = \mathbf{G}_{\varphi(x)}$ for all $x \in \mathsf{nodes}\,\mathbf{F}$.

Lemma 3.2. *Suppose that* $\mathbf{F}$ *is a foliage tree and* X *is a space.*

(a) $\mathbf{F}$ *is a locally strict foliage* ω,ω*-tree with strict branches* iff $\mathbf{F}$ *is isomorphic to* $\mathbf{S}$.

(b) $\mathbf{F}$ *is a Baire foliage tree on* X iff *there exist an isomorphism* (φ,ψ) *between* $\mathbf{F}$ *and* $\mathbf{S}$ *and a topology* τ *on* ${}^{\omega}\omega$ *such that*
  - ψ *is a homeomorphism from* X *onto* $({}^{\omega}\omega,\tau)$ *and*
  - $\mathbf{S}$ *is a Baire foliage tree on* $({}^{\omega}\omega,\tau)$.

(c) $\mathbf{F}$ *is a* π*-tree on* X iff *there exist an isomorphism* (φ,ψ) *between* $\mathbf{F}$ *and* $\mathbf{S}$ *and a topology* τ *on* ${}^{\omega}\omega$ *such that*
  - ψ *is a homeomorphism from* X *onto* $({}^{\omega}\omega,\tau)$ *and*
  - $\mathbf{S}$ *is a* π*-tree on* $({}^{\omega}\omega,\tau)$.

Proof. (a) Suppose that $\mathbf{F}$ is a locally strict foliage ω,ω-tree with strict branches. Let φ be an order isomorphism from $\mathsf{skeleton}\,\mathbf{F}$ onto the tree $({}^{<\omega}\omega,\subset) = \mathsf{skeleton}\,\mathbf{S}$. For each $p \in {}^{\omega}\omega$, the set $\{x \in {}^{<\omega}\omega : x \subseteq p\}$ is a branch in $\mathbf{S}$, so since $\mathbf{F}$ has strict branches it follows that there is a point $\chi(p)$ in $\mathsf{flesh}\,\mathbf{F}$ such that

$$\{\chi(p)\} = \bigcap\{\mathbf{F}_{\varphi^{-1}(x)} : x \in {}^{<\omega}\omega \text{ and } x \subseteq p\}.$$

Then it is not hard to prove that the function $\chi\colon {}^{\omega}\omega \to \mathsf{flesh}\,\mathbf{F}$ is a bijection and (φ,χ^{-1}) is an isomorphism between $\mathbf{F}$ and $\mathbf{S}$. The $\leftarrow$ direction follows from (b) of Lemma 2.6.

(b) Suppose that $\mathbf{F}$ is a Baire foliage tree on X. Let (φ, ψ) be an isomorphism between $\mathbf{F}$ and $\mathbf{S}$, which exists by (a). Then ψ is a bijection from X onto ${}^{\omega}\omega$. Put

$$\tau := \{\psi[U] : U \text{ is an open subset of } X\};$$

clearly, τ is a topology on ${}^{\omega}\omega$ and ψ is a homeomorphism from X onto $({}^{\omega}\omega, \tau)$. It follows that $\mathbf{S}$ is a Baire foliage tree on $({}^{\omega}\omega, \tau)$ because $\mathbf{F}$ is a Baire foliage tree on X. The $\leftarrow$ direction is similar. Part (c) can be proved by the same argument. □

Corollary 3.3. *Suppose that* $\mathbf{F}$ *is a Baire foliage tree on a space* X *and* $p \in X$.

(a) $\mathbf{F}$ *is nonincreasing,* $\mathsf{flesh}\,\mathbf{F} = \mathbf{F}_{0_\mathbf{F}}$, *and* $\mathsf{height}\,\mathbf{F} = \omega$;
(b) $\mathbf{F}_v$ *is closed-and-open in* X *and* $|\mathbf{F}_v| = 2^{\omega}$ *for all* $v \in \mathsf{nodes}\,\mathbf{F}$;
(c) $\mathsf{scope}_\mathbf{F}(p)$ *is a branch in* $\mathbf{F}$;
(d) $\forall n{\in}\omega\ \exists! v{\in}\mathsf{scope}_\mathbf{F}(p)\ \big[\ \mathsf{height}_\mathbf{F}(v) = n\big]$.

Proof. This corollary is a consequence of (b) of Lemma 3.2 and (c) of Lemma 2.6. □

Now we introduce terminology that we need to formulate Theorems 4.1 and 5.1.

Definition 3.4. Suppose $\mathbf{F}$ is a foliage tree and X is a space.

✎ $\mathsf{span}_\mathbf{F}(p, U) := \{\ \mathsf{height}_\mathbf{F}(v) : v \in \mathsf{scope}_\mathbf{F}(p) \text{ and } \mathsf{shoot}_\mathbf{F}(v) \gg \{U\}\};$
✎ $\mathsf{spectrum}_\mathbf{F}(X) := \{\ \mathsf{span}_\mathbf{F}(p, U) : p \in X \text{ and } U \in \mathsf{nbhds}(p, X)\}.$

Example 3.5. $\mathsf{span}_\mathbf{S}(p, \mathbf{S}_{p\restriction n}) = \omega \setminus n$ for all $p \in {}^{\omega}\omega$ and $n \in \omega$.

Lemma 3.6. *Suppose that* $\mathbf{F}$ *is a foliage tree and* X *is a space.*

(a) $\mathbf{F}$ *grows into* X iff $\varnothing \notin \mathsf{spectrum}_\mathbf{F}(X)$.
(b) *If* $\mathbf{F}$ *is a* π*-tree on* X *and* $p \in X$, *then*
 (b1) *the family* $\{\ \mathsf{span}_\mathbf{F}(p, U) : U \in \mathsf{nbhds}(p, X)\}$ *has the* FIP,
 (b2) $\bigcap\{\ \mathsf{span}_\mathbf{F}(p, U) : U \in \mathsf{nbhds}(p, X)\} = \varnothing$, *and*
 (b3) $\mathsf{span}_\mathbf{F}(p, U) \in [\omega]^{\omega}$ *for all* $U \in \mathsf{nbhds}(p, X)$.

Proof. Part (a) is trivial.

(b1) We must show that

$$\text{if} \quad \varepsilon \in [\mathsf{nbhds}(p, X)]^{<\omega} \setminus \{\varnothing\}, \quad \text{then} \quad \bigcap_{U \in \varepsilon} \mathsf{span}_\mathbf{F}(p, U) \neq \varnothing.$$

For each $U \in \varepsilon$, we have $\mathsf{span}_\mathbf{F}(p, U) \supseteq \mathsf{span}_\mathbf{F}(p, \bigcap\varepsilon)$ because $U \supseteq \bigcap\varepsilon \neq \varnothing$. Therefore

$$\bigcap_{U \in \varepsilon} \mathsf{span}_\mathbf{F}(p, U) \supseteq \mathsf{span}_\mathbf{F}(p, \textstyle\bigcap\varepsilon)$$

and it follows from (a) that $\mathsf{span}_\mathbf{F}(p, \bigcap\varepsilon) \neq \varnothing$ since $\bigcap\varepsilon \in \mathsf{nbhds}(p, X)$.

(b2) By (c) of Lemma 3.2, there exist an isomorphism (φ, ψ) between $\mathbf{F}$ and $\mathbf{S}$ and a topology τ on ${}^{\omega}\omega$ such that ψ is a homeomorphism from X onto $({}^{\omega}\omega, \tau)$ and $\mathbf{S}$ is a π-tree on $({}^{\omega}\omega, \tau)$. Put $q := \psi(p)$. For each $U \subseteq X$, we have $\mathsf{span}_{\mathbf{F}}(p, U) = \mathsf{span}_{\mathbf{S}}\big(q, \psi[U]\big)$ and

$$U \in \mathsf{nbhds}(p, X) \quad\leftrightarrow\quad \psi[U] \in \mathsf{nbhds}\big(q, ({}^{\omega}\omega, \tau)\big).$$

Then it is enough to show that

$$\text{the set} \quad M_q \;:=\; \bigcap\Big\{\mathsf{span}_{\mathbf{S}}(q, V) : V \in \mathsf{nbhds}\big(q, ({}^{\omega}\omega, \tau)\big)\Big\} \quad \text{is empty.}$$

It follows from Lemma 2.6 that $\mathbf{S}_{q\restriction n} \in \mathsf{nbhds}\big(q, ({}^{\omega}\omega, \tau)\big)$ for all $n \in \omega$, so using Example 3.5 we have

$$M_q \;\subseteq\; \bigcap\big\{\mathsf{span}_{\mathbf{S}}(q, \mathbf{S}_{q\restriction n}) : n \in \omega\big\} \;=\; \bigcap\{\omega \smallsetminus n : n \in \omega\} \;=\; \varnothing.$$

(b3) It follows from (b1)–(b2) that the set $\mathsf{span}_{\mathbf{F}}(p, U)$ is infinite for all $U \in \mathsf{nbhds}(p, X)$, and $\mathsf{span}_{\mathbf{F}}(p, U) \subseteq \omega$ because $\mathsf{height}\,\mathbf{F} = \omega$ by (a) of Corollary 3.3. □

4. The First Theorem

Theorem 4.1. *Suppose that $\mathbf{H}(\lambda)$ is a π-tree on a space X_λ for every $\lambda \in \Lambda$, where $2 \leqslant |\Lambda| \leqslant \omega$. Suppose also that for each finite nonempty $I \subseteq \Lambda$,*

- ➢ *if $R_i \in \mathsf{spectrum}_{\mathbf{H}(i)}(X_i)$ for all $i \in I$,*
- ➢ *then $\bigcap_{i\in I} R_i$ is infinite.*

Then the product $\prod_{\lambda\in\Lambda} X_\lambda$ has a π-tree.

Corollary 4.2. *Suppose that $\mathbf{H}(\lambda)$ is a π-tree on a space X_λ and $\mathsf{cofin}\,\omega \gg \mathsf{spectrum}_{\mathbf{H}(\lambda)}(X_\lambda)$ for all $\lambda \in \Lambda$, where $1 \leqslant |\Lambda| \leqslant \omega$. Suppose also that a space Y has a π-tree. Then the product $Y \times \prod_{\lambda\in\Lambda} X_\lambda$ also has a π-tree.*

Proof. Let $\mathbf{G}$ be a π-tree on Y and $I \subseteq \Lambda$ be finite and nonempty. Now, if $R \in \mathsf{spectrum}_{\mathbf{G}}(Y)$ and $R_i \in \mathsf{spectrum}_{\mathbf{H}(i)}(X_i)$ for every $i \in I$, then $R \in [\omega]^{\omega}$ by (b3) of Lemma 3.6 and it follows from (a) of Lemma 3.6 that $\bigcap_{i\in I} R_i \supseteq \omega \smallsetminus n$ for some $n \in \omega$. Therefore $R \cap \bigcap_{i\in I} R_i$ is infinite. □

Proof of Theorem 4.1. We may assume that $2 \leqslant \Lambda \in \omega \cup \{\omega\}$. By (c) of Lemma 3.2, for each $n \in \Lambda$, there exist an isomorphism (φ_n, ψ_n) between $\mathbf{H}(n)$ and $\mathbf{S}$ and a topology τ_n on ${}^{\omega}\omega$ such that ψ_n is a homeomorphism from X_n onto $({}^{\omega}\omega, \tau_n)$ and $\mathbf{S}$ is a π-tree on $({}^{\omega}\omega, \tau_n)$. It follows that

$$\mathsf{spectrum}_{\mathbf{H}(n)}(X_n) = \mathsf{spectrum}_{\mathbf{S}}\big(({}^{\omega}\omega, \tau_n)\big) \qquad \text{for all } n \in \Lambda.$$

Now, for every $k \in \Lambda$, we have the following:

$$\text{if } R_i \in \mathsf{spectrum}_{\mathbf{S}}\big(({}^{\omega}\omega, \tau_i)\big) \text{ for every } i \in k+1, \text{ then } \textstyle\bigcap_{i\in k+1} R_i \text{ is infinite.} \tag{4.1}$$

And we must prove that the space $\prod_{n\in\Lambda}({}^{\omega}\omega, \tau_n)$ has a π-tree.

In this proof we use several specific notations. First, $E \cdot F := \{e \cup f : e \in E, f \in F\}$. We use this operation in situations when $E \subseteq {}^{A}C$ and $F \subseteq {}^{B}C$ with $A \cap B = \varnothing$, so that

$$E \cdot F \;=\; \big\{\, p \in {}^{A\cup B}C \;:\; p{\restriction}A \in E \text{ and } p{\restriction}B \in F \,\big\};$$

in particular, when $B = \varnothing$, we have $E \cdot {}^{\varnothing}C = E$ because ${}^{\varnothing}C = \{\varnothing\}$. Recall that

$$\textstyle\prod_{i\in I} D_i \;:=\; \big\{\, \langle p_i\rangle_{i\in I} \in {}^{I}\big(\bigcup_{i\in I} D_i\big) \;:\; p_i \in D_i \text{ for all } i \in I \,\big\}.$$

When $v \in {}^{<\omega}\omega$ and $m \in \omega$, we put

$$\widetilde{\mathbf{S}}^{m}_{v} \;:=\; \bigcup\{\mathbf{S}_{v\hat{}\langle l\rangle} : l \in \omega \setminus m\}. \tag{4.2}$$

Note that $\{\widetilde{\mathbf{S}}^{m}_{v} : m \in \omega\} \gg \mathsf{shoots}_{\mathbf{S}}(v)$ for all $v \in {}^{<\omega}\omega$.

We build a π-tree on the space $\prod_{n\in\Lambda}({}^{\omega}\omega, \tau_n) = \big({}^{\Lambda}({}^{\omega}\omega), \tau\big)$, where τ is the Tychonoff product topology, by using Lemma 4.3. This lemma states that there exists an indexed family

$$\big\langle\, a(n,v,i) \;:\; n\in\omega,\ v\in{}^{2n}\omega,\ i\in\Lambda\cap(n{+}1) \,\big\rangle$$

such that

(a1) $\forall n\in\omega\ \forall v\in{}^{2n}\omega\ \forall i\in\Lambda\cap(n{+}1)\ \big[a(n,v,i) \in {}^{n}\omega\big]$;

(a2) $\forall n\in\omega\ \forall v\in{}^{2n}\omega\ \forall m\in\omega$

$$\Big(\Big(\prod_{i\in\Lambda\cap(n+1)} \widetilde{\mathbf{S}}^{m}_{a(n,v,i)}\Big) \setminus \Big(\prod_{i\in\Lambda\cap(n+1)} \widetilde{\mathbf{S}}^{m+1}_{a(n,v,i)}\Big)\Big) \cdot {}^{\Lambda\cap\{n+1\}}({}^{\omega}\omega) \;\equiv\; \bigsqcup_{l\in\omega} \prod_{i\in\Lambda\cap(n+2)} \mathbf{S}_{a(n+1,v\hat{}\langle m,l\rangle,i)}\,.$$

Let $\mathbf{G}(\Lambda)$ be a foliage tree with $\mathsf{skeleton}\,\mathbf{G}(\Lambda) := ({}^{<\omega}\omega, \subset)$ and with leaves defined as follows:

(b1) $\forall n\in\omega\ \forall v\in{}^{2n}\omega$

$$\mathbf{G}(\Lambda)_v \;:=\; \Big(\prod_{i\in\Lambda\cap(n+1)} \mathbf{S}_{a(n,v,i)}\Big) \cdot {}^{\Lambda\setminus(n+1)}({}^{\omega}\omega);$$

(b2) $\forall n\in\omega\ \forall v\in{}^{2n}\omega\ \forall m\in\omega$

$$\mathbf{G}(\Lambda)_{v\hat{}\langle m\rangle} \;:=\; \Big(\Big(\prod_{i\in\Lambda\cap(n+1)} \widetilde{\mathbf{S}}^{m}_{a(n,v,i)}\Big) \setminus \Big(\prod_{i\in\Lambda\cap(n+1)} \widetilde{\mathbf{S}}^{m+1}_{a(n,v,i)}\Big)\Big) \cdot {}^{\Lambda\setminus(n+1)}({}^{\omega}\omega).$$

Notice that the construction of $\mathbf{G}(\Lambda)$ doesn't depend on topologies τ_n, $n \in \Lambda$; it depends only on the cardinality of Λ.

To complete the proof, we show that $\mathbf{G}(\Lambda)$ is indeed a π-tree on $\big({}^{\Lambda}({}^{\omega}\omega), \tau\big)$:

☛ $\mathbf{G}(\Lambda)$ is a foliage ω,ω-tree.

☛ $\mathbf{G}(\Lambda)_{0_{\mathbf{G}(\Lambda)}} = {}^{\Lambda}({}^{\omega}\omega)$.

We have $0_{\mathbf{G}(\Lambda)} = \langle\rangle$, clause (b1) with $n = 0$ says that

$$\mathbf{G}(\Lambda)_{\langle\rangle} = {}^{\{0\}}\mathbf{S}_{a(0,\langle\rangle,0)} \cdot {}^{\Lambda\setminus 1}({}^{\omega}\omega),$$

so using (4.8) (see the proof of Lemma 4.3) we have

$$\mathbf{G}(\Lambda)_{0_{\mathbf{G}(\Lambda)}} = {}^{\{0\}}\mathbf{S}_{\langle\rangle} \cdot {}^{\Lambda\setminus 1}({}^{\omega}\omega) = {}^{\{0\}}({}^{\omega}\omega) \cdot {}^{\Lambda\setminus\{0\}}({}^{\omega}\omega) = {}^{\Lambda}({}^{\omega}\omega).$$

☛ $\mathbf{G}(\Lambda)$ is open in $\big({}^{\Lambda}({}^{\omega}\omega), \tau\big)$.

By (b) of Corollary 3.3, every set $\mathbf{S}_v$ is closed-and-open in each of spaces $({}^{\omega}\omega, \tau_n)$, and the formula

$$\widetilde{\mathbf{S}}^m_v = \mathbf{S}_v \setminus \bigcup\{\mathbf{S}_{v\hat{}\langle l\rangle} : l \in m\}$$

(which follows from (4.2)) implies that every set $\widetilde{\mathbf{S}}^m_v$ is closed-and-open in each of $({}^{\omega}\omega, \tau_n)$ too. Therefore every leaf of $\mathbf{G}(\Lambda)$ is open in $\big({}^{\Lambda}({}^{\omega}\omega), \tau\big)$.

☛ $\mathbf{G}(\Lambda)$ is locally strict.

Let $t \in \mathsf{nodes}\,\mathbf{G}(\Lambda)$. First, suppose that $t \in {}^{2n}\omega$ for some $n \in \omega$. Since $\mathbf{S}_u = \widetilde{\mathbf{S}}^0_u$ for all $u \in {}^{<\omega}\omega$, then by (b1) we have

$$\mathbf{G}(\Lambda)_t = \Big(\prod_{i\in\Lambda\cap(n+1)} \widetilde{\mathbf{S}}^0_{a(n,t,i)}\Big) \cdot {}^{\Lambda\setminus(n+1)}({}^{\omega}\omega).$$

Note that for each $u \in {}^{<\omega}\omega$, the $\langle\widetilde{\mathbf{S}}^m_u\rangle_{m\in\omega}$ is a strictly decreasing sequence of sets and $\bigcap_{m\in\omega}\widetilde{\mathbf{S}}^m_u = \varnothing$. Then it follows from (b2) that

$$\mathbf{G}(\Lambda)_t \equiv \bigsqcup_{m\in\omega} \mathbf{G}(\Lambda)_{t\hat{}\langle m\rangle}.$$

Now suppose that $t \in {}^{2n+1}\omega$ for some $n \in \omega$, so that $t = u\hat{}\langle m\rangle$ for some $u \in {}^{2n}\omega$, $m \in \omega$. Then by (b2) we have

$$\mathbf{G}(\Lambda)_t = \mathbf{G}(\Lambda)_{u\hat{}\langle m\rangle} =$$

$$\bigg(\Big(\prod_{i\in\Lambda\cap(n+1)} \widetilde{\mathbf{S}}^m_{a(n,u,i)}\Big) \setminus \Big(\prod_{i\in\Lambda\cap(n+1)} \widetilde{\mathbf{S}}^{m+1}_{a(n,u,i)}\Big)\bigg) \cdot {}^{\Lambda\cap\{n+1\}}({}^{\omega}\omega) \cdot {}^{\Lambda\setminus(n+2)}({}^{\omega}\omega),$$

so (a2) implies

$$\mathbf{G}(\Lambda)_t \equiv \bigsqcup_{l\in\omega}\bigg(\Big(\prod_{i\in\Lambda\cap(n+2)} \mathbf{S}_{a(n+1,u\hat{}\langle m,l\rangle,i)}\Big) \cdot {}^{\Lambda\setminus(n+2)}({}^{\omega}\omega)\bigg),$$

and then using (b1) with $v = u\hat{}\langle m,l\rangle \in {}^{2(n+1)}\omega$ we have

$$\mathbf{G}(\Lambda)_t \equiv \bigsqcup_{l\in\omega} \mathbf{G}(\Lambda)_{u\hat{}\langle m,l\rangle}, \quad \text{that is,} \quad \mathbf{G}(\Lambda)_t \equiv \bigsqcup_{l\in\omega} \mathbf{G}(\Lambda)_{t\hat{}\langle l\rangle}.$$

☛ $\mathbf{G}(\Lambda)$ has strict branches.

Suppose that B is a branch in $\mathsf{skeleton}\,\mathbf{G}(\Lambda)$, which means that $B = \{z{\restriction}n : n \in \omega\}$ for some $z \in {}^\omega\omega$. Since $\mathbf{G}(\Lambda)$ is a locally strict foliage ω,ω-tree, then $\mathbf{G}(\Lambda)$ is nonincreasing, so

$$\bigcap_{b\in B} \mathbf{G}(\Lambda)_b \;=\; \bigcap_{n\in\omega} \mathbf{G}(\Lambda)_{z\restriction 2n} \tag{4.3}$$

because the chain $\{z{\restriction}2n : n \in \omega\}$ is cofinal in $(B, \subset)$. By (b1) we have

$$\mathbf{G}(\Lambda)_{z\restriction 2n} \;=\; \Big(\prod_{i\in\Lambda\cap(n+1)} \mathbf{S}_{a(n,z\restriction 2n,i)}\Big) \cdot {}^{\Lambda\smallsetminus(n+1)}({}^\omega\omega) \quad \text{for all } n \in \omega. \tag{4.4}$$

Since $\mathbf{G}(\Lambda)$ is nonincreasing, it follows from (4.4) and (a1) that

$$\mathbf{S}_{a(n,z\restriction 2n,i)} \supset \mathbf{S}_{a(n+1,z\restriction 2(n+1),i)} \quad \text{for all } n \in \omega \text{ and } i \in \Lambda \cap (n+1)$$

— that is, for all $i \in \Lambda$ and $n \in \omega \smallsetminus i$. This implies

$$a(n, z{\restriction}2n, i) \subset a\big(n{+}1, z{\restriction}2(n{+}1), i\big) \quad \text{for all } i \in \Lambda \text{ and } n \in \omega \smallsetminus i,$$

and then, for every $i \in \Lambda$, there is $y_i \in {}^\omega\omega$ such that $a(n, z{\restriction}2n, i) \subset y_i$ for all $n \in \omega \smallsetminus i$. Then

$$\bigcap_{n\in\omega\smallsetminus i} \mathbf{S}_{a(n,z\restriction 2n,i)} = \{y_i\} \qquad \text{for all } i \in \Lambda. \tag{4.5}$$

Put $y := \langle y_i \rangle_{i\in\Lambda} \in {}^\Lambda({}^\omega\omega)$. Now (4.4) and (4.5) imply $\bigcap_{n\in\omega} \mathbf{G}(\Lambda)_{z\restriction 2n} = \{y\}$, so the $\bigcap_{b\in B} \mathbf{G}(\Lambda)_b$ is a singleton by (4.3).

☛ $\mathbf{G}(\Lambda)$ grows into $\big({}^\Lambda({}^\omega\omega), \tau\big)$.

Suppose that

$$p = \langle p_i \rangle_{i\in\Lambda} \in {}^\Lambda({}^\omega\omega) \quad \text{and} \quad U \in \mathsf{nbhds}\Big(p, \big({}^\Lambda({}^\omega\omega), \tau\big)\Big).$$

We may assume that

$$U \;=\; \Big(\prod_{i\in k+1} U_i\Big) \cdot {}^{\Lambda\smallsetminus(k+1)}({}^\omega\omega) \tag{4.6}$$

for some $k \in \Lambda$ and some $U_i \in \mathsf{nbhds}\big(p_i, ({}^\omega\omega, \tau_i)\big)$ for every $i \in k+1$. Put $R_i := \mathsf{span}_{\mathbf{S}}(p_i, U_i)$ for every $i \in k+1$. Then $\bigcap_{i\in k+1} R_i$ is infinite by (4.1), so there is some $\bar{n} \in \bigcap_{i\in k+1} R_i$ such that $\bar{n} \geqslant k$. By definition of $\mathsf{span}_{\mathbf{S}}(p_i, U_i)$, for each $i \in k+1$, there is $v_i \in \mathsf{scope}_{\mathbf{S}}(p_i)$ such that

$$\mathsf{height}_{\mathbf{S}}(v_i) = \bar{n} \quad \text{and} \quad \mathsf{shoots}_{\mathbf{S}}(v_i) \gg \{U_i\}.$$

This means that for each $i \in k+1$, we have $p_i \in \mathbf{S}_{v_i}$, $v_i \in {}^{\bar{n}}\omega$ (hence $v_i = p_i{\restriction}\bar{n}$), and $\widetilde{\mathbf{S}}^{m_i}_{v_i} = \widetilde{\mathbf{S}}^{m_i}_{p_i\restriction\bar{n}} \subseteq U_i$ for some $m_i \in \omega$. Let $\bar{m}$ be the maximal element of $\{m_i : i \in k+1\}$. Then $\widetilde{\mathbf{S}}^{\bar{m}}_{p_i\restriction\bar{n}} \subseteq U_i$ for all $i \in k+1$, and hence

$$\prod_{i\in k+1} \widetilde{\mathbf{S}}^{\bar{m}}_{p_i\restriction\bar{n}} \;\subseteq\; \prod_{i\in k+1} U_i.$$

Note that $k+1 = \Lambda \cap (k+1)$ because $k \in \Lambda$, so using (4.6) we get

$$\Big(\prod_{i\in\Lambda\cap(k+1)} \widetilde{\mathbf{S}}^{\bar m}_{p_i\restriction\bar n}\Big)\cdot{}^{\Lambda\smallsetminus(k+1)}({}^{\omega}\omega) \subseteq U. \tag{4.7}$$

We already know that $\mathbf{G}(\Lambda)$ is a Baire foliage tree on $\big({}^{\Lambda}({}^{\omega}\omega),\tau\big)$, so using (d) of Corollary 3.3 we can take node $\bar v \in \mathsf{scope}_{\mathbf{G}(\Lambda)}(p)$ such that $\bar v \in {}^{2\bar n}\omega$. Then, using (b1) with $n=\bar n$ and $v=\bar v$, we have

$$p = \langle p_i\rangle_{i\in\Lambda} \in \mathbf{G}(\Lambda)_{\bar v} = \Big(\prod_{i\in\Lambda\cap(\bar n+1)} \mathbf{S}_{a(\bar n,\bar v,i)}\Big)\cdot{}^{\Lambda\smallsetminus(\bar n+1)}({}^{\omega}\omega),$$

so $p_i \in \mathbf{S}_{a(\bar n,\bar v,i)}$ for all $i\in\Lambda\cap(\bar n+1)$, and hence using (a1) we get $a(\bar n,\bar v,i) = p_i\restriction\bar n$ for all $i\in\Lambda\cap(\bar n+1)$. Using (b2) and inequality $\bar n \geqslant k$ we can write

$$\mathbf{G}(\Lambda)_{\bar v\frown\langle m\rangle} \subseteq \Big(\prod_{i\in\Lambda\cap(\bar n+1)} \widetilde{\mathbf{S}}^{m}_{p_i\restriction\bar n}\Big)\cdot{}^{\Lambda\smallsetminus(\bar n+1)}({}^{\omega}\omega) \subseteq$$
$$\Big(\prod_{i\in\Lambda\cap(k+1)} \widetilde{\mathbf{S}}^{m}_{p_i\restriction\bar n}\Big)\cdot{}^{\Lambda\smallsetminus(k+1)}({}^{\omega}\omega) \qquad \text{for all } m\in\omega.$$

Therefore using (4.7) we get $\mathbf{G}(\Lambda)_{\bar v\frown\langle m\rangle} \subseteq U$ for all $m\in\omega\smallsetminus\bar m$. This means that we have found $\bar v\in\mathsf{scope}_{\mathbf{G}(\Lambda)}(p)$ such that $\mathsf{shoot}_{\mathbf{G}(\Lambda)}(\bar v) \gg \{U\}$. $\square$

Lemma 4.3. *For each* $\Lambda\in\big(\omega\cup\{\omega\}\big)\smallsetminus 2$, *there is a family* $\big\langle\, a(n,v,i) : n\in\omega,\ v\in{}^{2n}\omega,\ i\in\Lambda\cap(n+1)\,\big\rangle$ *such that*

(a1) $\forall n\in\omega\ \forall v\in{}^{2n}\omega\ \forall i\in\Lambda\cap(n+1)\ \big[a(n,v,i)\in{}^{n}\omega\big]$;

(a2) $\forall n\in\omega\ \forall v\in{}^{2n}\omega\ \forall m\in\omega$

$$\bigg(\Big(\prod_{i\in\Lambda\cap(n+1)} \widetilde{\mathbf{S}}^{m}_{a(n,v,i)}\Big)\smallsetminus\Big(\prod_{i\in\Lambda\cap(n+1)} \widetilde{\mathbf{S}}^{m+1}_{a(n,v,i)}\Big)\bigg)\cdot{}^{\Lambda\cap\{n+1\}}({}^{\omega}\omega) \equiv$$
$$\bigsqcup_{l\in\omega}\ \prod_{i\in\Lambda\cap(n+2)} \mathbf{S}_{a(n+1,v\frown\langle m,l\rangle,i)}\,.$$

Proof. We construct this indexed family by recursion on $n\in\omega$ as follows:

When $n=0$, we have ${}^{2n}\omega = {}^{n}\omega = {}^{0}\omega = \big\{\langle\rangle\big\}$ and $\Lambda\cap(n+1)=\{0\}$ because $\Lambda\geqslant 2$, so (a1) with $n=0$ just says

$$a\big(0,\langle\rangle,0\big) = \langle\rangle. \tag{4.8}$$

When $n=1$, we must choose $a(1,v,i)\in{}^{1}\omega$ (for all $v\in{}^{2}\omega$ and $i\in\Lambda\cap 2$) in such a way that (a2) with $n=0$ is satisfied. Since $\Lambda\geqslant 2$, then $\Lambda\cap 1=\{0\}$ and $\Lambda\cap 2=\{0,1\}$, so (a2) with $n=0$ says that

$$\Big({}^{\{0\}}\widetilde{\mathbf{S}}^{m}_{a(0,\langle\rangle,0)}\smallsetminus{}^{\{0\}}\widetilde{\mathbf{S}}^{m+1}_{a(0,\langle\rangle,0)}\Big)\cdot{}^{\{1\}}({}^{\omega}\omega) \equiv \bigsqcup_{l\in\omega}\prod_{i\in\{0,1\}} \mathbf{S}_{a(1,\langle\rangle\frown\langle m,l\rangle,i)} \text{ for all } m\in\omega.$$

Using (4.2) and (4.8), this can be simplified to

$${}^{\{0\}}\mathbf{S}_{\langle m\rangle}\cdot{}^{\{1\}}({}^{\omega}\omega) \equiv \bigsqcup_{l\in\omega}\ \prod_{i\in\{0,1\}} \mathbf{S}_{a(1,\langle m,l\rangle,i)} \qquad \text{for all } \forall m\in\omega.$$

Then we can take $a\big(1,\langle m,l\rangle,0\big) \coloneqq \langle m\rangle$ and $a\big(1,\langle m,l\rangle,1\big) \coloneqq \langle l\rangle$ for every $m,l\in\omega$.

When $n\geqslant 2$, the choice of $a(n,v,i)$ can be carried out similar to the case $n=1$ if we note that

$$^{\omega}\omega \equiv \bigsqcup_{a\in{}^{h}\omega}\mathbf{S}_a \qquad \text{for all } h\in\omega,$$

and that for every $k\geqslant 2$, every $a=\langle a_i\rangle_{i\in k}\in{}^{k}({}^{2n}\omega)$, and every $m\in\omega$,

$$\Big(\prod_{i\in k}\widetilde{\mathbf{S}}^{m}_{a_i}\Big)\smallsetminus\Big(\prod_{i\in k}\widetilde{\mathbf{S}}^{m+1}_{a_i}\Big) = \Big(\prod_{i\in k}\ \bigcup_{l\in\omega\smallsetminus m}\mathbf{S}_{a_i\hat{}\langle l\rangle}\Big)\smallsetminus\Big(\prod_{i\in k}\ \bigcup_{l\in\omega\smallsetminus(m+1)}\mathbf{S}_{a_i\hat{}\langle l\rangle}\Big) =$$

$$\bigcup\Big\{\prod_{i\in k}\mathbf{S}_{a_i\hat{}\langle l_i\rangle} : l=\langle l_i\rangle_{i\in k}\in{}^{k}(\omega\smallsetminus m)\Big\}\smallsetminus\bigcup\Big\{\prod_{i\in k}\mathbf{S}_{a_i\hat{}\langle l_i\rangle} : l=\langle l_i\rangle_{i\in k}\in{}^{k}\big(\omega\smallsetminus(m+1)\big)\Big\} \equiv$$

$$\bigsqcup\Big\{\prod_{i\in k}\mathbf{S}_{a_i\hat{}\langle l_i\rangle} : l=\langle l_i\rangle_{i\in k}\in{}^{k}(\omega\smallsetminus m)\smallsetminus{}^{k}\big(\omega\smallsetminus(m+1)\big)\Big\}$$

and the set ${}^{k}(\omega\smallsetminus m)\smallsetminus{}^{k}\big(\omega\smallsetminus(m+1)\big)$ is infinite. □

5. The Second Theorem

Theorem 5.1.

(a) *Suppose that* $\mathbf{F}(\alpha)$ *is a* π*-tree on a space* X_α *for every* $\alpha\in A$, *where* $1\leqslant|A|\leqslant\omega$. *Suppose also that for each* $\alpha\in A$, *there is* $\gamma_\alpha\subseteq$ $\mathsf{power.set}(\omega)$ *such that*

- ➢ $|\gamma_\alpha|\leqslant\omega$,
- ➢ γ_α *has the* FIP, *and*
- ➢ $\gamma_\alpha\gg\mathsf{spectrum}_{\mathbf{F}(\alpha)}(X_\alpha)$.

Then the product $\prod_{\alpha\in A}X_\alpha$ *has a* π*-tree.*

(b) *Suppose, in addition to* (a), *that* $\mathbf{G}$ *is a* π*-tree on a space* Y *and* $\mathsf{spectrum}_{\mathbf{G}}(Y)$ *has the* FIP. *Then the product* $Y\times\prod_{\alpha\in A}X_\alpha$ *also has a* π*-tree.*

Lemma 5.2. *Suppose that* $2\leqslant\Lambda\in\omega\cup\{\omega\}$ *and for each* $n\in\Lambda$, $\varnothing\neq\delta_n\subseteq$ $\mathsf{power.set}(\omega)\smallsetminus\{\varnothing\}$ *and* $\bigcap\delta_n=\varnothing$. *Suppose also that* δ_0 *has the* FIP *and for each* $n\in\Lambda\smallsetminus\{0\}$, *there is* $\gamma_n\subseteq\mathsf{power.set}(\omega)$ *such that*

- ➢ $|\gamma_n|\leqslant\omega$,
- ➢ γ_n *has the* FIP, *and*
- ➢ $\gamma_n\gg\delta_n$.

Then there exists a sequence $\langle\alpha_n\rangle_{n\in\Lambda}$ *of strictly increasing functions* $\alpha_n\colon\omega\to\omega$ *such that*

(♣) *if* $k\in\Lambda$ *and* $A_i\in\big\{\alpha_i[D] : D\in\delta_i\big\}$ *for all* $i\leqslant k$, *then* $\bigcap_{i\leqslant k}A_i$ *is infinite.*

Lemma 5.3. *Suppose that* $\mathbf{F}$ *is a* π*-tree on a space* X *and* $\alpha{:}\,\omega \to \omega$ *is a strictly increasing function. Then there exists a* π*-tree* $\mathbf{H}$ *on* X *such that*

($\heartsuit$) $\alpha\big[\mathsf{span}_{\mathbf{F}}(p,U)\big] \subseteq \mathsf{span}_{\mathbf{H}}(p,U)$ *for all* $p \in X$ *and* $U \in \mathsf{nbhds}(p,X)$.

Proof of Theorem 5.1. Note that part (a) follows from part (b). Indeed, let $\beta \in A$, $\mathbf{G} := \mathbf{F}(\beta)$, and $Y := X_\beta$. Since γ_β has the FIP, $\gamma_\beta \gg \mathsf{spectrum}_{\mathbf{G}}(Y)$, and (by (a) of Lemma 3.6) $\varnothing \notin \mathsf{spectrum}_{\mathbf{G}}(Y)$, then $\mathsf{spectrum}_{\mathbf{G}}(Y)$ also has the FIP. The case when $|A| = 1$ is trivial, so we may assume that $A \smallsetminus \{\beta\} \neq \varnothing$, and then the space

$$\prod_{\alpha \in A} X_\alpha \;=\; Y \times \prod_{\alpha \in A \smallsetminus \{\beta\}} X_\alpha$$

has a π-tree by (b).

To prove (b) it is convenient to assume that $A = \Lambda \smallsetminus \{0\}$ and $2 \leqslant \Lambda \in \omega \cup \{\omega\}$. Put $X_0 := Y$ and $\mathbf{F}(0) := \mathbf{G}$; then we must prove that the space $\prod_{n \in \Lambda} X_n$ has a π-tree. Let $\delta_n := \mathsf{spectrum}_{\mathbf{F}(n)}(X_n)$ for every $n \in \Lambda$. Then using Lemma 3.6 we see that $\delta_n \subseteq \mathsf{power.set}(\omega) \smallsetminus \{\varnothing\}$ and $\bigcap \delta_n = \varnothing$ for all $n \in \Lambda$, so we can apply Lemma 5.2. Then we get a sequence $\langle \alpha_n \rangle_{n \in \Lambda}$ of strictly increasing functions $\alpha_n{:}\,\omega \to \omega$ such that condition ($\clubsuit$) holds. Next, applying Lemma 5.3 to $\mathbf{F}_n$, X_n, and α_n for every $n \in \Lambda$, we obtain a sequence $\big\langle \mathbf{H}(n) \big\rangle_{n \in \Lambda}$ such that for every $n \in \Lambda$, $\mathbf{H}(n)$ is a π-tree on X_n and

$$\begin{aligned} &\alpha_n\big[\,\mathsf{span}_{\mathbf{F}(n)}(p,U)\big] \subseteq \mathsf{span}_{\mathbf{H}(n)}(p,U) \\ &\text{for all } \; p \in X_n \; \text{ and } \; U \in \mathsf{nbhds}(p,X_n)\,. \end{aligned} \tag{5.1}$$

Now we can use Theorem 4.1 to show that the product $\prod_{n \in \Lambda} X_n$ has a π-tree. Suppose that $I \subseteq \Lambda$ is finite and nonempty; then $I \subseteq k+1$ for some $k \in \Lambda$. Let $R_i \in \mathsf{spectrum}_{\mathbf{H}(i)}(X_i)$ for every $i \in k+1$; we must show that the set $\bigcap_{i \in I} R_i$ is infinite. For each $i \in k+1$, $R_i = \mathsf{span}_{\mathbf{H}(i)}(p_i, U_i)$ for some $p_i \in X_i$ and $U_i \in \mathsf{nbhds}(p_i, X_i)$. Put $A_i := \alpha_i\big[\,\mathsf{span}_{\mathbf{F}(i)}(p_i,U_i)\big]$ for every $i \in k+1$; then by (5.1) we have $A_i \subseteq R_i$. Now,

$$A_i \in \big\{\alpha_i[R] : R \in \mathsf{spectrum}_{\mathbf{F}(i)}(X_i)\big\} = \big\{\alpha_i[D] : D \in \delta_i\big\} \quad \text{for all } \; i \in k+1,$$

so (by ($\clubsuit$) of Lemma 5.2) the $\bigcap_{i \in k+1} A_i$ is infinite, hence the $\bigcap_{i \in k+1} R_i$ is infinite, and then the $\bigcap_{i \in I} R_i$ is infinite too. $\square$

Proof of Lemma 5.2. It is not hard to show that each γ_n is not empty, so we may assume that $\gamma_n = \big\{G_i(n) : i \in \omega\big\}$ for every $n \in \Lambda \smallsetminus \{0\}$. Since δ_0 has the FIP and $\bigcap \delta_0 = \varnothing$, then

$$\bigcap \varepsilon \text{ is infinite} \qquad \text{for all } \; \varepsilon \in [\delta_0]^{<\omega} \smallsetminus \{\varnothing\}. \tag{5.2}$$

Also $\bigcap \gamma_n = \varnothing$ for all $n \in \Lambda \setminus \{0\}$ (because $\gamma_n \gg \delta_n$, $\varnothing \notin \delta_n \neq \varnothing$, and $\bigcap \delta_n = \varnothing$), so by the same reasons we get

$$\bigcap_{j \leqslant i} G_j(n) \text{ is infinite} \qquad \text{for all } n \in \Lambda \setminus \{0\} \text{ and } i \in \omega. \tag{5.3}$$

Now, using (5.3), for every $n \in \Lambda \setminus \{0\}$ and $i \in \omega$, we can choose $f_i(n) \in \bigcap_{j \leqslant i} G_j(n)$ in such a way that

$$f_{i+1}(n) > f_i(n) \qquad \text{for all } n \in \Lambda \setminus \{0\} \text{ and } \forall i \in \omega. \tag{5.4}$$

Put $F(n) := \{f_i(n) : i \in \omega\}$ for every $n \in \Lambda \setminus \{0\}$; then $\{F(n) \setminus m : m \in \omega\} \gg \gamma_n$ for all $n \in \Lambda \setminus \{0\}$, and hence

$$\{F(n) \setminus m : m \in \omega\} \gg \delta_n \qquad \text{for all } n \in \Lambda \setminus \{0\}. \tag{5.5}$$

Let $F(0) \in \delta_0$; then $F(0)$ is infinite by (5.2), so we may assume that $F(0) = \{f_i(0) : i \in \omega\}$ and $f_{i+1}(0) > f_i(0)$ for all $i \in \omega$. Put $h_{-1} := -1$ and $f_{-1}(n) := -1$ for every $n \in \Lambda$. By recursion on $i \in \omega$, we can build a strictly increasing sequence $\langle h_i \rangle_{i \in \omega}$ of natural numbers in such a way that

$$h_i > h_{i-1} + f_{i-j}(j) - f_{i-j-1}(j) \quad \text{for all } i \in \omega \text{ and } j \in \Lambda \cap (i+1). \tag{5.6}$$

Let $\langle \beta_n \rangle_{n \in \Lambda}$ be a sequence of functions with

$$\mathsf{domain}\,\beta_n := F(n) \cup \{-1\} = \{f_l(n) : l \in \omega \cup \{-1\}\}$$

and such that

$$\beta_n\big(f_l(n)\big) = h_{n+l} \qquad \text{for all } n \in \Lambda \text{ and } l \in \omega \cup \{-1\}. \tag{5.7}$$

Note that (5.7) implies

$$\beta_n\big[F(n)\big] = \{h_j : j \in \omega \setminus n\} \qquad \text{for all } n \in \Lambda. \tag{5.8}$$

Now, for all $n \in \Lambda$ and $l \in \omega$, (5.6) with $i = n + l$, $j = n$ says that

$$h_{n+l} - h_{n+l-1} > f_{n+l-n}(n) - f_{n+l-n-1}(n) = f_l(n) - f_{l-1}(n),$$

so by (5.7) we have

$$\beta_n\big(f_l(n)\big) - \beta_n\big(f_{l-1}(n)\big) > f_l(n) - f_{l-1}(n) \qquad \text{for all } n \in \Lambda \text{ and } l \in \omega.$$

This means that for each $n \in \Lambda$, we can choose a strictly increasing function $\alpha_n : \omega \to \omega$ such that $\alpha_n \restriction F(n) = \beta_n \restriction F(n)$.

Now we prove that condition (♣) is satisfied. Suppose that $k \in \Lambda$ and for every $i \in k+1$, $A_i = \alpha_i\big[D(i)\big]$ for some $D(i) \in \delta_i$. Using (5.5), for each $i \in (k+1) \setminus \{0\}$, we can choose some $m_i \in \omega$ such that $D(i) \supseteq F(i) \setminus m_i$. Therefore, by (5.8), for each $i \in (k+1) \setminus \{0\}$, we can find some $l_i \in \omega$ such that $A_i \supseteq \{h_j : j \in \omega \setminus l_i\}$. It follows that

$$\bigcap_{i \in (k+1) \setminus \{0\}} A_i \supseteq \{h_j : j \in \omega \setminus l\} \qquad \text{for some } l \in \omega.$$

Now, $D(0) \cap F(0)$ is infinite by (5.2), $\alpha_0\big[F(0)\big] = \{h_j : j \in \omega\}$ by (5.8), and α_0 is injective, therefore $A_0 \cap \{h_j : j \in \omega\}$ is infinite. This means that $\bigcap_{i \in k+1} A_i$ is infinite too. $\square$

Proof of Lemma 5.3. In this proof we apply the foliage hybrid operation, see details in Section 8. Put $\alpha(-1) := -1$. Suppose that $v \in \mathsf{nodes}\,\mathbf{F}$. Set

$$k(v) \;:=\; \alpha\big(\mathsf{height}_{\mathbf{F}}(v)\big) - \alpha\big(\mathsf{height}_{\mathbf{F}}(v) - 1\big);$$

then $k(v) \in \omega \setminus \{0\}$. Let $\mathcal{T}(v)$ be a tree isomorphic to the tree $({}^{<k(v)+1}\omega, \subset)$ and such that $0_{\mathcal{T}(v)} = v$ and $\max \mathcal{T}(v) = \mathsf{sons}_{\mathbf{F}}(v)$. Let $\mathbf{G}(v)$ be a foliage tree with $\mathsf{skeleton}\,\mathbf{G}(v) := \mathcal{T}(v)$ and with leaves defined by recursion on $i \in k(v) + 1$ as follows:

(base) If $i = 0$ and $t \in \mathsf{level}_{\mathcal{T}(v)}\big(k(v) - i\big)$ (that is, if $t \in \max \mathcal{T}(v)$), then $\mathbf{G}(v)_t := \mathbf{F}_t$.

(step) If $1 \leqslant i \leqslant k(v)$ and $t \in \mathsf{level}_{\mathcal{T}(v)}\big(k(v) - i\big)$, then $\mathbf{G}(v)_t := \bigcup\big\{\mathbf{G}(v)_s : s \in \mathsf{sons}_{\mathcal{T}(v)}(t)\big\}$.

It is not hard to show the following (we use here the terminology of Definition 8.2):

(c1) $0_{\mathbf{G}(v)} = v$ and $\max \mathbf{G}(v) = \mathsf{sons}_{\mathbf{F}}(v) = \mathsf{level}_{\mathbf{G}(v)}\big(k(v)\big)$;
(c2) $\mathbf{G}(v)_v = \mathbf{F}_v$;
(c3) $\mathbf{G}(v)$ is a foliage graft for $\mathbf{F}$;
(c4) $\mathsf{cut}\big(\mathbf{F}, \mathbf{G}(v)\big) = \varnothing$;
(c5) $\mathbf{G}(v)$ is ω-branching, locally strict, open in X, and has bounded chains;
(c6) $\mathsf{height}\,\mathbf{G}(v) = k(v) + 1$;
(c7) $\mathsf{shoot}_{\mathbf{G}(v)}(t) \gg \mathsf{shoot}_{\mathbf{F}}(v)$ for all $t \in \mathsf{nodes}\,\mathbf{G}(v) \setminus \max \mathbf{G}(v)$;
(c8) $\mathsf{explant}\big(\mathbf{F}, \mathbf{G}(v)\big) = \varnothing$.

Now let $\varphi := \big\{\mathbf{G}(v) : v \in \mathsf{nodes}\,\mathbf{F}\big\}$. We may assume that

$$\mathsf{implant}\,\mathbf{G}(v) \cap \mathsf{implant}\,\mathbf{G}(u) = \varnothing \qquad \text{for all } v \neq u \in \mathsf{nodes}\,\mathbf{F},$$

so φ is a consistent family of foliage grafts for $\mathbf{F}$. Let $\mathbf{H} := \mathsf{fol.hybr}(\mathbf{F}, \varphi)$; note that $\mathsf{loss}(\mathbf{F}, \varphi) = \varnothing$ by (c4). By induction on $\mathsf{height}_{\mathbf{F}}(v)$, we can prove that

$$\mathsf{height}_{\mathbf{H}}(v) \;=\; \alpha\big(\mathsf{height}_{\mathbf{F}}(v) - 1\big) + 1 \qquad \text{for all } v \in \mathsf{nodes}\,\mathbf{F}. \tag{5.9}$$

Indeed, if $\mathsf{height}_{\mathbf{F}}(v) = 0$, then $v = 0_{\mathbf{F}}$, so $v = 0_{\mathbf{H}}$, and hence

$$\mathsf{height}_{\mathbf{H}}(v) \;=\; 0 \;=\; \alpha\big(\mathsf{height}_{\mathbf{F}}(v) - 1\big) + 1.$$

If $\mathsf{height}_{\mathbf{F}}(v) \geqslant 1$, then let t be the node in $\mathbf{F}$ such that $v \in \mathsf{sons}_{\mathbf{F}}(t)$, and then inductively we can write

$$\mathsf{height}_{\mathbf{H}}(v) = \mathsf{height}_{\mathbf{H}}(t)+\mathsf{height}_{\mathbf{G}(t)}(v) = \mathsf{height}_{\mathbf{H}}(t)+\big(\mathsf{height}\,\mathbf{G}(t)-1\big) =$$
$$\mathsf{height}_{\mathbf{H}}(t) + \big(k(t)+1\big) - 1 = \alpha\big(\mathsf{height}_{\mathbf{F}}(t)-1\big)+1+k(t) =$$
$$\alpha\big(\mathsf{height}_{\mathbf{F}}(t)-1\big)+1+\alpha\big(\mathsf{height}_{\mathbf{F}}(t)\big)-\alpha\big(\mathsf{height}_{\mathbf{F}}(t)-1\big) =$$
$$\alpha\big(\mathsf{height}_{\mathbf{F}}(t)\big)+1 = \alpha\big(\mathsf{height}_{\mathbf{F}}(v)-1\big)+1.$$

Now, (c4)–(c6) with Lemma 8.4 say that $\mathbf{H}$ is a Baire foliage tree on X and (c7)–(c8) imply that each $\mathbf{G}(v)$ preserves shoots of $\mathbf{F}$ (see Definition 8.5), so $\mathbf{H}$ grows into X by Lemmas 8.6 and 8.7. Therefore $\mathbf{H}$ is a π-tree on X.

Let us show that (♥) holds. Suppose that $p \in X$, $U \in \mathsf{nbhds}(p, X)$, and $r \in \mathsf{span}_{\mathbf{F}}(p, U)$. Then $r = \mathsf{height}_{\mathbf{F}}(v)$ for some node $v \in \mathsf{scope}_{\mathbf{F}}(p)$ such that $\mathsf{shoot}_{\mathbf{F}}(v) \gg \{U\}$. Let s be the node in $\mathsf{sons}_{\mathbf{F}}(v)$ such that $p \in \mathbf{F}_s$ and let t be the node in $\mathbf{G}(v)$ such that $s \in \mathsf{sons}_{\mathbf{G}(v)}(t)$. Then $t \in \mathsf{scope}_{\mathbf{H}}(p)$ and using (c7) and (a) of Proposition 8.3 we obtain

$$\mathsf{shoot}_{\mathbf{H}}(t) = \mathsf{shoot}_{\mathbf{G}(v)}(t) \gg \mathsf{shoot}_{\mathbf{F}}(v), \tag{5.10}$$

so $\mathsf{shoot}_{\mathbf{H}}(t) \gg \{U\}$, and hence $\mathsf{height}_{\mathbf{H}}(t) \in \mathsf{span}_{\mathbf{H}}(p, U)$. Therefore to complete the proof it is enough to show that $\alpha(r) = \mathsf{height}_{\mathbf{H}}(t)$. Indeed, using (c6) and (5.9) we have

$$\mathsf{height}_{\mathbf{H}}(t) = \mathsf{height}_{\mathbf{H}}(v)+\mathsf{height}_{\mathbf{G}(v)}(t) = \mathsf{height}_{\mathbf{H}}(v)+\mathsf{height}_{\mathbf{G}(v)}(s)-1 =$$
$$\mathsf{height}_{\mathbf{H}}(v) + \big(\mathsf{height}\,\mathbf{G}(v)-1\big)-1 = \mathsf{height}_{\mathbf{H}}(v)+\big(k(v)+1\big)-2 =$$
$$\alpha\big(\mathsf{height}_{\mathbf{F}}(v)-1\big)+1+\alpha\big(\mathsf{height}_{\mathbf{F}}(v)\big)-\alpha\big(\mathsf{height}_{\mathbf{F}}(v)-1\big)-1 =$$
$$\alpha\big(\mathsf{height}_{\mathbf{F}}(v)\big) = \alpha(r). \qquad \square$$

6. Nice π-Tree for a Co-Countable Subspace

In this section we prove Corollary 6.2, which states that if a space X has a "very nice" π-tree (that is, a π-tree $\mathbf{F}$ such that $\mathsf{cofin}\,\omega \gg \mathsf{spectrum}_{\mathbf{F}}(X)$) and if $A \subseteq X$ is at most countable, then the subspace $X \setminus A$ has a "nice" π-tree — that is, a π-tree that satisfies the conditions of Theorem 5.1. This result allows to apply Theorem 5.1 to co-countable subspaces of the Sorgenfrey line, see (c) of Lemma 7.1 and Corollary 7.2 in Section 7.

Proposition 6.1. *Suppose that $\mathbf{F}$ is a Baire foliage tree on a space X and $A \subseteq X$ is at most countable. Then there exists a Baire foliage tree $\mathbf{H}$ on the subspace $X \setminus A$ such that for every $p \in X \setminus A$, there is a strictly increasing function $f_p\colon \omega \to \omega$ with a property*

($\blacklozenge$) $\{2n+1 \ : \ n \in \omega$ and $f_p(n) \in \mathsf{span}_{\mathbf{F}}(p,U)\} \ \subseteq \ \mathsf{span}_{\mathbf{H}}(p, U \smallsetminus A)$ *for all* $U \in \mathsf{nbhds}(p,X)$.

Corollary 6.2. *Suppose that* $\mathbf{F}$ *is a* π*-tree on a space* X *such that* $\mathsf{cofin}\,\omega \gg \mathsf{spectrum}_{\mathbf{F}}(X)$ *and* $A \subseteq X$ *is at most countable. Then there exists a* π*-tree* $\mathbf{H}$ *on the subspace* $X \smallsetminus A$ *such that*

($\spadesuit$) $\mathsf{cofin}\{2n+1 : n \in \omega\} \gg \mathsf{spectrum}_{\mathbf{H}}(X \smallsetminus A)$.

Remark 6.3. In statements of Proposition 6.1 and Corollary 6.2 the sequence $\langle 2n+1\rangle_{n\in\omega}$ can be replaced by an arbitrary sequence $\langle k_n\rangle_{n\in\omega}$ of natural numbers such that $k_0 \geqslant 1$ and $k_{n+1} > k_n + 1$ for all $n \in \omega$.

Proof of Corollary 6.2. Let $\mathbf{H}$ be a Baire foliage tree on the subspace $X \smallsetminus A$ from Proposition 6.1. First we show that condition ($\spadesuit$) is satisfied. Suppose that $D \in \mathsf{spectrum}_{\mathbf{H}}(X \smallsetminus A)$; that is, $D = \mathsf{span}_{\mathbf{H}}(p, U \smallsetminus A)$ for some $p \in X \smallsetminus A$ and $U \in \mathsf{nbhds}(p,X)$. Let $f_p{:}\,\omega \to \omega$ be a function that satisfies condition ($\blacklozenge$) of Proposition 6.1. Since $\mathbf{F}$ is a π-tree on X, then $\mathsf{span}_{\mathbf{F}}(p,U) \neq \varnothing$ by (a) of Lemma 3.6, so it follows from $\mathsf{cofin}\,\omega \gg \mathsf{spectrum}_{\mathbf{F}}(X)$ that there is some $\bar m \in \omega$ such that $\omega \smallsetminus \bar m \subseteq \mathsf{span}_{\mathbf{F}}(p,U)$. Therefore, since f_p is strictly increasing, there is some $\bar n \in \omega$ such that $f_p(n) \in \mathsf{span}_{\mathbf{F}}(p,U)$ for all $n \geqslant \bar n$. Then by ($\blacklozenge$) we have

$$\{2n+1 : n \in \omega \smallsetminus \bar n\} \ \subseteq \ \mathsf{span}_{\mathbf{H}}(p, U \smallsetminus A) \ = \ D,$$

hence ($\spadesuit$) is satisfied.

It follows from the above reasoning that $D \neq \varnothing$, so $\varnothing \notin \mathsf{spectrum}_{\mathbf{H}}(X \smallsetminus A)$, and hence $\mathbf{H}$ grows into $X \smallsetminus A$ by (a) of Lemma 3.6. This means that $\mathbf{H}$ is a π-tree on $X \smallsetminus A$. $\square$

In the following lemma we use terminology of the foliage hybrid operation, see Definition 8.2 in Section 8.

Lemma 6.4. *Suppose that* $\mathbf{F}$ *is a Baire foliage tree on a space* X, $p \in X$, *and* $v \in \mathsf{scope}_{\mathbf{F}}(p)$. *Then there exists a foliage tree* $\mathbf{G}$ *such that*

(d1) $0_{\mathbf{G}} = v$ *and* $\max \mathbf{G} = \mathsf{sons}_{\mathbf{G}}(0_{\mathbf{G}})$;
(d2) $\mathbf{G}_v \ = \ \mathbf{F}_v \smallsetminus \{p\} \ \equiv \ \bigsqcup_{m \in \max \mathbf{G}} \mathbf{F}_m$;
(d3) $\mathbf{G}$ *is a foliage graft for* $\mathbf{F}$;
(d4) $\mathsf{implant}\,\mathbf{G} = \varnothing$;
(d5) $\mathbf{G}$ *is* ω*-branching, locally strict, open in* X, *has bounded chains, and* $\mathsf{height}\,\mathbf{G} = 2$.

Proof. Put

$$B \ := \ \bigcup\bigl\{\mathsf{sons}_{\mathbf{F}}(u) : u \in \mathsf{scope}_{\mathbf{F}}(p) \cap v{\downarrow}_{\mathbf{F}}\bigr\} \quad \text{and} \quad \mathsf{MAX} \ := \ B \smallsetminus \mathsf{scope}_{\mathbf{F}}(p).$$

Let $\mathcal{T}$ be a partial order such that

$$\mathsf{nodes}\,\mathcal{T} \ := \ \{v\} \cup \mathsf{MAX} \quad \text{and} \quad <_{\mathcal{T}} \ := \ \bigl\{(v,m) : m \in \mathsf{MAX}\bigr\}.$$

Then $\mathcal{T}$ is a tree, $0_{\mathcal{T}} = v$, $\max \mathcal{T} = \mathsf{MAX}$, and $\mathcal{T}$ is a graft for $\mathsf{skeleton}\,\mathbf{F}$.

Now let $\mathbf{G}$ be a foliage tree with $\mathsf{skeleton}\,\mathbf{G} := \mathcal{T}$ and with leaves $\mathbf{G}_v := \mathbf{F}_v \smallsetminus \{p\}$ and $\mathbf{G}_m := \mathbf{F}_m$ for all $m \in \max\mathcal{T}$. Then using (b) of Lemma 3.2 and Corollary 3.3 it is not hard to verify that clauses (d1)–(d5) are satisfied. $\square$

Proof of Proposition 6.1. We may assume that $A = \{p_i : i \in |A|\}$ and $p_i \neq p_j$ for all $i \neq j \in |A|$. First we build sequences $\langle M_i \rangle_{i\in|A|}$, $\langle z_i \rangle_{i\in|A|}$, and $\langle \mathbf{G}(i) \rangle_{i\in|A|}$ by recursion on $i \in |A|$:

(e1) $M_0 := \{0_{\mathbf{F}}\}$;
(e2) $z_i :=$ the node in $\mathbf{F}$ such that $\{z_i\} = M_i \cap \mathsf{scope}_{\mathbf{F}}(p_i)$;
(e3) $\mathbf{G}(i) :=$ the foliage tree $\mathbf{G}$ from Lemma 6.4 with $p = p_i$ and $v = z_i$;
(e4) $M_{i+1} := \big(M_i \smallsetminus \{z_i\}\big) \cup \bigcup\big\{ \mathsf{sons}_{\mathbf{F}}(m) : m \in \max \mathbf{G}(i)\big\}$.

The correctness of clause (e2) follows from (f3), see below.

It is not hard to verify that for each $i \in |A|$, the following conditions are satisfied:

(f1) M_i is an antichain in $\mathbf{F}$
(that is, $u \not<_{\mathbf{F}} v$ and $v \not<_{\mathbf{F}} u$ for all $u, v \in M_i$);
(f2) $M_{i+1} \subseteq (M_i){\downarrow}_{\mathbf{F}}$;
(f3) $X \smallsetminus \{p_j : j \in i\} \equiv \bigsqcup_{m\in M_i} \mathbf{F}_m$;
(f4) $0_{\mathbf{G}(i)} \in M_i$;
(f5) $\mathsf{cut}\big(\mathbf{F}, \mathbf{G}(i)\big) = \{p_i\}$;
(f6) $\big\{\mathbf{G}(j) : j \in i+1\big\}$ is a consistent family of foliage grafts for $\mathbf{F}$.

Now it follows from (f6) that $\varphi := \big\{\mathbf{G}(i) : i \in |A|\big\}$ is a consistent family of foliage grafts for $\mathbf{F}$, so we can define $\mathbf{H} := \mathsf{fol.hybr}(\mathbf{F}, \varphi)$. Then $\mathbf{H}$ is a Baire foliage tree on $X \smallsetminus A$ by Lemma 8.4, (f5), and (d5). Also $\mathbf{H}$ satisfies the following:

(g1) $\mathbf{H}$ has nonempty leaves.
This follows from (b) of Corollary 3.3.
(g2) $\mathsf{nodes}\,\mathbf{H} \subseteq \mathsf{nodes}\,\mathbf{F}$.
This follows from (d4).
(g3) $\mathbf{H}_u = \mathbf{F}_u \smallsetminus A$ for all $u \in \mathsf{nodes}\,\mathbf{H}$.
This also follows from (d4).
(g4) $\mathsf{height}_{\mathbf{H}}(u) \in \{2n : n \in \omega\}$ for all $u \in M_i$ and $i \in |A|$.
We prove (g4) by induction on $i \in |A|$. Obviously, $\mathsf{height}_{\mathbf{H}}(u)$ is even when $u \in M_0$. Assume as inductive hypothesis that the assertion of (g4) holds for all $u \in \bigcup_{i\leqslant k} M_i$ and prove it for an arbitrary $u \in M_{k+1}$. If $u \in M_k$, then $\mathsf{height}_{\mathbf{H}}(u)$ is even by the inductive hypothesis. If $u \notin M_k$, then (e4) implies that $u \in \mathsf{sons}_{\mathbf{F}}(m)$ for some $m \in \max \mathbf{G}(k)$. We have

$$\max \mathbf{G}(k) = \mathsf{sons}_{\mathbf{G}(k)}(0_{\mathbf{G}(k)}) \quad \text{and} \quad \mathsf{sons}_{\mathbf{G}(k)}(0_{\mathbf{G}(k)}) = \mathsf{sons}_{\mathbf{H}}(0_{\mathbf{G}(k)})$$

by (d1) and by (a) of Proposition 8.3, so it follows from (f4) and from the inductive hypothesis that $\mathsf{height}_{\mathbf{H}}(m)$ is odd and then, using (f4) and the inductive hypothesis again, we have $m \notin \{0_{\mathbf{G}(j)} : j \in k+1\}$. Let us show that $m \notin \{0_{\mathbf{G}(j)} : j \in |A|\}$. If not, then

$$m \in \{0_{\mathbf{G}(j)} : j \in |A| \setminus (k+1)\}, \quad \text{so} \quad m \in \bigcup \{M_j : j \in |A| \setminus (k+1)\}$$

by (f4). Then it follows from (f2) that $m \in (M_{k+1})\downarrow_{\mathbf{F}}$ — that is, $m \geqslant_{\mathbf{F}} t$ for some $t \in M_{k+1}$. Since $u \in \mathsf{sons}_{\mathbf{F}}(m)$, then $u >_{\mathbf{F}} m$, therefore $u >_{\mathbf{F}} t$. This contradicts (f1) because $t, u \in M_{k+1}$. Now it follows from (d4) and (a) of Proposition 8.3 that $\mathsf{sons}_{\mathbf{F}}(m) = \mathsf{sons}_{\mathbf{H}}(m)$, so $u \in \mathsf{sons}_{\mathbf{H}}(m)$. Then $\mathsf{height}_{\mathbf{H}}(u) \in \{2n : n \in \omega\}$ because $\mathsf{height}_{\mathbf{H}}(m)$ is odd.

Now suppose that $p \in X \setminus A$; we must find a strictly increasing function $f_p{:}\omega \to \omega$ that satisfies (♦). First, using (d) of Corollary 3.3, for each $n \in \omega$, we define $m(p,n)$ to be the node in $\mathsf{scope}_{\mathbf{H}}(p)$ such that $\mathsf{height}_{\mathbf{H}}\big(m(p,n)\big) = 2n+1$. Using (g3) we have

$$\forall n \in \omega \ \big[m(p,n) \in \mathsf{scope}_{\mathbf{F}}(p)\big]. \tag{6.1}$$

Next, using (g2), we can define

$$f_p(n) := \mathsf{height}_{\mathbf{F}}\big(m(p,n)\big) \quad \text{for every } n \in \omega;$$

then $f_p{:}\omega \to \omega$ by (a) of Corollary 3.3. If $n'' > n'$, then $m(p,n'') >_{\mathbf{H}} m(p,n')$ (because $\mathsf{scope}_{\mathbf{H}}(p)$ is a chain in $\mathbf{H}$ by (c) of Corollary 3.3), so $m(p,n'') >_{\mathbf{F}} m(p,n')$ by (g2) and (b) of Lemma 8.3. This implies that f_p is strictly increasing. Now, using (f4) and (g4), for every $n \in \omega$, we have

$$m(p,n) \notin \{0_{\mathbf{G}(i)} : i \in |A|\}, \quad \text{so} \quad \mathsf{sons}_{\mathbf{H}}\big(m(p,n)\big) = \mathsf{sons}_{\mathbf{F}}\big(m(p,n)\big)$$

by (d4) and (a) of Proposition 8.3. Then (g3) and (g1) imply

$$\mathsf{shoot}_{\mathbf{H}}\big(m(p,n)\big) \gg \mathsf{shoot}_{\mathbf{F}}\big(m(p,n)\big) \quad \text{for all } n \in \omega. \tag{6.2}$$

To complete the proof it remains to verify (♦); suppose that

$$U \in \mathsf{nbhds}(p,X), \quad n \in \omega, \quad \text{and} \quad f_p(n) \in \mathsf{span}_{\mathbf{F}}(p,U).$$

The last formula means that $f_p(n) = \mathsf{height}_{\mathbf{F}}(v)$ for some $v \in \mathsf{scope}_{\mathbf{F}}(p)$ such that $\mathsf{shoot}_{\mathbf{F}}(v) \gg \{U\}$. Then $v = m(p,n)$ by (d) of Corollary 3.3, by (6.1), and by definition of $f_p(n)$. It follows that

$$\mathsf{shoot}_{\mathbf{F}}\big(m(p,n)\big) \gg \{U\}, \quad \text{so} \quad \mathsf{shoot}_{\mathbf{H}}\big(m(p,n)\big) \gg \{U\}$$

by (6.2). Then (g3) implies $\mathsf{shoot}_{\mathbf{H}}\big(m(p,n)\big) \gg \{U \setminus A\}$, therefore

$$2n+1 \ = \ \mathsf{height}_{\mathbf{H}}\big(m(p,n)\big) \ \in \ \mathsf{span}_{\mathbf{H}}(p, U \setminus A)$$

by definition of $\mathsf{span}_{\mathbf{H}}(p, U \setminus A)$. □

7. New Examples of Spaces with a π-Tree

Recall that $\mathcal{N}$ is the Baire space, $\mathcal{R}_{\mathcal{S}}$ is the Sorgenfrey line, and $\mathcal{I}_{\mathcal{S}} := \mathcal{R}_{\mathcal{S}} \smallsetminus \mathbb{Q}$ is the irrational Sorgenfrey line.

Lemma 7.1.

(a) *$\mathcal{N}$ has a π-tree $\mathbf{F}$ such that* $\mathsf{cofin}\,\omega \gg \mathsf{spectrum}_{\mathbf{F}}(\mathcal{N})$.
(b) *$\mathcal{R}_{\mathcal{S}}$ has a π-tree $\mathbf{G}$ such that* $\mathsf{cofin}\,\omega \gg \mathsf{spectrum}_{\mathbf{G}}(\mathcal{R}_{\mathcal{S}})$.
(c) *If $X \subseteq \mathcal{R}_{\mathcal{S}}$ and $\mathcal{R}_{\mathcal{S}} \smallsetminus X$ is at most countable, then X has a π-tree $\mathbf{H}$ such that* $\mathsf{cofin}\{2n+1 : n\in\omega\} \gg \mathsf{spectrum}_{\mathbf{H}}(X)$.

Proof. Part (a) follows from (b) of Lemma 2.6 and Example 3.5; part (b) can be derived from the proof of Lemma 3.6 in [8]; part (c) follows from part (b) and Corollary 6.2. □

Using the above lemma and Theorem 5.1 we obtain the following statement:

Corollary 7.2. *Suppose that $1 \leqslant |A| \leqslant \omega$ and for each $\alpha \in A$,*

$$\textit{either} \quad X_\alpha = \mathcal{N} \quad \textit{or} \quad X_\alpha \subseteq \mathcal{R}_{\mathcal{S}} \ \textit{with} \ |\mathcal{R}_{\mathcal{S}} \smallsetminus X_\alpha| \leqslant \omega.$$

Then the product $\prod_{\alpha\in A} X_\alpha$ has a π-tree. □

Corollary 7.3.

(a) *${\mathcal{R}_{\mathcal{S}}}^n$ and ${\mathcal{I}_{\mathcal{S}}}^n$ have a π-tree for all $n \in \omega \smallsetminus \{0\}$.*
(b) *${\mathcal{R}_{\mathcal{S}}}^\omega$ and ${\mathcal{I}_{\mathcal{S}}}^\omega$ have a π-tree.* □

Note that if $X \subseteq \mathcal{N}$ with $|\mathcal{N} \smallsetminus X| \leqslant \omega$, then X is homeomorphic to $\mathcal{N}$ (this can be easily derived from the Alexandrov-Urysohn characterization of the Baire space and from the characterization of its Polish subspaces — see Theorems 3.11 and 7.7 in [6]). Notice also that $\mathcal{N}^n$ is homeomorphic to $\mathcal{N}$ for all $n \in \omega \smallsetminus \{0\}$ and $\mathcal{N}^\omega$ is also homeomorphic to $\mathcal{N}$.

Corollary 7.4. *If a space X has a π-tree, then $X \times \mathcal{N}$, $X \times \mathcal{R}_{\mathcal{S}}$, and $X \times {\mathcal{R}_{\mathcal{S}}}^\omega$ also have a π-tree.*

Proof. This statement follows from Corollary 4.2 and Lemma 7.1. □

8. Appendix. The Foliage Hybrid Operation

In the proofs of Lemma 5.3 and Proposition 6.1 we employ the foliage hybrid operation, which was introduced in [10]. For completeness of exposition we list here definitions and results that we use. The definition of graft, which we give below, slightly differs from the definition of graft in [10], but these two definitions are easily seen to be equivalent. The same can be said about our definition of $\mathsf{hybrid}(\mathcal{T}, \gamma)$, see details in [10, Remark 20]. To ease comprehension of notions from Definition 8.2, you can look at pictures that illustrate this definition in [9].

Notation 8.1.

✎ $\forall x \neq y \in A\ \varphi(x,y) \quad :\longleftrightarrow \quad \forall x, y \in A\ \big[x \neq y \to \varphi(x,y)\big]$;

✎ $x \parallel_{\mathcal{P}} y \quad :\longleftrightarrow \quad x \nleq_{\mathcal{P}} y$ and $x \ngtr_{\mathcal{P}} y$.

Definition 8.2 ([10, definitions 15, 17, 19, and 25–27 and Remark 20])**.** Suppose that $\mathcal{T}, \mathcal{G}$ are trees and $\mathbf{F}, \mathbf{G}$ are nonincreasing foliage trees.

✎ $\mathcal{G}$ is a **graft** for $\mathcal{T}$ $\quad :\longleftrightarrow$
- ➢ $|\,\mathsf{nodes}\,\mathcal{G}| > 1$,
- ➢ $\mathcal{G}$ has the least node,
- ➢ $\mathsf{nodes}\,\mathcal{G} \cap \mathsf{nodes}\,\mathcal{T} = \{0_{\mathcal{G}}\} \cup \max \mathcal{G}$, and
- ➢ $\forall x, y \in \mathsf{nodes}\,\mathcal{G} \cap \mathsf{nodes}\,\mathcal{T}\ [x <_{\mathcal{G}} y \leftrightarrow x <_{\mathcal{T}} y]$.

✎ If $\mathcal{G}$ is a graft for $\mathcal{T}$, then:
- ➢ $\mathsf{implant}\,\mathcal{G} \ :=\ \mathsf{nodes}\,\mathcal{G} \smallsetminus \big(\{0_{\mathcal{G}}\} \cup \max \mathcal{G}\big)$;
- ➢ $\mathsf{explant}(\mathcal{T}, \mathcal{G}) \ :=\ (0_{\mathcal{G}}){\downarrow}_{\mathcal{T}} \smallsetminus (\max \mathcal{G}){\downarrow}_{\mathcal{T}}$.

✎ γ is a **consistent** family of grafts for $\mathcal{T}$ $\quad :\longleftrightarrow$
- ➢ $\forall \mathcal{G} \in \gamma\ [\ \mathcal{G}$ is a graft for $\mathcal{T}\ ]$,
- ➢ $\forall \mathcal{D} \neq \mathcal{E} \in \gamma\ [\ \mathsf{implant}\,\mathcal{D} \cap \mathsf{implant}\,\mathcal{E} = \varnothing\ ]$, and
- ➢ $\forall \mathcal{D} \neq \mathcal{E} \in \gamma\ \big[0_{\mathcal{D}} \parallel_{\mathcal{T}} 0_{\mathcal{E}}$ or $0_{\mathcal{D}} \in (\max \mathcal{E}){\downarrow}_{\mathcal{T}}$ or $0_{\mathcal{E}} \in (\max \mathcal{D}){\downarrow}_{\mathcal{T}}\big]$.

✎ If γ is a consistent family of grafts for $\mathcal{T}$, then:
- ➢ $\mathsf{support}(\mathcal{T}, \gamma) \ :=\ \mathsf{nodes}\,\mathcal{T} \smallsetminus \bigcup_{\mathcal{G} \in \gamma} \mathsf{explant}(\mathcal{T}, \mathcal{G})$;
- ➢ $\mathsf{hybrid}(\mathcal{T}, \gamma) \ :=$ the pair $(H, <)$ (actually, a tree) such that
  $H \ :=\ \mathsf{support}(\mathcal{T}, \gamma) \cup \bigcup_{\mathcal{G} \in \gamma} \mathsf{implant}\,\mathcal{G}$ and
  $< \ :=$ the transitive closure of relation $(<_{\mathcal{T}} \cup \bigcup_{\mathcal{G} \in \gamma} <_{\mathcal{G}}) \cap (H \times H)$.

✎ $\mathbf{G}$ is a **foliage graft** for $\mathbf{F}$ $\quad :\longleftrightarrow$
- ➢ $\mathbf{G}$ is nonincreasing,
- ➢ $\mathsf{skeleton}\,\mathbf{G}$ is a graft for $\mathsf{skeleton}\,\mathbf{F}$,
- ➢ $\mathbf{G}_{0_{\mathbf{G}}} \subseteq \mathbf{F}_{0_{\mathbf{G}}}$, and
- ➢ $\forall m \in \max \mathbf{G}\ [\mathbf{G}_m = \mathbf{F}_m]$.

✎ If $\mathbf{G}$ is a foliage graft for $\mathbf{F}$, then
- ➢ $\mathsf{cut}(\mathbf{F}, \mathbf{G}) := \mathbf{F}_{0_{\mathbf{G}}} \smallsetminus \mathbf{G}_{0_{\mathbf{G}}}$.

✎ φ is a **consistent** family of foliage grafts for $\mathbf{F}$ $\quad :\longleftrightarrow$
- ➢ $\forall \mathbf{G} \in \varphi\ [\mathbf{G}$ is a foliage graft for $\mathbf{F}]$,
- ➢ $\forall \mathbf{D} \neq \mathbf{E} \in \varphi\ [\mathsf{skeleton}\,\mathbf{D} \neq \mathsf{skeleton}\,\mathbf{E}]$, and
- ➢ $\{\mathsf{skeleton}\,\mathbf{G} : \mathbf{G} \in \varphi\}$ is a consistent family of grafts for $\mathsf{skeleton}\,\mathbf{F}$.

✎ If φ is a consistent family of foliage grafts for $\mathbf{F}$, then:
- ➢ $\mathsf{loss}(\mathbf{F}, \varphi) \ :=\ \bigcup_{\mathbf{G} \in \varphi} \mathsf{cut}(\mathbf{F}, \mathbf{G})$;
- ➢ $\mathsf{fol.hybr}(\mathbf{F}, \varphi) \ :=\ $ the **foliage hybrid** of $\mathbf{F}$ and φ $\ :=$
  the foliage tree $\mathbf{H}$ such that

$$\mathsf{skeleton}\,\mathbf{H} := \mathsf{hybrid}\big(\mathsf{skeleton}\,\mathbf{F}, \{\mathsf{skeleton}\,\mathbf{G} : \mathbf{G} \in \varphi\}\big) \quad \text{and}$$
$$\mathbf{H}_x := \begin{cases} \mathbf{G}_x \setminus \mathsf{loss}(\mathbf{F},\varphi), & \text{if } x \in \mathsf{implant}\,\mathbf{G} \text{ for some } \mathbf{G} \in \varphi; \\ \mathbf{F}_x \setminus \mathsf{loss}(\mathbf{F},\varphi), & \text{otherwise.} \end{cases}$$

Lemma 8.3 ([10, Lemma 21 and Proposition 23])**.** *Suppose that γ is a consistent family of grafts for a tree $\mathcal{T}$, $\mathcal{H} = \mathsf{hybrid}(\mathcal{T},\gamma)$, and $\mathcal{G} \in \gamma$.*

(a) $\mathsf{nodes}\,\mathcal{G} \subseteq \mathsf{nodes}\,\mathcal{H}$ *and* $\forall x, y \in \mathsf{nodes}\,\mathcal{G}\ [x <_{\mathcal{H}} y \leftrightarrow x <_{\mathcal{G}} y]$.

(b) $\mathsf{support}(\mathcal{T},\gamma) = \mathsf{nodes}\,\mathcal{H} \cap \mathsf{nodes}\,\mathcal{T}$ *and* $\forall x, y \in \mathsf{support}(\mathcal{T},\gamma)\ [x <_{\mathcal{H}} y \leftrightarrow x <_{\mathcal{T}} y]$.

(c) *For each* $x \in \mathsf{nodes}\,\mathcal{H}$,
$$\mathsf{sons}_{\mathcal{H}}(x) = \begin{cases} \mathsf{sons}_{\mathcal{G}}(x), & \textit{if } x \in \{0_{\mathcal{G}}\} \cup \mathsf{implant}\,\mathcal{G} \textit{ for some } \mathcal{G} \in \gamma; \\ \mathsf{sons}_{\mathcal{T}}(x), & \textit{else}\big(\textit{i.e., when } x \in \mathsf{support}(\mathcal{T},\gamma) \setminus \{0_{\mathcal{G}} : \mathcal{G} \in \gamma\}\big). \end{cases}$$

Lemma 8.4 ([10, Lemma 30])**.** *Suppose that $\mathbf{F}$ is a Baire foliage tree on a space X and φ is a consistent family of foliage grafts for $\mathbf{F}$ such that every $\mathbf{G}$ in φ is ω-branching, locally strict, open in X, has bounded chains, and has $\mathsf{height}\,\mathbf{G} \leqslant \omega$. Then the foliage hybrid of $\mathbf{F}$ and φ is a Baire foliage tree on $X \setminus \mathsf{loss}(\mathbf{F},\varphi)$.*

Definition 8.5 ([10, Definition 31 and Definition 33])**.** Suppose that $\mathbf{H}, \mathbf{F}$ are nonincreasing foliage trees and $\mathbf{G}$ is a foliage graft for $\mathbf{F}$.

✎ **H shoots into F** $:\longleftrightarrow$
$\forall p \in \mathsf{flesh}\,\mathbf{H}\ \forall y \in \mathsf{scope}_{\mathbf{F}}(p)\ \exists x \in \mathsf{scope}_{\mathbf{H}}(p)\ \big[\mathsf{shoot}_{\mathbf{H}}(x) \gg \mathsf{shoot}_{\mathbf{F}}(y)\big]$.

✎ **G preserves shoots** of **F** $:\longleftrightarrow$
$\forall p \in \mathsf{flesh}\,\mathbf{G}\ \forall y \in \mathsf{scope}_{\mathbf{F}}(p) \cap \big(\{0_{\mathbf{G}}\} \cup \mathsf{explant}(\mathbf{F},\mathbf{G})\big)$
$\exists x \in \mathsf{scope}_{\mathbf{G}}(p) \cap \big(\{0_{\mathbf{G}}\} \cup \mathsf{implant}\,\mathbf{G}\big)\ \big[\mathsf{shoot}_{\mathbf{G}}(x) \gg \mathsf{shoot}_{\mathbf{F}}(y)\big]$.

Lemma 8.6 ([10, Lemma 34])**.** *Suppose that $\mathbf{F}$ is a nonincreasing foliage tree, φ is a consistent family of foliage grafts for $\mathbf{F}$, the foliage hybrid of $\mathbf{F}$ and φ has nonempty leaves, and each $\mathbf{G} \in \varphi$ preserves shoots of $\mathbf{F}$. Then the foliage hybrid of $\mathbf{F}$ and φ shoots into $\mathbf{F}$.*

Lemma 8.7 ([10, Lemma 32])**.** *Suppose that a foliage tree $\mathbf{H}$ shoots into a foliage tree $\mathbf{F}$ and $\mathbf{F}$ grows into a space X. Then $\mathbf{H}$ grows into the subspace $X \cap \mathsf{flesh}\,\mathbf{H}$ of X.*

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Krasovskii Institute of Mathematics and Mechanics of UB RAS; 16 Sofia Kovalevskaya Street; 620990, Yekaterinburg, Russia and Ural Federal University; 19 Mira Street; 620002, Yekaterinburg, Russia

E-mail address: patrakeev@mail.ru