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by

Koichi Motooka

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Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
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WEAK SELECTIONS AND COUNTABLE COMPACTNESS

KOICHI MOTOOKA

Abstract. We prove that every countably compact Hausdorff space with a continuous weak selection is weakly orderable, which answers a question of Buhagiar and Gutev affirmatively. We also prove that every feebly compact regular space with a continuous weak selection is suborderable.

1. Introduction

All spaces in this paper are assumed to be Hausdorff topological spaces. For a space X, let $\mathcal{F}_2(X) = \{F \subset X : 1 \leq |F| \leq 2\}$, where $|F|$ is the cardinality of F. The set $\mathcal{F}_2(X)$ is assumed to have the *Vietoris topology* τ_V which has a base consisting of all sets of the form

$$\langle \mathcal{V} \rangle = \{S \in \mathcal{F}_2(X) : S \subset \bigcup \mathcal{V} \text{ and } S \cap V \neq \emptyset \text{ for each } V \in \mathcal{V}\},$$

where $\mathcal{V}$ runs over all finite families of open subsets of X. (It suffices to take only $|\mathcal{V}| \leq 2$ here.) We say that a function $\sigma : \mathcal{F}_2(X) \to X$ is a *weak selection* on the space X if $\sigma(F) \in F$ for every $F \in \mathcal{F}_2(X)$. A weak selection on the space X is said to be *continuous* if it is continuous with respect to the Vietoris topology on $\mathcal{F}_2(X)$ and the topology of X.

For a linear order $\preceq$ on a set X, let $\tau_{\preceq}$ be the order topology generated by $\preceq$. A space (X, τ) is *orderable* (respectively, *weakly orderable*) if $\tau_{\preceq} = \tau$ (respectively, $\tau_{\preceq} \subset \tau$) for some linear ordering $\preceq$ on the set X.

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A space (X, τ) is *suborderable* if it is a subspace of an orderable space. A suborderable space is also called a generalized ordered space (*GO-space*). The following implications hold for every space:

$$\begin{array}{ll} & \text{orderable} \to \text{suborderable} \to \text{weakly orderable} \\ (1) & \\ & \qquad\qquad\qquad\qquad\quad \to \text{admits a continuous weak selection.} \end{array}$$

Studying the relation between continuous weak selections and orderability properties of a space traces back to [9, Lemma 7.2]. In 1981, van Mill and Wattel proved that all implications in (1) are reversible for compact spaces.

Theorem 1.1 ([10, Theorem 1.1])**.** *A compact space is orderable if and only if it has a continuous weak selection.*

In 1984, van Mill and Wattel characterized suborderability of Tychonoff spaces in terms of weak selections.

Theorem 1.2 ([11, Theorem 3.1])**.** *A Tychonoff space is suborderable if and only if it has a locally uniform weak selection.*

For the definition of local uniformity, see Section 2.

Applying Theorem 1.2, Artico, Marconi, Pelant, Rotter and Tkachenko [1] proved that the last two implications in (1) remain reversible for countably compact Tychonoff spaces.

Theorem 1.3 ([1, Corollary 1.6])**.** *A countably compact Tychonoff space is suborderable if and only if it has a continuous weak selection.*

Buhagiar and Gutev gave an example of a non-regular (Hausdorff) countably compact space with a continuous weak selection, which is not suborderable [2, Example 3.9]. Motivated by this, they posed the following question.

Question 1.4 ([2, Question 1])**.** Let X be a countably compact space which has a (separately) continuous weak selection. Then, is it true that X is weakly orderable? What about if X is regular?

The purpose of this paper is to answer Question 1.4 for the case of a countably compact space with a continuous weak selection.

Theorem 1.5. *Every countably compact space with a continuous weak selection is weakly orderable.*

Under the additional assumption of regularity, we provide a much stronger conclusion than what Question 1.4 asks.

Theorem 1.6. *Every countably compact regular space with a continuous weak selection is suborderable.*

Since every suborderable space is hereditary normal, Theorem 1.6 implies the following.

Corollary 1.7. *Every countably compact regular space with a continuous weak selection is hereditary normal.*

The proofs of Theorems 1.5 and 1.6 are given in Section 3.

The part of Question 1.4 asking whether every countably compact (regular) space with a separately continuous weak selection is weakly orderable remains open.

2. Preliminaries

Every weak selection $\sigma : \mathcal{F}_2(X) \to X$ determines a natural order-like relation $\preceq_\sigma$ on the set X defined by letting $x \preceq_\sigma y$ if and only if $\sigma(\{x, y\}) = x$ for $\{x, y\} \in \mathcal{F}_2(X)$. The relation $\preceq_\sigma$ is total and antisymmetric, but it could fail to be transitive. We write $x \prec_\sigma y$ if $x \preceq_\sigma y$ and $x \neq y$.

Let σ be a weak selection on a space X. For disjoint subsets $A, B \subset X$, we will write $A \prec_\sigma B$ if $a \prec_\sigma b$ for every $(a, b) \in A \times B$. For a point $x \in X$, we use $A \prec_\sigma x$ and $x \prec_\sigma B$ instead of $A \prec_\sigma \{x\}$ and $\{x\} \prec_\sigma B$, respectively. We define

$$(\leftarrow, x)_\sigma = \{y \in X : y \prec_\sigma x\} \text{ and } (x, \rightarrow)_\sigma = \{y \in X : x \prec_\sigma y\}.$$

The topology $\tau_{\preceq_\sigma}$ on the space X having all $\preceq_\sigma$-open intervals $(\leftarrow, x)_\sigma$ and $(x, \rightarrow)_\sigma$ with $x \in X$ as a subbase is called the *selection topology* determined by σ [6].

We use the following theorems.

Theorem 2.1 ([7, Corollary 2.3])**.** *For every weak selection σ on a space X, the space $(X, \tau_{\preceq_\sigma})$ is regular.*

Note that Hrušák and Martínez-Ruiz [8] proved that the space $(X, \tau_{\preceq_\sigma})$ is Tychonoff for every weak selection σ on a space X.

Theorem 2.2 ([6, Theorem 3.5])**.** *If σ is a continuous weak selection on a space (X, τ), then $\tau_{\preceq_\sigma} \subset \tau$.*

Theorem 2.3 ([3, Theorem 2])**.** *Every countably compact space with a continuous weak selection is sequentially compact.*

A weak selection σ on a space X is said to be *locally uniform* [11] if for every $x \in X$ and for every neighborhood U of x there exists a neighborhood V of x which is contained in U and such that for all $p \in X \backslash U$ and $y \in V$,

$$\sigma(\{p, y\}) = p \text{ if and only if } \sigma(\{p, x\}) = p.$$

For two topologies τ and τ' on a set X, we say that a weak selection σ on the space X is (τ', τ)-*locally uniform* if for every $x \in X$ and for every τ'-neighborhood U of x there exists a τ-neighborhood V of x such that for all $p \in X \setminus U$, either

$$p \prec_\sigma V \text{ or } V \prec_\sigma p.$$

The notion of $(\tau_{\preceq_\sigma}, \tau)$-local uniformity was introduced under the name selection-local uniformity in [12].

Remark 2.4. A weak selection σ on a space (X, τ) is locally uniform if and only if σ is (τ, τ)-locally uniform [12, Proposition 2.1(ii)].

According to [12, Proposition 2.1 (i), (iii)], the following implications hold for every weak selection σ on a space (X, τ).

(2) $\qquad$ locally uniform $\rightarrow$ $(\tau_{\preceq_\sigma}, \tau)$-locally uniform $\rightarrow$ continuous.

Note that none of the implications in (2) are reversible in general; see [12, Remark 3.6].

We also use the following theorems.

Theorem 2.5 ([12, Theorem 3.1]). *If a space (X, τ) has a $(\tau_{\preceq_\sigma}, \tau)$-locally uniform weak selection σ, then X is weakly orderable.*

Theorem 2.6 ([12, Theorem 3.2]). *If a space X has a locally uniform weak selection, then X is suborderable.*

The symbol ω will denote the first infinite ordinal.

3. Proofs of Theorems 1.5 and 1.6

First, we prove the following lemma needed in the sequel.

Lemma 3.1. *Let (X, τ) be a countably compact space and σ a continuous weak selection on the space (X, τ). Assume that a topology τ' on the set X satisfies the following conditions:*

(a) *(X, τ') is regular;*

(b) *$\tau_{\preceq_\sigma} \subset \tau' \subset \tau$.*

Then σ is (τ', τ)-locally uniform.

Proof. Our proof is based on the argument in [1, Lemma 1.3]. By Theorem 2.3, (X, τ) is sequentially compact. Suppose to the contrary that the weak selection σ is not (τ', τ)-locally uniform. Then we can find $x \in X$ and τ'-neighborhood U of x satisfying the following property:

(3) for each τ-neighborhood V of x, there are $p_V \in X \setminus U$ and $y_V \in V$ such that $(p_V \prec_\sigma y_V$ and $x \prec_\sigma p_V)$ or $(y_V \prec_\sigma p_V$ and $p_V \prec_\sigma x)$.

For a subset V of X, $\overline{V}$ and $\overline{V}^{\tau'}$ denote the closure of V with respect to τ and τ', respectively.

Claim 1. For each $n \in \omega$, there exist a τ'-neighborhood V_n of x and points p_{V_n}, y_{V_n} satisfying the following conditions:

(i) $p_{V_n} \in X \setminus U$ and $y_{V_n} \in V_n$;
(ii) $(p_{V_n} \prec_\sigma y_{V_n}$ and $x \prec_\sigma p_{V_n})$ or $(y_{V_n} \prec_\sigma p_{V_n}$ and $p_{V_n} \prec_\sigma x)$;
(iii) $\overline{V_0} \subset U$ and $\overline{V_{n+1}} \subset V_n$;
(iv) $\overline{V_{n+1}} \prec_\sigma p_{V_n} \Longleftrightarrow x \prec_\sigma p_{V_n}$.

Proof of Claim 1. We choose V_n, p_{V_n} and y_{V_n} by induction on n. Since $x \in U \in \tau'$, by (a), there exists a τ'-neighborhood V_0 of x such that $V_0 \subset \overline{V_0}^{\tau'} \subset U$. Since $\overline{V_0} \subset \overline{V_0}^{\tau'}$ by (b), we have $\overline{V_0} \subset U$. Assume that V_n has been obtained. By (b), V_n is a τ-neighborhood of x. Thus, by (3), there exist $p_{V_n} \in X \setminus U$ and $y_{V_n} \in V_n$ satisfying (ii).

In case of $x \prec_\sigma p_{V_n}$, since $(\leftarrow, p_{V_n})_\sigma \in \tau_{\preceq_\sigma} \subset \tau'$ by (b), $(\leftarrow, p_{V_n})_\sigma \cap V_n$ is a τ'-neighborhood of x. By (a), there exists a τ'-neighborhood V_{n+1} of x such that $V_{n+1} \subset \overline{V_{n+1}}^{\tau'} \subset (\leftarrow, p_{V_n})_\sigma \cap V_n$. Since $\overline{V_{n+1}} \subset \overline{V_{n+1}}^{\tau'}$ by (b), we have that $\overline{V_{n+1}} \subset V_n$ and $\overline{V_{n+1}} \prec_\sigma p_{V_n}$.

In case of $p_{V_n} \prec_\sigma x$, a similar argument allows to find a τ'-neighborhood V_{n+1} of x such that $\overline{V_{n+1}} \subset V_n$ and $p_{V_n} \prec_\sigma \overline{V_{n+1}}$. Thus, we can find p_{V_n}, y_{V_n} and V_{n+1} which satisfy the conditions (i)–(iv). $\square$

Since $\{n \in \omega : x \prec_\sigma p_{V_n}\} \cup \{n \in \omega : p_{V_n} \prec_\sigma x\} = \omega$, without loss of generality, we may assume that $|\{n \in \omega : x \prec_\sigma p_{V_n}\}| = \omega$, that is, $p_{V_n} \prec_\sigma y_{V_n}$ and $x \prec_\sigma p_{V_n}$ for every $n \in \omega$.

Since the space X is sequentially compact, $X \times X$ is also sequentially compact [4, Theorem 3.10.35]. Therefore, the sequence $\{(p_{V_n}, y_{V_n})\}_{n\in\omega}$ has a subsequence $\{(p_{V_{n_m}}, y_{V_{n_m}})\}_{m\in\omega}$ converging to a point $(u, v) \in X \times X$. Since $u \in \overline{\{p_{V_{n_m}} : m \in \omega\}} \subset X \setminus U$ and $v \in \overline{\{y_{V_{n_m}} : m \in \omega\}} \subset \overline{V_0} \subset U$, we have $u \neq v$. Since $p_{V_{n_m}} \prec_\sigma y_{V_{n_m}}$ for every $m \in \omega$, by the continuity of σ, we have $u \prec_\sigma v$.

On the other hand, since $v \in \overline{\{y_{V_{n_k}} : k \in \omega \text{ and } k \geq m+1\}} \subset \overline{V_{n_{m+1}}}$ and $\overline{V_{n_{m+1}}} \subset \overline{V_{n_m+1}} \prec_\sigma p_{V_{n_m}}$ for every $m \in \omega$, we have that $v \prec_\sigma p_{V_{n_m}}$ for every $m \in \omega$. Thus, by the continuity of σ, we have $v \prec_\sigma u$. However, this contradicts the fact $u \prec_\sigma v$. The attained contradiction means that σ is (τ', τ)-locally uniform. $\square$

Proposition 3.2. *Every continuous weak selection σ on a countably compact space (X, τ) is $(\tau_{\preceq_\sigma}, \tau)$-locally uniform.*

Proof. Let σ be a continuous weak selection on a countably compact space (X, τ). Define $\tau' = \tau_{\preceq_\sigma}$. By Theorems 2.1 and 2.2, τ' satisfies items (a) and (b) of Lemma 3.1, respectively. Thus, by Lemma 3.1, σ is $(\tau_{\preceq_\sigma}, \tau)$-locally uniform. □

By Proposition 3.2 and Theorem 2.5, we have Theorem 1.5.

Proposition 3.3. *Every continuous weak selection on a countably compact regular space is locally uniform.*

Proof. Let σ be a continuous weak selection on a countably compact regular space (X, τ). Define $\tau' = \tau$. Since (X, τ) is regular, τ' satisfies item (a) of Lemma 3.1. By Theorem 2.2, τ' satisfies item (b) of Lemma 3.1. Thus, by Lemma 3.1, σ is (τ, τ)-locally uniform, which means that σ is locally uniform by Remark 2.4. □

By Proposition 3.3 and Theorem 2.6, we have Theorem 1.6.

4. On extension to feebly compact spaces

A space X is said to be *feebly compact (or, lightly compact)* if every locally finite family of open sets in X is finite. The following implications hold for every space:

$$\text{countably compact} \to \text{feebly compact} \to \text{pseudocompact}. \tag{4}$$

Here, a space X is said to be *pseudocompact* if every continuous real-valued function on X is bounded. Note that every pseudocompact Tychonoff space is feebly compact and none of the implications in (4) are reversible in general; see [13, 1.11(d), 1P(3) and 1U].

By the same argument as in [5] and [1], we can generalize Theorem 1.6 to feebly compact regular spaces as follows:

Theorem 4.1. *Every feebly compact regular space with a continuous weak selection is suborderable.*

Proof. Let X be a feebly compact regular space with a continuous weak selection σ. By the same argument as in [5, Theorem 2.3], we can prove that $X \times X$ is feebly compact. This and the same argument as in [1, Lemma 1.3] yield that σ is locally uniform. Thus, by Theorem 2.6, X is suborderable. □

Comparing Theorems 1.6 and 4.1, we can ask the following question.

Question 4.2. Is every feebly compact space with a continuous weak selection weakly orderable?

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Graduate School of Science and Engineering, Ehime University, Matsuyama, 790-8577, Japan

Email address: `k-motooka0426@outlook.jp`