http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 53, 2019

Pages 209–218

http://topology.nipissingu.ca/tp/

On a construction of Malykhin

by

C. A. Martínez-Ranero and U. A. Ramos-García

Electronically published on January 18, 2019

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 53 (2019)
Pages 209-218

E-Published on January 18, 2019

ON A CONSTRUCTION OF MALYKHIN

C. A. MARTÍNEZ-RANERO AND U. A. RAMOS-GARCÍA

Abstract. We construct, using $\diamondsuit$, a nondiscrete Hausdorff extremally disconnected topological group of size ω_1 where every countable subset is closed and discrete.

1. Introduction

A topological space is called *extremally disconnected* [8], if the closure of any open set in this space is open (or, equivalently, the closures of any two disjoint open sets are disjoint). In 1967, Arhangel'skii posed the problem of the existence in ZFC of a nondiscrete Hausdorff extremally disconnected topological group [1]. Recently Reznichenko and Sipacheva in [6] have announced a proof that the existence of a countable nondiscrete Hausdorff extremally disconnected group implies the existence of a rapid ultrafilter; hence, such a group cannot be constructed in ZFC because the nonexistence of rapid ultrafilters is consistent with ZFC (see [5]). The general case is still open. In fact, the uncountable version of Arhangel'skii's problem remains largely unexplored. Among the few results that exist, we can find a forcing construction of Malykhin of a nondiscrete Hausdorff extremally disconnected group in which all countable subsets are closed and discrete [4]. More explicitly, he introduced a σ-close forcing notion

2010 *Mathematics Subject Classification.* Primary 22A05, 03E35; Secondary 54G05.

Key words and phrases. Uncountable topological group, extremally disconnected, Jensen's diamond principle.

The authors gratefully acknowledge the support from PAPIIT grant IA100517, and Fondecyt Iniciación grant 11130490.

that forces a nondiscrete Hausdorff extremally disconnected *linear* (*i.e.*, with a neighborhood base at neutral element consisting of subgroups) group topology on a *Boolean* (*i.e.*, each element has order 2) group of cardinality ω_1 in which all countable subsets are closed and discrete. Recall that every Boolean group can be naturally viewed as vector space over the two element field $\mathbb{F}_2$. Then each Boolean group is isomorphic to $([\kappa]^{<\omega}, \triangle)$ (the finite subsets of κ equipped with the operation $\triangle$ of symmetric difference) for some cardinal κ. In particular, every Boolean group of size ω_1 is isomorphic to $([\omega_1]^{<\omega}, \triangle)$.

The purpose of this note is to present a construction of such a group using Jensen's diamond principle $\diamondsuit$. Here we show:

Theorem 1.1. *$\diamondsuit$ implies that there exists a nondiscrete Hausdorff extremally disconnected linear group topology on $([\omega_1]^{<\omega}, \triangle)$ where every countable subset is closed and discrete.*

Our construction can be considered a cleaner version of the original forcing construction. Moreover, the construction brings to light an important property which is hidden in the original construction of Malykhin. At first sight, it does not have any relationship with the countable version of the problem and also does not require the existence of ultrafilters on ω_1 with strong combinatorial properties. However, it follows from our construction that this group admits a countable quotient group which is a nondiscrete Hausdorff extremally disconnected group (see Section 3).

2. The construction

Throughout the rest of this paper, we will write $+$ to denote the group operation of a Boolean group. In particular, we write $([\omega_1]^{<\omega}, +)$ instead of $([\omega_1]^{<\omega}, \triangle)$.

The following lemma although elemental is a key ingredient for the construction to work.

Lemma 2.1 (Diagonalization)**.** *Let $\mathbb{G}$ be a nondiscrete Hausdorff Boolean topological group with $\langle U_n \colon n < \omega\rangle$ a neighborhood base at $0_\mathbb{G}$. Then for every open set U with $0_\mathbb{G} \in \overline{U}$ there exists a linearly independent set $\{x_n \colon n < \omega\} \subset \mathbb{G}$ such that*

(1) *$span\{x_n \colon n < \omega\} \setminus \{0_\mathbb{G}\} \subset U$, and*
(2) *$span\{x_k \colon k \geqslant n\} \subset U_n$ for every $n < \omega$.*

In particular, the filter generated by $\langle span\{x_k \colon k \geqslant n\} \colon n < \omega\rangle$ extends the neighborhood filter at $0_\mathbb{G}$, which generates a Hausdorff linear group topology on $\mathbb{G}$.

Proof. Construct recursively a sequence $\langle x_n : n \in \omega \rangle \subset \mathbb{G}$ and $\langle V_n : n \in \omega \rangle$ a sequence of open neighborhoods of $0_{\mathbb{G}}$ such that

(i) $x_0 \in (U_0 \setminus \{0_{\mathbb{G}}\}) \cap U$, $V_0 \cap (x_0 + V_0) = \emptyset$ and $x_0 + V_0 \subset (U_0 \setminus \{0_{\mathbb{G}}\}) \cap U$;
(ii) $x_n \in (U_n \setminus \{0_{\mathbb{G}}\}) \cap V_{n-1}$, $V_n \cap (x_n + V_n) = \emptyset$ and $x_n + V_n \subset (U_n \setminus \{0_{\mathbb{G}}\}) \cap V_{n-1}$, for every $n > 0$.

The construction follows easily from the following elementary fact: given V an open neighborhood of $0_{\mathbb{G}}$ and given $x \in V \setminus \{0_{\mathbb{G}}\}$, there is an open neighborhood V_x of $0_{\mathbb{G}}$ such that $V_x \sqcup (x + V_x) \subset V$.[1] Indeed, put $W = V \cap (x + V)$. Notice that $\{0_{\mathbb{G}}, x\} \subset W$ and W is an open set. Then there exists an open neighborhood V_x of $0_{\mathbb{G}}$ such that $x \notin V_x + V_x \subset W$, and we are done.

By construction the set $\{x_n : n < \omega\}$ is as required. □

To prove Theorem 1.1 we will use a well known equivalent statement of $\diamondsuit$ which says: there is a sequence $\langle A_\alpha \subseteq \alpha \times \alpha : \alpha < \omega_1 \rangle$ such that for every $A \subseteq \omega_1 \times \omega_1$, the set $\{\alpha : A \cap \alpha \times \alpha = A_\alpha\}$ is stationary. The sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ is called a $\diamondsuit$-sequence on $\omega_1 \times \omega_1$ (*e.g.*, see [2] Exercise (51) on page 92).

Proof of the Theorem 1.1. Let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence on $\omega_1 \times \omega_1$. Put $C = \omega_1 \cap \mathsf{LIM}$ and fix an enumeration $f : \omega_1 \to [\omega_1]^{<\omega}$ such that $f''\alpha = [\alpha]^{<\omega}$ for every $\alpha \in C$.[2] We will recursively construct a sequence $\langle \mathcal{H}_\alpha : \alpha \in C \rangle$ such that

(i) (Group topology) $\mathcal{H}_\alpha = \{H^\alpha_\beta \leqslant [\alpha]^{<\omega} : \beta < \alpha\}$ forms a filter base on $[\alpha]^{<\omega}$ and $\bigcap \mathcal{H}_\alpha = \{\varnothing\}$ which generates a nondiscrete Hausdorff linear group topology τ_α on $[\alpha]^{<\omega}$.
(ii) (Coherence and continuity) For all $\beta < \delta \leqslant \alpha$ with $\delta, \alpha \in C$
$$H^\alpha_\beta = H^\delta_\beta + [\alpha \setminus \delta]^{<\omega}.$$
(iii) (Small subsets are closed and discrete) $[\alpha]^{<\omega} \cap H^{\alpha+\omega}_\alpha = \{\varnothing\}$ for every $\alpha \in C$.
(iv) (Full seal) For every $\alpha \in C$, if $\varnothing \in \overline{\bigcup_{(\gamma,\beta)\in A_\alpha} f(\gamma) + H^\alpha_\beta}^{\tau_\alpha}$ then
$$H^{\alpha+\omega}_\alpha \setminus \{\varnothing\} \subset \bigcup_{(\gamma,\beta)\in A_\alpha} f(\gamma) + H^{\alpha+\omega}_\beta.$$

$\mathsf{Base\ step}$: If $\alpha = \omega$, let $H^\omega_n = [\omega \setminus n]^{<\omega}$ for every $n < \omega$. Clearly, $\mathcal{H}_\omega = \{H^\omega_n : n < \omega\}$ satisfies (i) and (ii). The rest of clauses are vacuous for this step.

[1]Here $\sqcup$ denotes the disjoint union.
[2]Here LIM denotes the class of limit ordinals.

Successor-limit step: Assume $\alpha = \delta + \omega$ with $\delta \in C$, and suppose $\mathcal{H}_\delta$ has been constructed satisfying the clauses (i) to (iv). Put

$$U_\delta = \bigcup_{(\gamma,\beta)\in A_\delta} f(\gamma) + H^\delta_\beta,$$

and then applying Lemma 2.1 to U_δ if $\varnothing \in \overline{U_\delta}^{\tau_\delta}$ (or to $[\delta]^{<\omega} \setminus \{\varnothing\}$ if $\varnothing \notin \overline{U_\delta}^{\tau_\delta}$) we can find a linearly independent set $\{x_n : n < \omega\} \subset [\delta]^{<\omega}$ satisfying the clauses of the lemma. Set

$$H^\alpha_\beta = \begin{cases} H^\delta_\beta + [\alpha \setminus \delta]^{<\omega} & \text{if } \beta < \delta; \\ \operatorname{span}\{\{\delta + k\} + x_k : k \geqslant n\} & \text{if } \beta = \delta + n \text{ for some } n. \end{cases}$$

We need to verify that $\mathcal{H}_\alpha = \{H^\alpha_\beta : \beta < \alpha\}$ satisfies the above four clauses. To see (i), first we check that $\mathcal{H}_\alpha$ forms a filter base on $[\alpha]^{<\omega}$. For this, fix β, $\beta' < \alpha$. We have to distinguish three cases.

Case 1. β, $\beta' < \delta$. Then $H^\alpha_\beta = H^\delta_\beta + [\alpha \setminus \delta]^{<\omega}$ and $H^\alpha_{\beta'} = H^\delta_{\beta'} + [\alpha \setminus \delta]^{<\omega}$. Since H^δ_β and $H^\delta_{\beta'}$ are subsets of $[\delta]^{<\omega}$, it easily follows that $H^\alpha_\beta \cap H^\alpha_{\beta'} = \left(H^\delta_\beta \cap H^\delta_{\beta'}\right) + [\alpha \setminus \delta]^{<\omega}$. Now, there exists $\beta'' < \delta$ such that $H^\delta_{\beta''} \subset H^\delta_\beta \cap H^\delta_{\beta'}$. Hence, $H^\alpha_{\beta''} \subset H^\alpha_\beta \cap H^\alpha_{\beta'}$.

Case 2. $\beta < \delta$ and $\beta' = \delta + n$ for some n. Then $H^\alpha_\beta = H^\delta_\beta + [\alpha \setminus \delta]^{<\omega}$ and $H^\alpha_{\beta'} = \operatorname{span}\{\{\delta + k\} + x_k : k \geqslant n\}$. From Lemma 2.1, we know that there is $m \geqslant n$ such that $\operatorname{span}\{x_k : k \geqslant m\} \subset H^\delta_\beta \cap \operatorname{span}\{x_k : k \geqslant n\}$. Then,

$$\operatorname{span}\{\{\delta+k\}+x_k : k \geqslant m\} \subset \left(H^\delta_\beta + [\alpha \setminus \delta]^{<\omega}\right) \cap \left(\operatorname{span}\{\{\delta + k\} + x_k : k \geqslant n\}\right),$$

that is, $H^\alpha_{\beta''} \subset H^\alpha_\beta \cap H^\alpha_{\beta'}$ where $\beta'' = \delta + m$.

Case 3. $\beta = \delta+m$ and $\beta' = \delta+n$ for some m and n. Then, $H^\alpha_\beta \cap H^\alpha_{\beta'} = H^\alpha_{\max\{\beta,\beta'\}}$, and we are done.

To prove $\bigcap \mathcal{H}_\alpha = \{\varnothing\}$, notice that

$$\bigcap_{n<\omega} H^\alpha_{\delta+n} = \{\varnothing\}.$$

Indeed, let $x \in [\alpha]^{<\omega} \setminus \{\varnothing\}$. Put $y = x \cap \delta$ and $z = x \setminus \delta$. Then, $x = y \sqcup z = y + z$ and there is n such that $z \subset \delta + n$. If $y = \varnothing$, then $z \neq \varnothing$ and $x = z \notin \operatorname{span}\{\{\delta + k\} + x_k : k \geqslant n\} = H^\alpha_{\delta+n}$. Otherwise, if $y \neq \varnothing$, then we can find $m \geqslant n$ such that $y \notin \operatorname{span}\{x_k : k \geqslant m\}$. Thus, $x \notin \operatorname{span}\{\{\delta + k\} + x_k : k \geqslant m\} = H^\alpha_{\delta+m}$.

To see (ii), let $\gamma \in C$ and $\beta < \gamma$ with $\gamma \leqslant \alpha = \delta+\omega$. If $\gamma = \alpha$, then the equality follows. Otherwise, necessarily $\gamma \leqslant \delta$, and so $H^\delta_\beta = H^\gamma_\beta + [\delta \setminus \gamma]^{<\omega}$.

Now, since $\beta < \gamma \leqslant \delta$, it follows that $H^\alpha_\beta = H^\delta_\beta + [\alpha \setminus \delta]^{<\omega}$. Therefore,

$$H^\alpha_\beta = \Big(H^\gamma_\beta + [\delta \setminus \gamma]^{<\omega}\Big) + [\alpha \setminus \delta]^{<\omega} = H^\gamma_\beta + [\alpha \setminus \gamma]^{<\omega}.$$

To check (iii), note that for every $x \in H^{\delta+\omega}_\delta = \operatorname{span}\{\{\delta+k\} + x_k \colon k < \omega\}$ with $x \neq \varnothing$, it follows that $x \setminus \delta \neq \varnothing$. Hence, $[\delta]^{<\omega} \cap H^{\delta+\omega}_\delta = \{\varnothing\}$.

Finally, to see (iv), assume $\varnothing \in \overline{U_\delta}^{\tau_\delta}$. Fix $x \in H^{\delta+\omega}_\delta \setminus \{\varnothing\}$. Put $y = x \cap \delta$ and $z = x \setminus \delta$. Then $y \in \operatorname{span}\{x_k \colon k < \omega\} \setminus \{\varnothing\}$, $z \in [\alpha \setminus \delta]^{<\omega}$, and $x = y \sqcup z = y + z$. Now, by Lemma 2.1 (1), we know that $\operatorname{span}\{x_k \colon k < \omega\} \setminus \{\varnothing\} \subset U_\delta$, so there exists $(\gamma, \beta) \in A_\delta$ such that $y \in f(\gamma) + H^\delta_\beta$. Thus,

$$x \in \big(f(\gamma) + H^\delta_\beta\big) + [\alpha \setminus \delta]^{<\omega} = f(\gamma) + \big(H^\delta_\beta + [\alpha \setminus \delta]^{<\omega}\big) = f(\gamma) + H^{\delta+\omega}_\beta.$$

Therefore, (iv) follows.

Limit-limit step: Assume α is a limit of limit ordinals, and suppose $\mathcal{H}_\delta$ has been constructed for every $\delta < \alpha$ (with $\delta \in C$) satisfying the clauses (i) to (iv). For every $\beta < \alpha$, set

$$H^\alpha_\beta = H^{\delta(\beta)}_\beta + [\alpha \setminus \delta(\beta)]^{<\omega},$$

where $\beta < \delta(\beta) \in \alpha \cap C$ and $\delta(\beta)$ is minimum with this property.

Let us verify that $\mathcal{H}_\alpha = \{H^\alpha_\beta \colon \beta < \alpha\}$ satisfies the clauses (i) to (iv). Indeed, to see (i), we need to check first that $\mathcal{H}_\alpha$ forms a filter base on $[\alpha]^{<\omega}$. For this, fix β, $\beta' < \alpha$. Then, $\delta = \max\{\delta(\beta), \delta(\beta')\} \in \alpha \cap C$ and β, $\beta' < \delta$. Thus, since $\mathcal{H}_\delta$ satisfies the clause (i), there is $\beta'' < \delta$ such that $H^\delta_{\beta''} \subset H^\delta_\beta \cap H^\delta_{\beta'}$. Then by clause (ii), we conclude that

$$H^{\delta(\beta'')}_{\beta''} + [\delta \setminus \delta(\beta'')]^{<\omega} \subset \Big(H^{\delta(\beta)}_\beta + [\delta \setminus \delta(\beta)]^{<\omega}\Big) \cap \Big(H^{\delta(\beta')}_{\beta'} + [\delta \setminus \delta(\beta')]^{<\omega}\Big).$$

From this, it easily follows that

$$H^{\delta(\beta'')}_{\beta''} + [\alpha \setminus \delta(\beta'')]^{<\omega} \subset \Big(H^{\delta(\beta)}_\beta + [\alpha \setminus \delta(\beta)]^{<\omega}\Big) \cap \Big(H^{\delta(\beta')}_{\beta'} + [\alpha \setminus \delta(\beta')]^{<\omega}\Big),$$

that is, $H^\alpha_{\beta''} \subset H^\alpha_\beta \cap H^\alpha_{\beta'}$.

To prove $\bigcap \mathcal{H}_\alpha = \{\varnothing\}$, let $x \in [\alpha]^{<\omega} \setminus \{\varnothing\}$. Then there exists $\delta \in \alpha \cap C$ such that $x \in [\delta]^{<\omega} \setminus \{\varnothing\}$. Hence, since $\bigcap \mathcal{H}_\delta = \{\varnothing\}$, we can find $\beta < \delta$ such that $x \notin H^\delta_\beta$. Now, by condition (ii), we have $H^\delta_\beta = H^{\delta(\beta)}_\beta + [\delta \setminus \delta(\beta)]^{<\omega}$. Next note that $x \notin H^\delta_\beta + [\alpha \setminus \delta]^{<\omega}$, and therefore

$$x \notin \Big(H^{\delta(\beta)}_\beta + [\delta \setminus \delta(\beta)]^{<\omega}\Big) + [\alpha \setminus \delta]^{<\omega} = H^\alpha_\beta.$$

To see (ii), let $\delta \in C$ and $\beta < \delta$ with $\delta \leqslant \alpha$. The case $\delta = \alpha$ is trivial. Suppose $\delta < \alpha$. By induction hypothesis, $H^\delta_\beta = H^{\delta(\beta)}_\beta + [\delta \setminus \delta(\beta)]^{<\omega}$.

Thus,

$$H_\beta^\delta + [\alpha \setminus \delta]^{<\omega} = \left(H_\beta^{\delta(\beta)} + [\delta \setminus \delta(\beta)]^{<\omega}\right) + [\alpha \setminus \delta]^{<\omega} = H_\beta^\alpha.$$

Finally, the clauses (iii) and (iv) hold for this step.

This completes the recursive construction.

Set $\mathcal{H} = \{\widetilde{H}_\beta \colon \beta < \omega_1\}$, where $\widetilde{H}_\beta = H_\beta^{\delta(\beta)} + [\omega_1 \setminus \delta(\beta)]^{<\omega}$ with $\delta(\beta) = \min\{\alpha \in C \colon \beta < \alpha\}$. Notice that $\widetilde{H}_\beta = H_\beta^\alpha + [\omega_1 \setminus \alpha]^{<\omega}$ for every $\alpha \in C$ with $\beta < \alpha$ (by clause (ii)). Also, from the clauses (i) and (ii), it easily follows that $\mathcal{H}$ forms a filter base on $[\omega_1]^{<\omega}$ and $\bigcap \mathcal{H} = \{\varnothing\}$. Let τ be the nondiscrete Hausdorff linear group topology on $([\omega_1]^{<\omega}, +)$ generated by $\mathcal{H}$.

We claim that τ is extremally disconnected where $[\alpha]^{<\omega}$ is closed and discrete for every $\alpha < \omega_1$.

Indeed, to see the second part, let $\alpha < \omega_1$. Without loss of generality, $\alpha \in C$. Then, by clause (iii), $[\alpha]^{<\omega} \cap H_\alpha^{\alpha+\omega} = \{\varnothing\}$. But $\widetilde{H}_\alpha = H_\alpha^{\alpha+\omega} + [\omega_1 \setminus (\alpha + \omega)]^{<\omega}$, so we have $[\alpha]^{<\omega} \cap \widetilde{H}_\alpha = \{\varnothing\}$.

In order to prove that τ is extremally disconnected, it suffices to show that for every τ-open set U with $\varnothing \in \overline{U}$ there exists $\alpha < \omega_1$ such that $\widetilde{H}_\alpha \subseteq \overline{U}$. For this, we need to introduce the following definition: given U a τ-open set we say that $A_U \subseteq \omega_1 \times \omega_1$ is a *code* for U if

$$U = \bigcup_{(\gamma,\beta) \in A_U} f(\gamma) + \widetilde{H}_\beta.$$

Clearly, a code for a τ-open set always exists.

Claim 2.2. Let U be a τ-open set such that $\varnothing \in \overline{U}$. Then, the set

$$C_U = \{\alpha \in C \colon \varnothing \in \overline{\bigcup_{(\gamma,\beta)\in A_U^\alpha} f(\gamma) + H_\beta^\alpha}^{\tau_\alpha} \text{ where } A_U^\alpha = A_U \cap (\alpha \times \alpha)\}$$

forms a club.

Proof of the claim. First, we check that C_U is closed. Suppose that $\langle \alpha_n \colon n < \omega \rangle$ is an increasing sequence in C_U, and let $\alpha = \sup \alpha_n$. Fix $\beta' < \alpha$. Then $\alpha \in C$ and there exists $n < \omega$ such that $\beta' < \alpha_n$. Thus, since $\alpha_n \in C_U$, we can find

$$x \in H_{\beta'}^{\alpha_n} \cap \bigcup_{(\gamma,\beta)\in A_U^{\alpha_n}} f(\gamma) + H_\beta^{\alpha_n}.$$

Now, since $H^{\alpha}_{\beta'} = H^{\alpha_n}_{\beta'} + [\alpha \setminus \alpha_n]^{<\omega}$, $H^{\alpha}_{\beta} = H^{\alpha_n}_{\beta} + [\alpha \setminus \alpha_n]^{<\omega}$ and $A^{\alpha_n}_U \subset A^{\alpha}_U$, it immediately follows that

$$x \in H^{\alpha}_{\beta'} \cap \bigcup_{(\gamma,\beta)\in A^{\alpha}_U} f(\gamma) + H^{\alpha}_{\beta}.$$

Therefore, $\alpha \in C_U$.

To see that C_U is unbounded we use a standard elementary submodel argument. Let $\alpha < \omega_1$ be given. Find a countable elementary submodel $M \prec H(\theta)$ (for some large enough θ) containing α, f, A_U, $\mathcal{H}$ as elements. Put $\delta = M \cap \omega_1$. Clearly $\alpha < \delta \in C$. Let us verify that $\delta \in C_U$. Indeed, let $\beta' < \delta$. Then we have $\widetilde{H}_{\beta'} \cap \bigcup_{(\gamma,\beta)\in A_U} f(\gamma) + \widetilde{H}_{\beta} \neq \varnothing$ (in $H(\theta)$), and by elementarity, this is true in M as well. Hence there is $x \in M$ witness to the fact that this intersection is nonempty. We claim that

$$x \in H^{\delta}_{\beta'} \cap \bigcup_{(\gamma,\beta)\in A^{\delta}_U} f(\gamma) + H^{\delta}_{\beta}.$$

To see this, first note that since x is a finite set, $x \subset M$. Thus $x \in [\delta]^{<\omega}$, but $\widetilde{H}_{\beta'} = H^{\delta}_{\beta'} + [\omega_1 \setminus \delta]^{<\omega}$ and $x \in \widetilde{H}_{\beta'}$, therefore it follows that $x \in H^{\delta}_{\beta'}$. On the other hand, we have there exits $(\gamma,\beta) \in A_U$ such that $x \in f(\gamma) + \widetilde{H}_{\beta}$ (in $H(\theta)$). Using again elementarity, there are γ, $\beta \in M \cap \omega_1 = \delta$ such that $(\gamma,\beta) \in A_U$ and $x \in f(\gamma) + \widetilde{H}_{\beta}$. Hence, $(\gamma,\beta) \in A^{\delta}_U$. Now, since $f(\gamma) \in [\delta]^{<\omega}$ and $\widetilde{H}_{\beta} = H^{\delta}_{\beta} + [\omega_1 \setminus \delta]^{<\omega}$, necessarily $x \in f(\gamma) + H^{\delta}_{\beta}$. Therefore,

$$x \in \bigcup_{(\gamma,\beta)\in A^{\delta}_U} f(\gamma) + H^{\delta}_{\beta}.$$

This completes the proof of C_U is unbounded. ∎

We are now in a position to prove that τ is extremally disconnected. Let U be a τ-open set such that $\varnothing \in \overline{U}$. By Claim 2.2, we know that C_U is a club. Thus, since $\{\alpha < \omega_1 \colon A_U \cap (\alpha \times \alpha) = A_\alpha\}$ is stationary, there is $\alpha \in C_U$ such that $A^{\alpha}_U = A_\alpha$. That is, $\varnothing \in \overline{\bigcup_{(\gamma,\beta)\in A_\alpha} f(\gamma) + H^{\alpha}_{\beta}}^{\tau_\alpha}$. Hence, by clause (iv), we have

$$H^{\alpha+\omega}_{\alpha} \setminus \{\varnothing\} \subset \bigcup_{(\gamma,\beta)\in A_\alpha} f(\gamma) + H^{\alpha+\omega}_{\beta}.$$

Now, since $\widetilde{H}_{\alpha} = H^{\alpha+\omega}_{\alpha} + [\omega_1 \setminus (\alpha+\omega)]^{<\omega}$ and $\widetilde{H}_{\beta} = H^{\alpha+\omega}_{\beta} + [\omega_1 \setminus (\alpha+\omega)]^{<\omega}$, it follows that

$$\widetilde{H}_{\alpha} \setminus \{\varnothing\} \subset \bigcup_{(\gamma,\beta)\in A_\alpha} f(\gamma) + \widetilde{H}_{\beta} \subseteq \bigcup_{(\gamma,\beta)\in A_U} f(\gamma) + \widetilde{H}_{\beta} = U.$$

That is, $\widetilde{H}_\alpha \subseteq \overline{U}$. Therefore τ is extremally disconnected. ■

3. Final remarks and questions

The group topology τ on $([\omega_1]^{<\omega}, +)$ constructed in Theorem 1.1 has the following property:

Claim 3.1. For every $\alpha \in C$ the subgroup $[\omega_1 \setminus \alpha]^{<\omega}$ is a nowhere dense closed set.

Proof of the claim. First, we check that $[\omega_1 \setminus \alpha]^{<\omega}$ is closed. Let $x \in [\omega_1]^{<\omega} \setminus [\omega_1 \setminus \alpha]^{<\omega}$. Put $y = x \cap \alpha$. So $y \in [\alpha]^{<\omega} \setminus \{\varnothing\}$. By clause (i), we can find $\beta < \alpha$ such that $y \notin H^\alpha_\beta$. Now, since $\widetilde{H}_\beta = H^\alpha_\beta + [\omega_1 \setminus \alpha]^{<\omega}$ and $x \in y + [\omega_1 \setminus \alpha]^{<\omega}$, we have $x + \widetilde{H}_\beta = \left(y + H^\alpha_\beta\right) + [\omega_1 \setminus \alpha]^{<\omega} \subset [\omega_1]^{<\omega} \setminus [\omega_1 \setminus \alpha]^{<\omega}$.

To see that $[\omega_1 \setminus \alpha]^{<\omega}$ is nowhere dense, notice that by construction in successor-limit step and by Lemma 2.1 (2), we can prove inductively that for all $\delta \in C$ and for each $\beta < \delta$ it follows that $H^\delta_\beta \restriction \omega = \{x \cap \omega \colon x \in H^\delta_\beta\}$ is a nontrivial subgroup of $[\omega]^{<\omega}$ and hence $\widetilde{H}_\beta \restriction \omega$ as well. Thus, $\widetilde{H}_\beta \nsubseteq [\omega_1 \setminus \alpha]^{<\omega}$ for all $\beta < \omega_1$. ■

Using the previous claim we can conclude that for every $\alpha \in C$ the topological quotient group $[\omega_1]^{<\omega}/[\omega_1 \setminus \alpha]^{<\omega}$ is a countable nondiscrete Hausdorff extremally disconnected group (being an open image of an extremally disconnected group). In fact, this quotient is topologically isomorphic to the topological group $([\alpha]^{<\omega}, +, \tau \restriction \alpha)$, where $\tau \restriction \alpha = \{U \restriction \alpha = \{x \cap \alpha \colon x \in U\} \colon U \in \tau\}$.

The above leaves the following open question.

Question 3.2. *Does there exist (consistently or in* ZFC*) a nondiscrete Hausdorff extremally disconnected group topology on* $([\omega_1]^{<\omega}, +)$ *such that all subgroups of the form* $[\omega_1 \setminus \alpha]^{<\omega}$ *are open (and therefore closed)?*

Note that for any such group, all countable subsets are closed and discrete. We would like to mention here that such a group topology must have a weight greater than ω_1, which makes guessing principles much harder to apply.

Proposition 3.3. *Let* τ *be a nondiscrete Hausdorff extremally disconnected group topology on* $([\omega_1]^{<\omega}, +)$ *such that all subgroups of the form* $[\omega_1 \setminus \alpha]^{<\omega}$ *are open. Then, the weight* $w(\tau) > \omega_1$*.*

Proof. It suffices to prove that the character $\chi(\tau, \varnothing) > \omega_1$. Let $\{U_\alpha \colon \alpha < \omega_1\}$ be a family of open neighborhoods at $\varnothing$. Since $[\omega_1 \setminus \alpha]^{<\omega}$ are open for every $\alpha < \omega_1$, we can choose recursively $\{x^0_\alpha, x^1_\alpha\} \in [U_\alpha]^2$ $(\alpha < \omega_1)$

so that $\gamma^0_\alpha = \min(x^0_\alpha) < \min(x^1_\alpha) = \gamma^1_\alpha$ and $\sup\{\gamma^1_\beta\colon \beta < \alpha\} < \gamma^0_\alpha$. Put $A_i = \{\gamma^i_\alpha\colon \alpha < \omega_1\}$ for each $i < 2$, and put $A_2 = \omega_1 \setminus \bigsqcup_{i<2} A_i$. Then we see that $[\omega_1]^{<\omega} \setminus \{\varnothing\} = \bigsqcup_{i<3} V_i$ where $V_i = \bigsqcup_{\gamma\in A_i}\{\gamma\} + [\omega_1 \setminus (\gamma+1)]^{<\omega}$ is an open set for all $i < 3$. Clearly $\min'' V_i = A_i$ for all $i < 3$. The extremal disconnectedness of τ implies that there exists $i < 3$ such that $V_i \cup \{\varnothing\}$ is an open neighborhood at $\varnothing$. Now there is $j < 2$ with $j \neq i$, and hence $\min'' V_j \cap \min'' V_i = \varnothing$. Thus, $\{U_\alpha\colon \alpha < \omega_1\}$ cannot form a neighborhood base at $\varnothing$. □

We do not know if the condition of all countable subsets are closed and discrete is compatible with all subgroups of the form $[\omega_1 \setminus \alpha]^{<\omega}$ are dense.

Question 3.4. *Does there exist (consistently or in* ZFC*) a nondiscrete Hausdorff extremally disconnected group topology on* $([\omega_1]^{<\omega}, +)$ *such that all subgroups of the form* $[\omega_1 \setminus \alpha]^{<\omega}$ *are dense and also all countable subsets are closed and discrete?*

Concerning Arhangel'skii's problem, it was known from the beginning that if κ is a measurable cardinal with $\mathcal{U}$ a uniform normal ultrafilter on κ then $\tau_{\mathcal{U}}$ the group topology (like Sirota's group topology on $([\omega]^{<\omega}, +)$, see [7]) generated by $\mathcal{U}^{<\omega} = \{[A]^{<\omega}\colon A \in \mathcal{U}\}$ on $([\kappa]^{<\omega}, +)$ forms a nondiscrete Hausdorff extremally disconnected linear group topology. In fact, similar to a result of Louveau on $([\omega]^{<\omega}, +)$ (see [3]), if the topological group $([\kappa]^{<\omega}, +, \tau_{\mathcal{U}})$ is extremally disconnected with $\mathcal{U}$ a uniform ultrafilter on κ uncountable then $\mathcal{U}$ is a Ramsey ultrafilter and hence κ is a measurable cardinal. Clearly, in these kinds of groups every subset of size $< \kappa$ is closed and discrete.

Question 3.5. *Does there exist (consistently or in* ZFC*) a nondiscrete Hausdorff extremally disconnected group topology on* $([\kappa]^{<\omega}, +)$ *with* $\kappa > \omega_1$ *a small uncountable cardinal (*e.g., $\omega_2, \omega_3, \ldots$*) such that all subsets of size* $< \kappa$ *are closed and discrete?*

Acknowledgements

The authors are very grateful to the anonymous referee for useful comments, which helped to considerably improve the exposition. They also thank Michael Hrušák for stimulating discussions on the subject.

References

1. Alexander Arhangelski, *Groupes topologiques extrémalement discontinus*, C. R. Acad. Sci. Paris Sér. A-B **265** (1967), A822–A825. MR 0222207
2. Kenneth Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam-New York, 1980, An introduction to independence proofs. MR 597342

3. Alain Louveau, *Sur un article de S. Sirota*, Bull. Sci. Math. (2) **96** (1972), 3–7. MR 0308326
4. V. I. Malykhin, *Extremally disconnected topological groups*, Uspekhi Mat. Nauk **34** (1979), no. 6(210), 59–66. MR 562819 (81c:54007)
5. Arnold W. Miller, *There are no Q-points in Laver's model for the Borel conjecture*, Proc. Amer. Math. Soc. **78** (1980), no. 1, 103–106. MR 548093
6. Evgenii Reznichenko and Ol'ga Sipacheva, *Discrete subsets in topological groups and countable extremally disconnected groups*, arXiv preprint arXiv:1608.03546 (2016).
7. S. M. Sirota, *A product of topological groups, and extremal disconnectedness*, Mat. Sb. (N.S.) **79 (121)** (1969), 179–192. MR 0242988
8. M. H. Stone, *Algebraic characterizations of special boolean rings*, Fundamenta Mathematicae **29** (1937), no. 1, 223–303.

Departamento de Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile.
Email address: cmartinezr@udec.cl

Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, Campus Morelia, Morelia, Michoacán, México 58089.
Email address, Corresponding author: ariet@matmor.unam.mx