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SOME TOPOLOGICAL GAMES, D-SPACES AND COVERING PROPERTIES OF HYPERSPACES

LIANG-XUE PENG*, YUAN SUN, AND SHANG-ZHI WANG

Abstract. We study topological games, D-spaces and covering properties of hyperspaces with the upper (lower) Vietoris topology $V^+(V^-)$. Let $\mathbf{C}$ be the class of all compact spaces. Let $\mathbf{W}$ be the class of all countable spaces. Let $\mathbf{1}$ denote the class of all one point spaces and empty set. We get the following conclusions:

If X is a $\mathbf{1}$-like T_1-space, then the hyperspace $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$ is a $\mathbf{1}$-like space. If X is a nc-$\mathbf{W}$-like T_1-space, then $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$ is a nc-$\mathbf{W}$-like space. If X is a $\mathbf{D1}$-like T_1-space, then $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$ is a $\mathbf{D1}$-like space. If X is a T_1-space and $(\mathcal{C}(X), V^+)$ is nc-$\mathbf{1}$-like, then X is $\mathbf{C}$-like. If X is a T_1-space and $(\mathcal{C}(X), V^-)$ is nc-$\mathbf{1}$-like, then X is $\mathbf{C}$-like. If X is a hemicompact Hausdorff space, then $(\mathcal{C}(X), V^+)$ is a nc-$\mathbf{1}$-like space. We finally show that if X is a Hausdorff topological space such that every closed compact subset of X is a G_δ-set of X, then $(\mathcal{C}(X), V^+)$ is a nc-$\mathbf{1}$-like space if and only if X is a hemicompact space. We point out that there exists a σ-compact ($\mathbf{1}$-like) T_2-space X such that the hyperspace $(\mathcal{C}(X), V^+)$ is not a nc-$\mathbf{1}$-like space.

If X is a T_1 D-space, then $(2^X, V^-)$ is a D-space. If X is a T_1-space, then X is a D-space if and only if $(\mathcal{C}(X), V^-)$ is a D-space. If X is a T_1-space and $(2^X, V^-)$ is a bD-space, then X is a bD-space. If X is a paracompact space, then $(2^X, V^-)$ is metacompact. If X_n is a T_1-space for each $n \in \mathbb{N}$ such that $\prod_{n\in\mathbb{N}} (\mathcal{C}(X_n), V_n^+)$ is Lindelöf, then $\prod_{n\in\mathbb{N}} X_n$ is Lindelöf.

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Introduction

Some properties of hyperspaces over a space X can be described by properties of the basic space. In [8], Maio and Kočinac studied covering properties of hyperspaces with the Vietoris topologies, as well as its lower and upper parts. The following notations and notions follow form [8] and [9].

Given a topological space, we define its hyperspaces as the following sets:

$2^X = \{A \subset X : A \text{ is closed and nonempty}\}$,

$\mathcal{C}(X) = \{A \in 2^X : A \text{ is compact}\}$.

2^X is topologized by the *Vietoris topology* defined as the topology generated by $\mathcal{B} = \{\langle U_1, \cdots, U_k\rangle : U_1, \cdots, U_k$ are open subsets of X, k is a positive integer$\}$, where $\langle U_1, \cdots, U_k\rangle = \{A \in 2^X : A \subset \bigcup_{j=1}^k U_j$ and $A \cap U_j \neq \emptyset$ for each $j \in \{1, \cdots, k\}\}$. The topology on 2^X generated by $\mathcal{B}$ is usually called the Vietoris topology, denoted V. If A is a subset of X and $\mathcal{A}$ is a family of subsets of X, then we write

$A^- = \{F \in 2^X : F \cap A \neq \emptyset\}$ and $\mathcal{A}^- = \{A^- : A \in \mathcal{A}\}$;

$A^+ = \{F \in 2^X : F \subset A\}$ and $\mathcal{A}^+ = \{A^+ : A \in \mathcal{A}\}$.

The *upper Vietoris topology* V^+ on 2^X is the topology whose base is the collection $\{U^+ : U \text{ is open in } X\}$. *The lower Vietoris topology* V^- is generated by all the sets U^-, $U \subset X$ nonempty, open. $(\mathcal{C}(X), V^+)$ denotes the hyperspace $\mathcal{C}(X)$ which is a subspace of $(2^X, V^+)$. $(\mathcal{C}(X), V^-)$ denotes the hyperspace $\mathcal{C}(X)$ which is a subspace of $(2^X, V^-)$. Let X be a topological space and $A \subset X$. In the hyperspaces $(\mathcal{C}(X), V^+)$ and $(\mathcal{C}(X), V^-)$, we denote $A^- = \{F \in \mathcal{C}(X) : F \cap A \neq \emptyset\}$ and $A^+ = \{F \in \mathcal{C}(X) : F \subset A\}$. In this note, we study topological games, D-spaces and covering properties of hyperspaces with the upper (lower) Vietoris topology $V^+(V^-)$.

Let $\mathbf{K}$ be a class of spaces which are hereditary with respect to closed subspaces. The notion of a $\mathbf{K}$-*like* space was introduced and studied by R. Telgársky in [12]. The notion of a *nc*-$\mathbf{K}$-like space was introduced by Peng and Shen in [11] is similar to the notion of a $\mathbf{K}$-like space. The difference of the two definitions is that the sets that player ONE chooses in a play of a *nc*-$\mathbf{K}$-like space may not be closed. Let $\mathbf{C}$ be the class of all compact spaces. Let $\mathbf{W}$ be the class of all countable spaces. Let $\mathbf{1}$ denote the class of all one point spaces and empty set. In this note, we study topological games, D-spaces and covering properties of hyperspaces with the upper (lower) Vietoris topology $V^+(V^-)$ and get the following conclusions:

If X is a $\mathbf{1}$-like T_1-space, then the hyperspace $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$ is a $\mathbf{1}$-like space. If X is a nc-$\mathbf{W}$-like T_1-space, then $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$ is a nc-$\mathbf{W}$-like space. If X is a $\mathbf{D1}$-like T_1-space, then $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$ is a $\mathbf{D1}$-like space. If X is a T_1-space and $(\mathcal{C}(X), V^+)$ is nc-$\mathbf{1}$-like, then X is $\mathbf{C}$-like. If X is a T_1-space and $(\mathcal{C}(X), V^-)$ is nc-$\mathbf{1}$-like, then X is $\mathbf{C}$-like. If X is a hemicompact Hausdorff space, then $(\mathcal{C}(X), V^+)$ is a nc-$\mathbf{1}$-like space. We finally show that if X is a Hausdorff topological space such that every closed compact subset of X is a G_δ-set of X, then $(\mathcal{C}(X), V^+)$ is a nc-$\mathbf{1}$-like space if and only if X is a hemicompact space. We point out that there exists a σ-compact ($\mathbf{1}$-like) T_2-space X such that the hyperspace $(\mathcal{C}(X), V^+)$ is not a nc-$\mathbf{1}$-like space.

If X is a T_1 D-space, then $(2^X, V^-)$ is a D-space. If X is a T_1-space, then X is a D-space if and only if $(\mathcal{C}(X), V^-)$ is a D-space. If X is a T_1-space and $(2^X, V^-)$ is a bD-space, then X is a bD-space. If X is a paracompact space, then $(2^X, V^-)$ is metacompact. If X_n is a T_1-space for each $n \in \mathbb{N}$ such that $\prod\limits_{n\in\mathbb{N}} (\mathcal{C}(X_n), V_n^+)$ is Lindelöf, then $\prod\limits_{n\in\mathbb{N}} X_n$ is Lindelöf.

The set of positive integers is denoted by $\mathbb{N}$ and $\omega = \mathbb{N} \cup \{0\}$. In notation and terminology we will follow [6].

Main results

Let $\mathbf{K}$ be a class of spaces which are hereditary with respect to closed subspaces. The notion of a $\mathbf{K}$-*like* space was introduced and studied by R. Telgársky in [12].

A sequence $(E_n : n \in \omega)$ of subsets of a space X is a *play* of $G(\mathbf{K}, X)$ if $E_0 = X$ and for each $n \in \omega$

(1) E_{2n+1} is the choice of player ONE;
(2) E_{2n+2} is the choice of player TWO;
(3) $E_{2n+1} \in \mathbf{K}$;
(4) $E_n \in 2^X \cup \{\emptyset\}$;
(5) $E_{2n+1} \subset E_{2n}$;
(6) $E_{2n+2} \subset E_{2n}$;
(7) $E_{2n+2} \cap E_{2n+1} = \emptyset$.

The player ONE wins the play if $\bigcap\{E_{2n} : n \in \omega\} = \emptyset$. A finite sequence $(E_m : m \leq n)$ is *admissible* for $G(\mathbf{K}, X)$ if $E_0, E_1, ..., E_n$ satisfy the above conditions (1)-(7). A function s is a *strategy* for player ONE if the domain of s consists of admissible sequences $(E_0, E_1, ..., E_n)$ with n even, and if $E_{n+1} = s(E_0, E_1, ..., E_n)$, then $(E_0, E_1, ..., E_n, E_{n+1})$ is admissible for $G(\mathbf{K}, X)$. The strategy s is a *winning strategy* for player ONE if it wins every play $(E_0, E_1, ...)$ of $G(\mathbf{K}, X)$, where $E_{2n+1} = s(E_0, E_1, ..., E_{2n})$, $n \in \omega$. If player ONE has a winning strategy in $G(\mathbf{K}, X)$, then X is said to be a $\mathbf{K}$-*like* space [12].

In [11], Peng and Shen introduced the notion of nc-$\mathbf{K}$-like which is similar to the notion of $\mathbf{K}$-like. The difference of the two definitions is that the sets that player ONE chooses in a play of a nc-$\mathbf{K}$-like space may not be closed.

A sequence $(E_n : n \in \omega)$ of subsets of a space X is a *play of* $G_{nc}(\mathbf{K}, X)$ if $E_0 = X$ and for each $n \in \omega$

(1) E_{2n+1} is the choice of player ONE;
(2) E_{2n+2} is the choice of player TWO;
(3) $E_{2n+1} \in \mathbf{K}$;
(4) $E_{2n} \in 2^X \cup \{\emptyset\}$;
(5) $E_{2n+1} \subset E_{2n}$;
(6) $E_{2n+2} \subset E_{2n}$;
(7) $E_{2n+2} \cap E_{2n+1} = \emptyset$.

Player ONE wins the play if $\bigcap\{E_{2n} : n \in \omega\} = \emptyset$. A function s is a *strategy for player ONE in* $G_{nc}(\mathbf{K}, X)$ if the domain of s consists of admissible sequences $(E_0, E_1, ..., E_n)$ with n even, and if $E_{n+1} = s(E_0, E_1, ..., E_n)$, then $(E_0, E_1, ..., E_n, E_{n+1})$ is admissible for $G_{nc}(\mathbf{K}, X)$. The strategy s is a *winning strategy for player ONE in* $G_{nc}(\mathbf{K}, X)$ if it wins every play $(E_0, E_1, ...)$ of $G_{nc}(\mathbf{K}, X)$, where $E_{2n+1} = s(E_0, E_1, ..., E_{2n})$, $n \in \omega$. If player ONE has a winning strategy in $G_{nc}(\mathbf{K}, X)$, then X is said to be a *nc-$\mathbf{K}$-like* space. Every $\mathbf{K}$-like space is a nc-$\mathbf{K}$-like space. In [11], it is pointed out that there is a space X which is a nc-$\mathbf{1}$-like space but not a $\mathbf{1}$-like space.

For a class $\mathbf{K}$ and a topological space $(X, \mathcal{T})$, let $\mathcal{B}$ be a base of $\mathcal{T}$. We denote $\mathcal{B}(x) = \{B \in \mathcal{B} : x \in B\}$ for each $x \in X$. The concept of weak $\mathbf{K}$-like was introduced by Peng in [10]. Let $WG(\mathbf{K}, X)$ be the following positional game with perfect information. There are two players, player ONE and player TWO. They choose alternatively consecutive terms of a sequence $(E_n : n \in \omega)$ of subsets of X, so that each player knows $\mathbf{K}, E_0, E_1, \ldots, E_n$ when he is choosing E_{n+1}.

A sequence $(E_n : n \in \omega)$ of subsets of X is a *play of* $WG(\mathbf{K}, X)$, if $E_0 = X$ and if for each $n \in \omega$

(1) E_{2n+1} is the choice of player ONE;
(2) E_{2n+2} is the choice of player TWO;
(3) $E_{2n+1} \in \mathbf{K}$;
(4) $E_n \in 2^X \cup \{\emptyset\}$;
(5) $E_{2n+1} \subset E_{2n}$;
(6) For each $x \in E_{2n+1}$, player TWO chooses a member $U(x) \in \mathcal{B}(x)$, and $E_{2n+2} = E_{2n} \setminus \bigcup\{U(x) : x \in E_{2n+1}\}$.

If $\bigcap\{E_{2n} : n \in \omega\} = \emptyset$, then the player ONE wins the play $(E_n : n \in \omega)$.

A finite sequence $(E_m : m \leq n)$ of subsets of X is *admissible for* $WG(\mathbf{K}, X)$ if the sequence $(E_0, E_1, \ldots, E_n, \emptyset, \ldots, \emptyset, \ldots)$ is a play of $WG(\mathbf{K}, X)$. A function s is a *strategy for player ONE in* $WG(\mathbf{K}, X)$ if the domain consists of admissible sequences $(E_0, \ldots, E_n)$ with n even, such that $s(E_0, \ldots, E_n) = E_{n+1}$ and $(E_0, \ldots, E_n, E_{n+1})$ is admissible for $WG(\mathbf{K}, X)$.

A strategy s is said to be *winning for player ONE in* $WG(\mathbf{K}, X)$, if player ONE wins each play of $WG(\mathbf{K}, X)$ by s. $WI(\mathbf{K}, X)$ denotes the set of all winning strategies of player ONE in $WG(\mathbf{K}, X)$. If $WI(\mathbf{K}, X) \neq \emptyset$, then X is called a *weak* $\mathbf{K}$*-like* space. If $\mathcal{B} = \mathcal{T}$ and X is weak $\mathbf{K}$-like with respect to $\mathcal{B}$, then X is a $\mathbf{K}$-like space. The definitions of $\mathbf{K}$-like spaces and weak $\mathbf{K}$-like spaces are very similar. In [10], Peng proved that $\mathbf{K}$-like and weak $\mathbf{K}$-like are equivalent. Some results on topological games can be found in [7], [13] and [15].

Lemma 1. *([10, Theorem 1]) A space X is a $\mathbf{K}$-like space if and only if X is a weak $\mathbf{K}$-like space.*

Similar to the concept of $WG(\mathbf{K}, X)$, we can give a notion of $WG^*(\mathbf{K}, X)$. Let $\mathcal{T}$ be the topology of the space X and let $\mathcal{B}$ be a base of $\mathcal{T}$. For each $x \in X$, let $\mathcal{B}^*(x)$ be a subfamily of $\mathcal{B}(x) = \{B \in \mathcal{B} : x \in B\}$ such that $\mathcal{B}^*(x)$ is a base of neighborhoods of the point x in X. If $\mathcal{B}(x)$ is replaced by $\mathcal{B}^*(x)$ in the definition of weak $\mathbf{K}$-like, then we denote the new game by $WG^*(\mathbf{K}, X)$. If player ONE has a winning strategy in $WG^*(\mathbf{K}, X)$, then X is said to be a $*\mathbf{K}$*-like* space. Similar to [10, Theorem 1], we have:

Lemma 2. *A space X is a $\mathbf{K}$-like space if and only if X is a $*\mathbf{K}$-like space.*

Similar to the notion of weak $\mathbf{K}$-like ($*\mathbf{K}$-like), we have a notion of *weak* nc-$\mathbf{K}$*-like* ($*nc$-$\mathbf{K}$*-like*). The difference of weak $\mathbf{K}$-like ($*\mathbf{K}$-like) and weak nc-$\mathbf{K}$-like ($*nc$-$\mathbf{K}$-like) is that the sets that player ONE chooses in a play of a weak nc-$\mathbf{K}$-like ($*nc$-$\mathbf{K}$-like) space may not be closed.

For a class $\mathbf{K}$ and a topological space $(X, \mathcal{T})$, let $\mathcal{B}$ be a base of $\mathcal{T}$. For each $x \in X$, let $\mathcal{B}^*(x)$ be a subfamily of $\mathcal{B}(x) = \{B \in \mathcal{B} : x \in B\}$ such that $\mathcal{B}^*(x)$ is a base of neighborhoods of the point x in X.

Let $WG_{nc}(\mathbf{K}, X)$ ($WG^*_{nc}(\mathbf{K}, X)$) be the following positional game with perfect information. There are two players, player ONE and player TWO. They choose alternatively consecutive terms of a sequence $(E_n : n \in \omega)$ of subsets of X, so that each player knows $\mathbf{K}, E_0, E_1, \ldots, E_n$ when he is choosing E_{n+1}.

A sequence $(E_n : n \in \omega)$ of subsets of X is a *play of* $WG_{nc}(\mathbf{K}, X)$ ($WG^*_{nc}(\mathbf{K}, X)$), if $E_0 = X$ and if for each $n \in \omega$

(1) E_{n+1} is the choice of player ONE;
(2) E_{2n+2} is the choice of player TWO;
(3) $E_{2n+1} \in \mathbf{K}$;
(4) $E_{2n} \in 2^X \cup \{\emptyset\}$;
(5) $E_{2n+1} \subset E_{2n}$;
(6) For each $x \in E_{2n+1}$, player TWO chooses a member $U(x) \in \mathcal{B}(x)$ ($U(x) \in \mathcal{B}^*(x)$), and $E_{2n+2} = E_{2n} \setminus \bigcup\{U(x) : x \in E_{2n+1}\}$.

If $\bigcap\{E_{2n} : n \in \omega\} = \emptyset$, then the player ONE wins the play $(E_n : n \in \omega)$.

A finite sequence $(E_m : m \leq n)$ of subsets of X is *admissible for* $WG_{nc}(\mathbf{K}, X)$ *(*$WG^*_{nc}(\mathbf{K}, X)$*)* if the sequence $(E_0, E_1, \ldots, E_n, \emptyset, \ldots, \emptyset, \ldots)$ is a play of $WG_{nc}(\mathbf{K}, X)$ ($WG^*_{nc}(\mathbf{K}, X)$). A function s is a *strategy for player ONE in* $WG_{nc}(\mathbf{K}, X)$ ($WG^*_{nc}(\mathbf{K}, X)$) if the domain consists of admissible sequences $(E_0, \ldots, E_n)$ with n even, such that $s(E_0, \ldots, E_n) = E_{n+1}$ and $(E_0, \ldots, E_n, E_{n+1})$ is admissible for $WG_{nc}(\mathbf{K}, X)$ ($WG^*_{nc}(\mathbf{K}, X)$).

A strategy s is said to be *winning for player ONE in* $WG_{nc}(\mathbf{K}, X)$ ($WG^*_{nc}(\mathbf{K}, X)$), if player ONE wins each play of $WG_{nc}(\mathbf{K}, X)$ ($WG^*_{nc}(\mathbf{K}, X)$) by s. If there is a strategy s which is winning for player ONE in $WG_{nc}(\mathbf{K}, X)$ ($WG^*_{nc}(\mathbf{K}, X)$), then X is called a *weak nc-*$\mathbf{K}$*-like (*nc-*$\mathbf{K}$*-like)* space. If $\mathcal{B} = \mathcal{T}$ and X is weak nc-$\mathbf{K}$-like with respect to $\mathcal{B}$, then X is a nc-$\mathbf{K}$-like space.

Similar to [10, Theorem 1], we have:

Lemma 3. *A space X is a nc-*$\mathbf{K}$*-like space if and only if X is a weak nc-*$\mathbf{K}$*-like space.*

Lemma 4. *A space X is a nc-*$\mathbf{K}$*-like space if and only if X is a $*$nc-*$\mathbf{K}$*-like space.*

Proposition 5. *If X is a T_1-space, then $\{\{x\}\}$ is closed in $(\mathcal{C}(X), V^-)$ $((2^X, V^-))$ for each $x \in X$.*

Proof. We just prove the case of $(\mathcal{C}(X), V^-)$. The proof of the case of $(2^X, V^-)$ is similar. Let $x \in X$ and let $C \in \mathcal{C}(X)$ such that $C \neq \{x\}$. Then there exists some $y \in C \setminus \{x\}$. Since X is a T_1-space, the set $\{x\}$ is closed in X. So $V = X \setminus \{x\}$ is open in X. Since $y \in C \setminus \{x\}$, the point $y \in C \cap V$. Thus $C \in V^-$ and $V^- \cap \{\{x\}\} = \emptyset$. So $\{\{x\}\}$ is closed in $(\mathcal{C}(X), V^-)$. □

If $X = \mathbb{R}$ with the usual topology, then $\{0\} \in \mathcal{C}(X)$ and $[-1, 1] \in \mathcal{C}(X)$. Since every open neighborhood of $[-1, 1]$ in $(\mathcal{C}(X), V^+)$ contains $\{0\}$, the set $\{\{0\}\}$ is not closed in $(\mathcal{C}(X), V^+)$.

If A is a countable subset of a T_1-space X, then $\{\{x\} : x \in A\}$ is a countable subset of $(2^X, V^-)$.

Theorem 6. *Let $(X, \mathcal{T})$ be a T_1-space and* $\mathbf{K}$ *be a class of sets such that if $E \subset X$ is in* $\mathbf{K}$*, then the set $\{\{x\} : x \in E\} \subset 2^X$ is in* $\mathbf{K}$*. If X is a* $\mathbf{K}$*-like space, then the space $(2^X, V^-)$ $(\mathcal{C}(X), V^-)$ is nc-*$\mathbf{K}$*-like.*

Proof. We just prove that $(2^X, V^-)$ is a nc-$\mathbf{K}$-like space. The proof of the case of $(\mathcal{C}(X), V^-)$ is similar. Let $\mathcal{B}$ be a base of $(2^X, V^-)$, which is generated by $\varphi = \{W^- : W \in \mathcal{T} \setminus \{\emptyset\}\}$. Since X is a T_1-space, $\{x\} \in 2^X$ for each $x \in X$. For each $x \in X$, let $\mathcal{B}^*(\{x\}) = \{V^- : x \in V \text{ and } V \in \mathcal{T}\}$. It is obvious that $\mathcal{B}^*(\{x\})$ is a base of neighborhoods of the point $\{x\}$ in $(2^X, V^-)$ and $\mathcal{B}^*(\{x\}) \subset \mathcal{B}(\{x\}) = \{B \in \mathcal{B} : \{x\} \in B\}$ for each $x \in X$. To prove $(2^X, V^-)$ is a nc-$\mathbf{K}$-like space, we just need to prove that $(2^X, V^-)$ is a $*nc$-$\mathbf{K}$-like space by Lemma 4.

Let s be a winning strategy for player ONE in $G(\mathbf{K}, X)$. Let $E_0 = X$ and let $A_0 = 2^X$. In what follows, we define a winning strategy for player ONE in $WG^*_{nc}(\mathbf{K}, 2^X)$.

Let $E_1 = s(E_0)$. Thus $E_1 \in \mathbf{K}$. So $\{\{x\} : x \in E_1\} \in \mathbf{K}$ and $\{\{x\} : x \in E_1\} \subset 2^X$. Define $t(A_0) = A_1 = \{\{x\} : x \in E_1\}$. For each $x \in E_1$, let V_x^- be any element of $\mathcal{B}^*(\{x\})$. If $A_2 = 2^X \setminus \bigcup\{V_x^- : x \in E_1\}$, then A_2 is closed in 2^X. If $y \in X \setminus \bigcup\{V_x : x \in E_1\}$, then $\{y\} \in A_2$. If $E_2 = X \setminus \bigcup\{V_x : x \in E_1\}$, then E_2 is closed in X. So (E_0, E_1, E_2) is admissiable for $G(K, X)$ and (A_0, A_1, A_2) is admissiable for $WG^*_{nc}(\mathbf{K}, 2^X)$.

Let $n \in \omega$ $(n \geq 1)$. Assume that we have an admissiable sequence $(E_0, \cdots, E_{2n})$ for $G(\mathbf{K}, X)$ and an admissiable sequence $(A_0, \ldots, A_{2n})$ for $WG^*_{nc}(\mathbf{K}, 2^X)$ with the following properties for each $k < n$:

(1) $E_{2k+1} = s(E_0, \ldots, E_{2k})$;
(2) $A_{2k+1} = t(A_0, \ldots, A_{2k}) = \{\{x\} : x \in E_{2k+1}\}$;
(3) For each $x \in E_{2k+1}$, $V_x^- \in \mathcal{B}^*(\{x\})$ and $A_{2k+2} = A_{2k} \setminus \bigcup\{V_x^- : x \in E_{2k+1}\}$;
(4) $E_{2k+2} = E_{2k} \setminus \bigcup\{V_x : x \in E_{2k+1}\}$;
(5) If $x \in E_{2k+2}$, then $\{x\} \in A_{2k+2}$.

If $E_{2n+1} = s(E_0, \ldots, E_{2n})$, then $E_{2n+1} \in \mathbf{K}$ and $\{\{x\} : x \in E_{2n+1}\} \in \mathbf{K}$. Since $E_{2n+1} \subset E_{2n}$, the set $\{\{x\} : x \in E_{2n+1}\} \subset A_{2n}$ by (5). If $A_{2n+1} = t(A_0, \ldots, A_{2n}) = \{\{x\} : x \in E_{2n+1}\}$, then $A_{2n+1} \in \mathbf{K}$ and $A_{2n+1} \subset A_{2n}$. For each $x \in E_{2n+1}$, let V_x^- be any element of $\mathcal{B}^*(\{x\})$. Thus $x \in V_x$ and V_x is open in X for each $x \in E_{2n+1}$. If $E_{2n+2} = E_{2n} \setminus (\bigcup\{V_x : x \in E_{2n+1}\})$, then E_{2n+2} is closed in X. If $A_{2n+2} = A_{2n} \setminus \bigcup\{V_x^- : x \in E_{2n+1}\}$, then A_{2n+2} is a closed subset of 2^X such that $A_{2n+2} \subset A_{2n}$ and $A_{2n+2} \cap A_{2n+1} = \emptyset$. Thus $(A_0, \ldots, A_{2n})$ is admissiable for $WG^*_{nc}(\mathbf{K}, 2^X)$ and $(E_0, \ldots, E_{2n})$ is admissiable for $G(\mathbf{K}, X)$.

If $y \in E_{2n+2}$, then $y \in E_{2n}$. Thus $\{y\} \in A_{2n}$. Since $y \in E_{2n+2}$, the point $y \notin V_x$ for each $x \in E_{2n+1}$. Thus $\{y\} \in A_{2n} \setminus \bigcup\{V_x^- : x \in E_{2n+1}\} = A_{2n+2}$.

So we get a play $(E_0, \ldots, E_{2n}, E_{2n+1}, \ldots)$ of $G(\mathbf{K}, X)$ and a play $(A_0, \ldots, A_{2n}, A_{2n+1}, \ldots)$ of $WG^*_{nc}(\mathbf{K}, 2^X)$ with the following properties:

(1) $E_{2n+1} = s(E_0, \ldots, E_{2n})$;
(2) $A_{2n+1} = t(A_0, \ldots, A_{2n}) = \{\{x\} : x \in E_{2n+1}\}$;
(3) $V_x^- \in \mathcal{B}^*(\{x\})$ for each $x \in E_{2n+1}$;
(4) $E_{2n+2} = E_{2n} \setminus \bigcup\{V_x : x \in E_{2n+1}\}$;
(5) $A_{2n+2} = A_{2n} \setminus \bigcup\{V_x^- : x \in E_{2n+1}\}$;
(6) If $x \in E_{2n}$, then $\{x\} \in A_{2n}$.

Since s is a winning strategy for player ONE in $G(\mathbf{K}, X)$, the set $\bigcap\limits_{n\in\omega} E_{2n} = \emptyset$. So $X = \bigcup\{\bigcup\{V_x : x \in E_{2n+1}\} : n \in \omega\}$. For any $E \in 2^X$, there exists some $n \in \omega$ such that $E \cap (\bigcup\{V_x : x \in E_{2n+1}\}) \neq \emptyset$. Thus $E \in V_x^-$ for some $x \in E_{2n+1}$. So $\bigcap\limits_{n\in\omega} A_{2n} = \emptyset$. Thus t is a winning strategy for player ONE in $WG^*_{nc}(\mathbf{K}, 2^X)$, Thus $(2^X, V^-)$ is a $*nc$-$\mathbf{K}$-like space. So X is a nc-$\mathbf{K}$-like space by Lemma 4. □

Similar to Theorem 6, we have the following Theorems 7 and 8:

Theorem 7. *Let $(X, \mathcal{T})$ be a T_1-space and $\mathbf{K}$ be a class of sets such that if $E \subset X$ is in $\mathbf{K}$ then $\{\{x\} : x \in E\} \subset 2^X$ is in $\mathbf{K}$ and closed in $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$. If X is a $\mathbf{K}$-like space, then the space $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$ is $\mathbf{K}$-like.*

Theorem 8. *Let $(X, \mathcal{T})$ be a T_1-space and $\mathbf{K}$ be a class of sets such that if $E \subset X$ is in $\mathbf{K}$, then $\{\{x\} : x \in E\} \subset 2^X$ is in $\mathbf{K}$. If X is a nc-$\mathbf{K}$-like space, then the space $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$ is nc-$\mathbf{K}$-like.*

Then by Proposition 5 and Theorem 7, we have:

Corollary 9. *If X is a $\mathbf{1}$-like T_1-space, then the hyperspace $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$ is a $\mathbf{1}$-like space.*

By Theorem 8, we have:

Corollary 10. *If $(X, \mathcal{T})$ is a nc-$\mathbf{W}$-like T_1 space, then the hyperspace $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$ is a nc-$\mathbf{W}$-like space.*

It is obvious that $(2^X, V^+)$ is a $\mathbf{1}$-like space for any space X. Proposition 31 in this article shows that there exists a $\mathbf{1}$-like Hausdorff space X such that $(\mathcal{C}(X), V^+)$ is not a nc-$\mathbf{1}$-like space.

Lemma 11. *Let X be a T_1-space. Then $D \subset X$ is a closed discrete subspace of X if and only if $D^* = \{\{d\} : d \in D\}$ is a closed discrete subspace of $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$.*

Proof. We just prove the case $(2^X, V^-)$, the proof of the other case is similar.

Assume that $D \subset X$ is a closed discrete subspace of X. Since X is a T_1-space, the set $D^* = \{\{d\} : d \in D\} \subset 2^X$. To show that D^* is closed discrete in $(2^X, V^-)$, it is sufficient to show that any subset $\mathcal{F}$ of D^* is closed in $(2^X, V^-)$. Let $\mathcal{F}$ be any subset of D^*. Then $\mathcal{F} = \{\{d\} : d \in F\}$ for some subset F of D. If $|F| = 1$, then $F = \{p\}$ for some $p \in D$. Thus $\mathcal{F} = \{\{p\}\}$ and $\mathcal{F}$ is closed in $(2^X, V^-)$ by Proposition 5. In what follows, we assume that $|F| > 1$. Let A be any point of $2^X \setminus \mathcal{F}$. Then A is a nonempty closed subset of X.

Case 1. $A \not\subset F$. Let $a \in A \setminus F$. Since $F \subset D$ and D is a closed discrete subspace of X, the set F is closed in X. If $O_a = X \setminus F$, then O_a is an open neighborhood of a in X. Thus O_a^- is an open neighborhood of the point A in $(2^X, V^-)$ such that $O_a^- \cap \mathcal{F} = \emptyset$.

Case 2. Now we assume that $A \subset F$. Since $A \notin \mathcal{F}$ and $A \subset F$, we know that $|A| \geq 2$. Let x, y be any distinct points of A. If $O_x = X \setminus (F \setminus \{x\})$ and $O_y = X \setminus (F \setminus \{y\})$, then O_x and O_y are open neighborhoods of x and y in X, respectively. Thus $O_A = O_x^- \cap O_y^-$ is an open neighborhood of the point A in $(2^X, V^-)$. Since $O_x \cap F = \{x\}$ and $O_y \cap F = \{y\}$, the set $O_A \cap \mathcal{F} = \emptyset$. Thus $\mathcal{F}$ is closed in $(2^X, V^-)$.

So D^* is a closed discrete subspace of $(2^X, V^-)$.

Now we prove the sufficiency. Assume that $D^* = \{\{d\} : d \in D\}$ is closed discrete in $(2^X, V^-)$. Let F be any subset of D and let x be any point of $X \setminus F$. Since $\{\{y\} : y \in F\}$ is closed discrete in $(2^X, V^-)$ and $\{x\} \notin \{\{y\} : y \in F\}$, there exists an open neighborhood V_x of x in X such that $V_x^- \cap \{\{y\} : y \in F\} = \emptyset$. So $V_x \cap F = \emptyset$. Thus F is closed in X. So D is closed discrete in X. □

Let **D1** be the class of discrete spaces. By Theorem 7 and Lemma 11, we have;

Corollary 12. *If X is a* **D1***-like T_1-space, then $(2^X, V^-)$ $(\mathcal{C}(X), V^-)$ is a* **D1***-like space.*

Proposition 13. *Let X be a T_1-space. If $i : X \to 2^X$ is a mapping such that $i(x) = \{x\}$ for each $x \in X$ and the topology of 2^X is V^-, then the mapping i is an embedding.*

Proposition 14. *Let X be a T_1-space. If $i : X \to 2^X$ is a mapping such that $i(x) = \{x\}$ for each $x \in X$ and the topology of 2^X is V^+, then the mapping i is an embedding and $i(X)$ is dense in 2^X.*

Proposition 15. *If X is a Hausdorff space, then X is homeomorphic to a closed subspace of $(2^X, V^-)$.*

Proof. Let $i : X \to (2^X, V^-)$ be a mapping such that $i(x) = \{x\}$ for each $x \in X$. Then X is homeomorphic to $i(X)$ by Proposition 13. In what follows we show that $i(X) = \{\{x\} : x \in X\}$ is closed in $(2^X, V^-)$ if X is a Hausdorff space. Let $E \in 2^X \setminus i(X)$. Then $|E| \geq 2$. Let $p, q \in E$ and $p \neq q$. Since X is a Hausdorff space, there are disjoint open subsets U, V of X such that $p \in U$ and $q \in V$. If $\mathcal{U} = U^- \cap V^-$, then $\mathcal{U}$ is an open neighborhood of E in $(2^X, V^-)$ such that $\mathcal{U} \cap i(X) = \emptyset$. Thus $i(X)$ is closed in $(2^X, V^-)$. $\square$

Corollary 16. *Let $\mathcal{P}$ be a topological property which is hereditary with respect to closed subsets. If X is a Hausdorff topological space such that $(2^X, V^-)$ has property $\mathcal{P}$, then X has property $\mathcal{P}$.*

In [11, Lemma 21], it is proved that if α is an ordinal then $\alpha + 1$ with the order topology is a **1**-like space. Thus $\omega_1 + 1$ is a **1**-like space. If $X = \{\alpha : \alpha < \omega_1\}$, then X is a subspace of $\omega_1 + 1$. Since X is not a Lindelöf space, X is not **1**-like. This shows that a subspace of a **1**-like space cannot be **1**-like. To study **K**-like properties of a space X by its hyperspace 2^X, we firstly study separation axioms of hyperspaces.

Proposition 17. *If $(X, \mathcal{T})$ is a T_1-space, then $(2^X, V^+)$ and $(\mathcal{C}(X), V^+)$ are T_0-spaces.*

Proof. We just prove that $(2^X, V^+)$ is a T_0-space. The proof of the other case is similar. Let A and B be any distinct points of $(2^X, V^+)$. Then $A \neq B$ and A, B are closed subsets of X. Thus there is some point $x \in A \setminus B$ or there is some point $y \in B \setminus A$. If there is some point $x \in A \setminus B$, then let $U_B = X \setminus \{x\}$. Thus $B \in U_B^+$ and U_B^+ is open in $(2^X, V^+)$ such that $A \notin U_B^+$. If there is some point $y \in B \setminus A$. Then let $U_A = X \setminus \{y\}$. Thus U_A^+ is an open neighborhood of A in $(2^X, V^+)$ and $B \notin U_A^+$. Thus $(2^X, V^+)$ is a T_0-space. $\square$

Proposition 18. *If $(X, \mathcal{T})$ is a topological space, then $(2^X, V^-)$ and $(\mathcal{C}(X), V^-)$ are T_0-spaces.*

Proof. If $A, B \in 2^X$ and $A \neq B$, then $A \setminus B \neq \emptyset$ or $B \setminus A \neq \emptyset$. If $A \setminus B \neq \emptyset$, then $U = X \setminus B$ is an open subset of X and $U \cap A \neq \emptyset$. Thus $A \in U^-$ and $B \notin U^-$. If $B \setminus A \neq \emptyset$, then $V = X \setminus A$ is open in X and $V \cap B \neq \emptyset$. Thus $B \in V^-$ and $A \notin V^-$. So $(2^X, V^-)$ is a T_0-space. Similarly, we can show that $(\mathcal{C}(X), V^-)$ is a T_0-space. $\square$

The spaces $(2^X, V^+)$ and $(\mathcal{C}(X), V^+)$ are not T_1-spaces. The reason is that if $A, B \in 2^X$ $(\mathcal{C}(X))$ and $A \subset B$ with $A \neq B$ then every open set U in $(2^X, V^+)$ $((\mathcal{C}(X), V^+))$ which contains B must contain A. Similarly, the spaces $(2^X, V^-)$ and $(\mathcal{C}(X), V^-)$ are not T_1-spaces.

Theorem 19. *Let $(X, \mathcal{T})$ be a T_1-space. If $(\mathcal{C}(X), V^+)$ is a nc-**1**-like space, then X is a **C**-like space.*

Proof. Since $(\mathcal{C}(X), V^+)$ is a nc-$\mathbf{1}$-like space, let s be a winning strategy for player ONE in $G_{nc}(\mathbf{1}, \mathcal{C}(X))$. Let $E_0 = \mathcal{C}(X)$ and let $A_0 = X$. In what follows we define a winning strategy t for player ONE in $G(\mathbf{C}, X)$. Let $E_1 = s(E_0) = \{B_1\}$, where $B_1 \in \mathcal{C}(X)$. Thus B_1 is a compact closed subset of X. Define $t(A_0) = A_1 = B_1$. Let A_2 be any closed subset of X such that $A_2 \cap A_1 = \emptyset$. Let $E_2 = \mathcal{C}(X) \setminus (X \setminus A_2)^+$, Then (E_0, E_1, E_2) is admissible for $G_{nc}(\mathbf{1}, \mathcal{C}(X))$ and (A_0, A_1, A_2) is admissible for $G(\mathbf{C}, X)$.

Let $n \in \mathbb{N}$. Assume that we have an admissible sequence $(A_0, \cdots, A_{2n})$ for $G(\mathbf{C}, X)$ and an admissible sequence $(E_0, \cdots, E_{2n})$ for $G_{nc}(\mathbf{1}, \mathcal{C}(X))$ with the following properties for each $k < n$:

(1) $E_{2k+1} = s(E_0, \cdots, E_{2k}) = \{B_{2k+1}\}$, where $B_{2k+1} \in \mathcal{C}(X)$;
(2) $A_{2k+1} = t(A_0, \cdots, A_{2k}) = B_{2k+1} \cap A_{2k} \neq \emptyset$;
(3) $U_{2k+1} = X \setminus A_{2k+2}$;
(4) $B_{2k+1} \in U^+_{2k+1}$;
(5) $E_{2k+2} = E_{2k} \setminus U^+_{2k+1}$.

Let $E_{2n+1} = s(E_0, \cdots, E_{2n}) = \{B_{2n+1}\}$, where $B_{2n+1} \in \mathcal{C}(X)$. So $E_{2n+1} \subset E_{2n}$ and hence $B_{2n+1} \in E_{2n}$. Since $E_{2n} = E_{2n-2} \setminus U^+_{2n-1} = E_{2n-2} \setminus (X \setminus A_{2n})^+$, the point $B_{2n+1} \notin (X \setminus A_{2n})^+$. So the set $B_{2n+1} \not\subset X \setminus A_{2n}$. Thus $B_{2n+1} \cap A_{2n} \neq \emptyset$. Since A_{2n} is closed in X and B_{2n+1} is a closed compact subset of X, the set $B_{2n+1} \cap A_{2n}$ is a closed compact subset of X. Define $A_{2n+1} = B_{2n+1} \cap A_{2n} = t(A_0, \cdots, A_{2n})$. Let A_{2n+2} be any closed subset of X such that $A_{2n+2} \subset A_{2n}$ and $A_{2n+2} \cap A_{2n+1} = \emptyset$. Then $(A_0, \cdots, A_{2n}, A_{2n+1}, A_{2n+2})$ is admissible for $G(\mathbf{C}, X)$. Let $U_{2n+1} = X \setminus A_{2n+2}$. Thus U_{2n+1} is an open subset of X such that $A_{2n+1} \subset U_{2n+1}$ and $B_{2n+1} \subset U_{2n+1}$. So $B_{2n+1} \in U^+_{2n+1}$. If $E_{2n+2} = E_{2n} \setminus U^+_{2n+1}$, then E_{2n+2} is closed in $\mathcal{C}(X)$ such that $E_{2n+2} \subset E_{2n}$ and $E_{2n+2} \cap E_{2n+1} = \emptyset$. Thus $(E_0, \cdots, E_{2n}, E_{2n+1}, E_{2n+2})$ is admissible for $G_{nc}(\mathbf{1}, \mathcal{C}(X))$.

So we get a play $(E_0, \cdots, E_{2n}, E_{2n+1}, \cdots)$ of $G_{nc}(\mathbf{1}, \mathcal{C}(X))$ and a play $(A_0, \cdots, A_{2n}, A_{2n+1}, \cdots)$ of $G(\mathbf{C}, X)$ with the following properties for each $n \in \omega$:

(1) $E_{2n+1} = s(E_0, \cdots, E_{2n}) = \{B_{2n+1}\}$, where $B_{2n+1} \in \mathcal{C}(X)$;
(2) $A_{2n+1} = t(A_0, \cdots, A_{2n}) = B_{2n+1} \cap A_{2n} \neq \emptyset$;
(3) $U_{2n+1} = X \setminus A_{2n+2}$;
(4) $B_{2n+1} \in U^+_{2n+1}$;
(5) $E_{2n+2} = E_{2n} \setminus U^+_{2n+1}$.

Since s is a winning strategy for player ONE in $G_{nc}(\mathbf{1}, \mathcal{C}(X))$. The set $\bigcap_{n \in \omega} E_{2n} = \emptyset$. For any $x \in X$, the set $\{x\}$ is closed in X following T_1-property of X. Thus $\{x\} \in \mathcal{C}(X)$. Let $m = \min\{n \in \omega : \{x\} \notin E_{2n}\}$. Thus $m \in \mathbb{N}$ and $\{x\} \in E_{2m-2} \setminus E_{2m}$. Since $E_{2m} = E_{2m-2} \setminus U^+_{2m-1}$, the point $\{x\} \in U^+_{2m-1}$. Thus $x \in U_{2m-1}$. So $x \notin X \setminus U_{2m-1} = A_{2m}$.

Thus $\bigcap\{A_{2n} : n \in \omega\} = \emptyset$. So t is a winning strategy for player ONE in $G(\mathbf{C}, X)$. Thus X is a $\mathbf{C}$-like space. $\square$

Theorem 20. *Let X be a T_1-space. If $(\mathcal{C}(X), V^-)$ is a nc-$\mathbf{1}$-like space, then X is a $\mathbf{C}$-like space.*

Proof. Since $(\mathcal{C}(X), V^-)$ is a nc-$\mathbf{1}$-like space, let s be a winning strategy for player ONE in $G_{nc}(\mathbf{1}, \mathcal{C}(X))$. In what follows, we define a winning strategy t for player ONE in $G(\mathbf{C}, X)$. Let $E_0 = \mathcal{C}(X)$ and let $A_0 = X$. Let $E_1 = s(E_0) = \{B_1\}$ for some $B_1 \in \mathcal{C}(X)$. Thus B_1 is a closed compact subset of X. Define $t(A_0) = A_1 = B_1$. Let A_2 be any closed subset of X such that $A_2 \cap A_1 = \emptyset$. If $V_1 = X \setminus A_2$, then V_1 is an open subset of X such that $A_1 \subset V_1$. Since $A_1 = B_1, B_1 \in V_1^-$. If $E_2 = \mathcal{C}(X) \setminus V_1^-$, then E_2 is a closed subset of $\mathcal{C}(X)$ and $E_2 \cap E_1 = \emptyset$. Thus (A_0, A_1, A_2) is admissible for $G(\mathbf{C}, X)$ and (E_0, E_1, E_2) is admissible for $G_{nc}(\mathbf{1}, \mathcal{C}(X))$.

Let $n \in \mathbb{N}$. Assume that we have an admissible sequence $(E_0, \cdots, E_{2n})$ in $G_{nc}(\mathbf{1}, \mathcal{C}(X))$ and an admissible sequence $(A_0, \cdots, A_{2n})$ in $G(\mathbf{C}, X)$ with the following properties for each $k < n$:

(1) $E_{2k+1} = s(E_0, \cdots, E_{2k}) = \{B_{2k+1}\}$, where $B_{2k+1} \in \mathcal{C}(X)$;
(2) $A_{2k+1} = t(A_0, \cdots, A_{2k}) = B_{2k+1}$;
(3) $V_{2k+1} = X \setminus A_{2k+2}$;
(4) $E_{2k+2} = E_{2k} \setminus V_{2k+1}^-$;
(5) If $C \in E_{2k+2}$, then $C \subset A_{2k+2}$.

Let $E_{2n+1} = s(E_0, \cdots, E_{2n}) = \{B_{2n+1}\}$. Thus B_{2n+1} is a closed compact subset of X and $B_{2n+1} \in E_{2n}$. Thus $B_{2n+1} \subset A_{2n}$. Define $A_{2n+1} = t(A_0, \cdots, A_{2n}) = B_{2n+1}$. Let A_{2n+2} be any closed subset of X such that $A_{2n+2} \subset A_{2n}$ and $A_{2n+2} \cap A_{2n+1} = \emptyset$. Let $V_{2n+1} = X \setminus A_{2n+2}$. Thus $A_{2n+1} \subset V_{2n+1}$. Since $B_{2n+1} = A_{2n+1}, B_{2n+1} \in V_{2n+1}^-$. If $E_{2n+2} = E_{2n} \setminus V_{2n+1}^-$, then $E_{2n+2} \subset E_{2n}$ is a closed subset of $\mathcal{C}(X)$ such that $E_{2n+2} \cap E_{2n+1} = \emptyset$. Thus $(E_0, \cdots, E_{2n+1}, E_{2n+2})$ is admissible for $G_{nc}(\mathbf{1}, C(X))$ and $(A_0, \cdots, A_{2n+1}, A_{2n+2})$ is admissible for $G(\mathbf{C}, X)$ and has the following property: If $C \in E_{2n+2}$, then $C \cap V_{2n+1} = \emptyset$ and $C \subset A_{2n+2}$.

So we get a play $(E_0, \cdots, E_{2n}, E_{2n+1}, \cdots)$ of $G_{nc}(\mathbf{1}, \mathcal{C}(X))$ and a play $(A_0, \cdots, A_{2n}, A_{2n+1}, \cdots)$ of $G(\mathbf{C}, X)$ with the following properties for each $n \in \omega$:

(1) $E_{2n+1} = s(E_0, \cdots, E_{2n}) = \{B_{2n+1}\}$, where $B_{2n+1} \in \mathcal{C}(X)$;
(2) $A_{2n+1} = t(A_0, \cdots, A_{2n}) = B_{2n+1}$;
(3) $V_{2n+1} = X \setminus A_{2n+2}$;
(4) $E_{2n+2} = E_{2n} \setminus V_{2n+1}^-$;
(5) If $C \in E_{2n+2}$, then $C \subset A_{2n+2}$.

Since s is a winning strategy for player ONE in $G_{nc}(\mathbf{1},\mathcal{C}(X))$, the set $\bigcap_{n\in\mathbb{N}} E_{2n}=\emptyset$. Let x be any element of X and let $n_x=\min\{m\in\omega:\{x\}\notin E_{2m}\}$. Thus $n_x>0$. So $\{x\}\in E_{2(n_x-1)}\setminus E_{2n_x}$.

Since $E_{2n_x}=E_{2(n_x-1)}\setminus V^-_{2n_x-1}$, the point $\{x\}\in V^-_{2n_x-1}$. Thus $x\in V_{2n_x-1}$. Since $V_{2n_x-1}=X\setminus A_{2n_x}$, the point $x\notin A_{2n_x}$. Thus $\bigcap_{n\in\omega} A_{2n}=\emptyset$. So the strategy t is a winning strategy for player ONE in $G(\mathbf{C},X)$. Thus X is a $\mathbf{C}$-like space. □

In what follows, we study the nc-$\mathbf{1}$-like property of the hyperspace $(\mathcal{C}(X),V^+)$ of a $\mathbf{C}$-like space X.

Recall that a space X is *hemicompact* if in the family of all compact subsets of X ordered by $\subset$ there exists a countable cofinal subfamily.

Theorem 21. *If X is hemicompact Hausdorff space, then $(\mathcal{C}(X),V^+)$ is a nc-$\mathbf{1}$-like space.*

Proof. By Lemma 3, we just need to prove that $(\mathcal{C}(X),V^+)$ is a weak nc-$\mathbf{1}$-like space with respect to a base $\mathcal{B}=\{U^+: U \text{ is open in } X\}$ of $(\mathcal{C}(X),V^+)$.

Since X is Hausdorff, $\mathcal{C}(X)$ is the family of all nonempty compact subsets of X. Since X is hemicompact, there exists a countable subfamily $\mathcal{A}=\{A_n:n\in\omega\}$ of $\mathcal{C}(X)$ such that for each $C\in\mathcal{C}(X)$ there exists some $m\in\omega$ such that $C\subset A_m$, we can assume that $A_n\subset A_{n+1}$ for each $n\in\omega$.

We define a winning strategy s for player ONE in $WG_{nc}(\mathbf{1},\mathcal{C}(X))$. Let $E_0=\mathcal{C}(X)$ and let $E_1=s(E_0)=\{A_0\}$ and let $m_1=0$. Let V_1 be any open subset of X such that $A_0\subset V_1$. Thus $A_{m_1}\in V_1^+$ and $V_1^+\in\mathcal{B}$. If $E_2=E_0\setminus V_1^+$, then E_2 is a closed subset of $\mathcal{C}(X)$ and $E_2\cap E_1=\emptyset$. Let $m_3=\min\{n\in\omega:A_n\not\subset V_1\}$. Thus $A_{m_3}\notin V_1^+$.

Let $n\in\omega$. Assume that we have an admissible sequence $(E_0,E_1,\cdots,E_{2n})$ for $WG_{nc}(\mathbf{1},\mathcal{C}(X))$ with the following properties for each $k<n$:

(1) $m_{2k+1}=\min\{n\in\omega:A_n\not\subset V_{2k-1}\}$;
(2) $E_{2k+1}=s(E_0,\cdots,E_{2n})=\{A_{m_{2k+1}}\}$;
(3) V_{2k+1} is any open subset of X such that $E_{2k+1}\subset V^+_{2k+1}$;
(4) $E_{2k+2}=E_{2k}\setminus V^+_{2k+1}$;
(5) If $i<j<n$, then $m_{2i+1}<m_{2j+1}$.

Thus $E_{2n-1}=\{A_{m_{2n-1}}\}$ and $A_{m_{2n-1}}\subset V_{2n-1}$.

Let $m_{2n+1}=\min\{k\in\omega:A_k\not\subset V_{2n-1}\}$. Thus $A_{m_{2n+1}}\notin V^+_{2n-1}$ and $m_{2n+1}>m_{2n-1}$. So if $k<n$ then $m_{2k+1}<m_{2n+1}$. Thus $A_{m_{2n+1}}\not\subset V_{2k+1}$ for each $k<n$. Thus $A_{m_{2n+1}}\in E_{2k+2}$ for each $k<n$.

So $A_{m_{2n+1}} \notin V^+_{2k+1}$ for each $k < n$. Thus $A_{m_{2n+1}} \in E_{2n} \setminus V^+_{2n-1}$. Define $s(E_0, \cdots, E_{2n}) = E_{2n+1} = \{A_{m_{2n+1}}\}$. Let V_{2n+1} be any open subset of X such that $A_{m_{2n+1}} \in V^+_{2n+1}$. Thus $E_{2n+2} = E_{2n} \setminus V^+_{2n+1}$ is a closed subset of $(\mathcal{C}(X), V^+)$ such that $E_{2n+2} \subset E_{2n}$ and $E_{2n+2} \cap E_{2n+1} = \emptyset$. So $(E_0, E_1, \cdots, E_{2n}, E_{2n+1}, E_{2n+2})$ is admissible for $WG_{nc}(\mathbf{1}, \mathcal{C}(X))$. In this way, we get a play $(E_0, \cdots, E_{2n}, E_{2n+1}, \cdots)$ of $WG_{nc}(\mathbf{1}, \mathcal{C}(X))$ with the following properties for each $n \in \omega$:

(1) $E_{2n+1} = s(E_0, \cdots, E_{2n}) = \{A_{m_{2n+1}}\}$;
(2) V_{2n+1} is any open subset of X such that $E_{2n+1} \subset V^+_{2n+1}$;
(3) $E_{2n+2} = E_{2n} \setminus V^+_{2n+1}$;
(4) $m_{2n+1} = \min\{k \in \omega : A_k \not\subset V_{2n-1}\}$;
(5) If $i < j$, then $m_{2i+1} < m_{2j+1}$.

Thus $\{m_{2n+1} : n \in \omega\}$ is a strict increasing sequence of $\mathbb{N}$. Thus $\{A_{m_{2n+1}} : n \in \omega\}$ is an increasing sequence of $\mathcal{A}$. So for any $C \in \mathcal{C}(X)$ there exists some $n \in \omega$ such that $C \subset A_{m_{2n+1}}$. Since $A_{m_{2n+1}} \subset V_{2n+1}$, the set $C \subset V_{2n+1}$. So $C \in V^+_{2n+1}$. Thus $\mathcal{C}(X) = \bigcup\{V^+_{2n+1} : n \in \omega\}$. Since $E_{2n+2} = E_{2n} \setminus V^+_{2n+1}$ for each $n \in \omega$, the set $\bigcap_{n\in\omega} E_{2n} = \emptyset$. Thus s is a winning strategy for player ONE in $WG_{nc}(\mathbf{1}, \mathcal{C}(X))$. So $(\mathcal{C}(X), V^+)$ is a weak nc-$\mathbf{1}$-like space. So $(\mathcal{C}(X), V^+)$ is a nc-$\mathbf{1}$-like space by Lemma 3. □

Since a $\mathbf{C}$-like space is Lindelöf and every Lindelöf locally compact Hausdorff space is hemicompact, we have:

Corollary 22. *If X is a $\mathbf{C}$-like locally compact Hausdorff space, then $(\mathcal{C}(X), V^+)$ is a nc-$\mathbf{1}$-like space.*

Let $n \in \mathbb{N}$ and let $\mathbb{R}$ be the set of reals with usual topology. By Theorem 21, we know that $(\mathcal{C}(\mathbb{R}^n), V^+)$ is a nc-$\mathbf{1}$-like space.

In what follows, we study the following questions:

Suppose that X is a σ-compact T_1-space. Is $(\mathcal{C}(X), V^+)$ is a nc-$\mathbf{1}$-like space?

Suppose that X is a T_1 $\mathbf{C}$-like space. Is $(\mathcal{C}(X), V^+)$ $((\mathcal{C}(X), V^-)$ a nc-$\mathbf{1}$-like space?

The following example shows that the converse of Theorem 20 does not hold.

Example 23. Let $X = [0, 3]$ be a space with the standard topology. Then X is $\mathbf{C}$-like, but $(\mathcal{C}(X), V^-)$ is not nc-$\mathbf{1}$-like.

Proof. Since X is compact, it is obvious that X is $\mathbf{C}$-like. Suppose that $(\mathcal{C}(X), V^-)$ is a nc-$\mathbf{1}$-like. Then let s be a winning strategy for player ONE in $G_{nc}(\mathbf{1}, \mathcal{C}(X))$. Let $E_0 = \mathcal{C}(X)$ and $E_1 = s(E_0)$. So $E_1 = \{B_1\}$ for some compact subset B_1 of X. Let $x_1 \in B_1$ and let $V_1 = \{y \in X : |y - x_1| < \frac{1}{2}\}$. Then V_1 is an open subset of X and $B_1 \in V^-_1$. Thus $E_2 = \mathcal{C}(X) \setminus V^-_1$ is closed in $\mathcal{C}(X)$ and (E_0, E_1, E_2) is an admissiable sequence in $G_{nc}(\mathbf{1}, \mathcal{C}(X))$. Let $n \in \omega$. Assume that we have an admissiable sequence $(E_0, E_1, \ldots, E_{2n})$ of $G_{nc}(\mathbf{1}, \mathcal{C}(X))$ with the following properties for each $k < n$:

(1) $E_{2k+1} = s(E_0, \ldots, E_{2k})$;
(2) $E_{2k+1} = \{B_{2k+1}\}$ for some compact subset B_{2k+1} of X;
(3) $x_{2k+1} \in B_{2k+1}$ and $V_{2k+1} = \{y \in X : |y - x_{2k+1}| < \frac{1}{2^{2k+1}}\}$;
(4) $E_{2k+2} = E_{2k} \setminus V_{2k+1}^-$.

Let $E_{2n+1} = s(E_0, \ldots, E_{2n})$. Then there exists some compact subset B_{2n+1} of X such that $E_{2n+1} = \{B_{2n+1}\}$. Let x_{2n+1} be any point of B_{2n+1} and let $V_{2n+1} = \{y \in X : |y - x_{2n+1}| < \frac{1}{2^{2n+1}}\}$. Thus $B_{2n+1} \in V_{2n+1}^-$. Denote $E_{2n+2} = E_{2n} \setminus V_{2n+1}^-$.

So we can get a play $(E_0, \ldots, E_{2n}, E_{2n+1}, \ldots)$ such that $E_{2n+1} = s(E_0, \ldots, E_{2n})$ and $E_{2n+2} = E_{2n} \setminus V_{2n+1}^-$, where $E_{2n+1} = \{B_{2n+1}\}$ for some compact subset B_{2n+1} of X, a point $x_{2n+1} \in B_{2n+1}$ and $V_{2n+1} = \{y \in X : |y - x_{2n+1}| < \frac{1}{2^{2n+1}}\}$. Since s is a winning strategy for player ONE in $G_{nc}(\mathbf{1}, \mathcal{C}(X))$. The set $\bigcap_{n\in\omega} E_{2n} = \emptyset$. But $\bigcup\{V_{2n+1} : n \in \omega\} \neq X$. If $a \in X \setminus \bigcup\{V_{2n+1} : n \in \omega\}$, then $\{a\} \in \mathcal{C}(X)$ and $\{a\} \in \mathcal{C}(X) \setminus \bigcup\{V_{2n+1}^- : n \in \omega\} = \bigcap_{n\in\omega} E_{2n}$. This contradicts that $\bigcap_{n\in\omega} E_{2n} = \emptyset$. Thus $(\mathcal{C}(X), V^-)$ is not a nc-$\mathbf{1}$-like space. □

The following example shows that there exists a T_1 $\mathbf{C}$-like space X such that player ONE has a winning strategy s and there exists a sequence $\{V_{2n+1} : n \in \omega\}$ of open subsets of X with the following properties:

(1) $E_1 = s(E_0) \subset V_1$, where $E_0 = X$;
(2) $E_{2n} = X \setminus \bigcup\{V_{2k+1} : k < n\}$ for each $n \in \mathbb{N}$;
(3) $E_{2n+1} = s(E_0, E_1, \ldots, E_{2n}) \subset V_{2n+1}$ for each $n \in \omega$;
(4) $X = \bigcup\{V_{2n+1} : n \in \omega\}$;
(5) $\mathcal{C}(X) \neq \bigcup\{V_{2n+1}^+ : n \in \omega\}$.

Example 24. Let $X = \mathbb{Q}$ be the set of rational numbers with the standard topology. The space X has the above properties.

Proof. Assume $X = \{q_n : n \in \mathbb{N}\}$. Let $E_0 = X$, $s(E_0) = \{q_1\} = E_1$. Denote $n_1 = 1$ and $V_1 = \{x \in X : |x - q_1| < 1\}$. Let $E_2 = X \setminus V_1$ and let $n_3 = \min\{l \in \mathbb{N} : q_l \in E_2\}$. Denote $s(E_0, E_1, E_2) = \{q_{n_3}\}$ and let $V_3 = \{x \in X : |x - q_{n_3}| < \frac{1}{3}\}$. Let $n \in \mathbb{N}$ and assume that we have a finite sequence $\{E_i : i \leq 2n\}$ of closed subsets of X with the following properties for each $k < n$:

(1) $E_0 = X$, $E_1 = s(E_0) = \{q_1\}$;
(2) $E_{2k+1} = s(E_0, \ldots, E_{2n}) = \{q_{m_{2k+1}}\}$;
(3) $V_{2k+1} = \{x \in X : |x - q_{m_{2k+1}}| < \frac{1}{2k+1}\}$;

(4) $E_{2k+2} = E_{2k} \setminus V_{2k+1}$;
(5) $m_{2k+1} = \min\{l \in \mathbb{N} : q_l \in E_{2k}\}$

Let $m_{2n+1} = \min\{l \in \mathbb{N} : q_l \in E_{2n}\}$ and let $s(E_0, \ldots, E_{2n}) = E_{2n+1} = \{q_{m_{2n+1}}\}$, $V_{2n+1} = \{x \in X : |x - q_{m_{2n+1}}| < \frac{1}{2n+1}\}$. Thus we have a play $(E_0, \ldots, E_{2n}, E_{2n+1}, \ldots)$ of $G(\mathbf{1}, X)$. We know that $\bigcap\limits_{n\in\omega} E_{2n} = \emptyset$. Thus s is a winning strategy for player ONE in $G(\mathbf{1}, X)$. So $X = \bigcup\{V_{2n+1} : n \in \omega\}$.

If $A = \{a_n : n \in \omega\}$, where $a_0 = 5$, $a_1 = 100$, $a_n = 5 - \frac{1}{n}$ for each $n \geq 2$. Thus $A \in \mathcal{C}(X)$. But $A \notin V^+_{2n+1}$ for each $n \in \omega$. $\square$

Now we study the question that whether $(\mathcal{C}(\mathbb{Q}), V^+)$ a nc-**1**-like space.

Lemma 25. *Let X be a topological space. Let A be a closed compact subset of X such that $A = \bigcap\{U_n : n \in \mathbb{N}\}$, where U_n is open in X for each $n \in \mathbb{N}$. Then for any $F \in \mathcal{C}(X)$ the set $F \setminus A \neq \emptyset$ if and only if $F \notin U^+_m$ for some $m \in \mathbb{N}$.*

Proof. Let $F \in \mathcal{C}(X)$. If $F \setminus A \neq \emptyset$, then let $b \in F \setminus A$. Since $A = \bigcap\{U_n : n \in \mathbb{N}\}$, there exists some $m \in \mathbb{N}$ such that $b \notin U_m$. So $F \notin U^+_m$. Now we assume that $F \notin U^+_m$ for some $m \in \mathbb{N}$. Thus $F \not\subset U_m$. Since $A \subset U_m$, the set $F \setminus A \neq \emptyset$. $\square$

Theorem 26. *Let X be a Hausdorff space. If $(\mathcal{C}(X), V^+)$ is a nc-**1**-like space and every compact subset of X is a G_δ-set of X, then X is hemicompact.*

Proof. Since X is a Hausdorff space, every compact subset of X is closed in X. Let s be a winning strategy for player ONE in $G_{nc}(\mathbf{1}, \mathcal{C}(X))$. Let $E_0 = \mathcal{C}(X)$. Then $E_1 = s(E_0) = \{B_1\}$ for some $B_1 \in \mathcal{C}(X)$. Denote $B_1 = B_1(m_0)$. Since every compact subset of X is a G_δ-set of X, there exists a sequence $\{U_{m_2} : m_2 \in \mathbb{N}\}$ of open subsets of X such that $B_1 = \bigcap\{U_{m_2} : m_2 \in \mathbb{N}\}$. For each $m_2 \in \mathbb{N}$, let $E_2(m_2) = \mathcal{C}(X) \setminus U^+_{m_2}$. Then $(E_0, E_1, E_2(m_2))$ is an admissible sequence for $G_{nc}(\mathbf{1}, \mathcal{C}(X))$. Let $n \in \omega$ and for each $k < n$ we have admissible sequences $(E_0, E_1, \ldots, E_{2k}(m_2, \ldots, m_{2k}))$ with the following properties:

(1) $E_{2k+1}(m_2, \ldots, m_{2k}) = s(E_0, E_1, E_2(m_2), \ldots, E_{2k}(m_2, \ldots, m_{2k}))$
$= \{B_{2k+1}(m_2, \ldots, m_{2k})\}$;
(2) $E_{2k}(m_2, \ldots, m_{2k}) \setminus \{C \in \mathcal{C}(X) : C \subset B_{2k+1}(m_2, \ldots, m_{2k})\}$
$= \bigcup\{E_{2k+2}(m_2, \ldots, m_{2k}, m_{2k+2}) : m_{2k+2} \in \mathbb{N}\}$,
where $E_{2k+2}(m_2, \ldots, m_{2k}, m_{2k+2})$ is closed in $(\mathcal{C}(X), V^+)$ and $(E_0, E_1, \ldots, E_{2k}(m_2, \ldots, m_{2k}), E_{2k+2}(m_2, \ldots, m_{2k}, m_{2k+2}))$ is admissible for $G_{nc}(\mathbf{1}, \mathcal{C}(X))$.

Thus $(E_0, E_1, \ldots, E_{2n}(m_2, \ldots, m_{2n}))$ is admissible for $G_{nc}(\mathbf{1}, \mathcal{C}(X))$.

So there exists some $B_{2n+1}(m_2,\ldots,m_{2n}) \in \mathcal{C}(X)$ such that
$s(E_0,E_1,\ldots,E_{2n}(m_2,\ldots,m_{2n}))$
$= E_{2n+1}(m_2,\ldots,m_{2n})$
$= \{B_{2n+1}(m_2,\ldots,m_{2n+1})\}$.

Since $B_{2n+1}(m_2,\ldots,m_{2n})$ is a G_δ-set of X, there exists a sequence $\{U_{2n+1}(m_2,\ldots,m_{2n},m_{2n+2}) : m_{2n+2} \in \mathbb{N}\}$ of open subsets of X such that
$B_{2n+1}(m_2,\ldots,m_{2n})$
$= \bigcap\{U_{2n+1}(m_2,\ldots,m_{2n},m_{2n+2}) : m_{2n+2} \in \mathbb{N}\}$.
Denote
$E_{2n+2}(m_2,\ldots,m_{2n},m_{2n+2})$
$= E_{2n}(m_2,\ldots,m_{2n}) \setminus U^+_{2n+1}(m_2,\ldots,m_{2n},m_{2n+2})$ for each $m_{2n+2} \in \mathbb{N}$.
Thus
$E_{2n}(m_2,\ldots,m_{2n}) \setminus \{C \in \mathcal{C}(X) : C \subset B_{2n+1}(m_2,\ldots,m_{2n})\}$
$= \bigcup\{E_{2n+2}(m_2,\ldots,m_{2n},m_{2n+2}) : m_{2n+2} \in \mathbb{N}\}$.
Thus
$(E_0,E_1,E_2(m_2),\ldots,E_{2n+1}(m_2,\ldots,m_{2n}),E_{2n+2}(m_2,\ldots,m_{2n+2}))$ is admissible for $G_{nc}(\mathbf{1},\mathcal{C}(X))$ and
$E_{2n}(m_1,\ldots,m_{2n}) \setminus \{C \in \mathcal{C}(X) : C \subset B_{2n+1}(m_2,\ldots,m_{2n})\}$
$= \bigcup\{E_{2n+2}(m_2,\ldots,m_{2n},m_{2n+2}) : m_{2n+2} \in \mathbb{N}\}$.

Thus we get plays
$(E_0,E_1,E_2(m_2),\ldots,E_{2n+1}(m_2,\ldots,m_{2n}),E_{2n+2}(m_2,\ldots,m_{2n+2}),\ldots)$
such that
$E_{2n+1}(m_2,\ldots,m_{2n})$
$= s(E_0,E_1,\ldots,E_{2n}(m_2,\ldots,m_{2n}))$
$= \{B_{2n+1}(m_2,\ldots,m_{2n})\}$.

So $\bigcap\limits_{n\in\omega} E_{2n}(m_2,\ldots,m_{2n}) = \emptyset$. In what follows, we show that $X = \bigcup\{B_{2n+1}(m_2,\ldots,m_{2n}) : n \in \omega\}$ and for every $C \in \mathcal{C}(X)$ there exists some $n \in \omega$ such that $C \subset B_{2n+1}(m_2,\ldots,m_{2n})$. Let C be any element of $\mathcal{C}(X)$. Suppose that $C \not\subset B_{2n+1}(m_2,\ldots,m_{2n})$ for each $n \in \omega$.

Since $C \not\subset B_1$ and there exists a sequence $\{U_{m_2} : m_2 \in \mathbb{N}\}$ of open subsets of X such that $B_1 = \bigcap\{U_{m_2} : m_2 \in \mathbb{N}\}$, there exists some $m_2 \in \mathbb{N}$ such that $C \notin U^+_{m_2}$ by Lemma 25. Since $E_2(m_2) = \mathcal{C}(X) \setminus U^+_{m_2}$, the point $C \in E_2(m_2)$.

Let $n \in \mathbb{N}$. Assume that there exists an admissible sequence
$(E_0,E_1,E_2(m_2),\ldots,E_{2n}(m_2,\ldots,m_{2n}))$ of $G_{nc}(\mathbf{1},\mathcal{C}(X))$ such that
$C \in \bigcap\{E_{2k}(m_2,\ldots,m_{2k}) : 1 \le k \le n\}$.

Thus
$E_{2n+1}(m_2,\ldots,m_{2n})$
$= s(E_0,\ldots,E_{2n}(m_2,\ldots,m_{2n}))$
$= \{B_{2n+1}(m_2,\ldots,m_{2n})\}$ for some closed compact subset
$B_{2n+1}(m_2,\ldots,m_{2n})$.

Since $C \not\subset B_{2n+1}(m_2, \ldots, m_{2n})$ and there exists a sequence $\{U_{2n+1}(m_2, \ldots, m_{2n}, m_{2n+2}) : m_{2n+2} \in \mathbb{N}\}$ of open subsets of X such that
$B_{2n+1}(m_2, \ldots, m_{2n})$
$= \bigcap\{U_{2n+1}(m_2, \ldots, m_{2n}, m_{2n+2}) : m_{2n+2} \in \mathbb{N}\}$, there exists some $m_{2n+2} \in \mathbb{N}$ such that $C \notin U^+_{2n+1}(m_2, \ldots, m_{2n}, m_{2n+2})$ by Lemma 25.
Thus
$C \in E_{2n}(m_2, \ldots, m_{2n}) \setminus U^+_{2n+1}(m_2, \ldots, m_{2n}, m_{2n+2})$
$= E_{2n+2}(m_2, \ldots, m_{2n}, m_{2n+2})$.

Then we can get a play
$(E_0, E_1, E_2(m_2), \ldots, E_{2n}(m_2, \ldots, m_{2n}), E_{2n+1}(m_2, \ldots, m_{2n}), \ldots)$ of $G_{nc}(\mathbf{1}, \mathcal{C}(X))$ such that
$C \in \bigcap\{E_{2n}(m_2, \ldots, m_{2n}) : n \in \omega\}$. A contradiction.

Thus for every $C \in \mathcal{C}(X)$ there exists some $n \in \omega$ such that $C \subset B_{2n+1}(m_2, \ldots, m_{2n})$. So $X = \bigcup\{B_{2n+1}(m_2, \ldots, m_{2n}) : m \in \omega\}$ and X is a hemicompact space. □

By Theorems 21 and 26, we have:

Corollary 27. *If X is a Hausdorff topological space such that every compact subset of X is a G_δ-set of X, then $(\mathcal{C}(X), V^+)$ is a nc-$\mathbf{1}$-like space if and only if X is hemicompact.*

The space $\mathbf{P}$ of the irrationals is identified with ω^ω. Given $p, q \in \mathbf{P}$ we denote by $p \leq q$ the fact that $p(n) \leq q(n)$ for any $n \in \omega$. A cover of a space X is compact if its elements are compact. A compact cover $\mathcal{K}$ of the space X is called $\mathbf{P}$-*ordered* if $\mathcal{K} = \{K_p : p \in \mathbf{P}\}$ and, for any $p, q \in \mathbf{P}$ with $p \leq q$ we have $K_p \subset K_q$ [14].

A space X is called *strongly* $\mathbf{P}$-*dominated* if it has a $\mathbf{P}$-ordered compact cover $\{K_p : p \in \mathbf{P}\}$ which swallows all compact subsets of X in the sense that, for any compact $K \subset X$ there is $p \in \mathbf{P}$ such that $K \subset K_p$ [14].

Let (X, ρ) be a metric space and let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of points of X. The sequence $\{x_i\}_{i\in\mathbb{N}}$ is called a *Cauchy sequence* in (X, ρ) if for every $\varepsilon > 0$ there exists a natural number k such that $\rho(x_i, x_k) \leq \varepsilon$ whenever $i \geq k$. A metric space (X, ρ) is *complete* if every Cauchy sequence in (X, ρ) is convergent to a point of X [6].

Theorem 28. *([4, Theorem 3.3]) A second countable space is strongly* $\mathbf{P}$*-dominated if and only if it is completely metrizable.*

The above conclusion is also cited in [14, Theorem 3.1].

Lemma 29. *([14, Proposition 3.3(a)]) Any hemicompact space is strongly* $\mathbf{P}$*-dominated.*

Let $X = \mathbb{Q}$ be the set of rational numbers with the standard topology. Let $x_n \in \mathbb{Q}$ such that $|x_n - \pi| < \frac{1}{n}$ for each $n \in \mathbb{N}$. Thus the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$. But $\{x_n\}_{n\in\mathbb{N}}$ does not converges to any point of $\mathbb{Q}$. Thus the space $\mathbb{Q}$ is not a complete metric space.

Thus we have:

Proposition 30. *If $X = \mathbb{Q}$, then X is not hemicompact.*

Proof. Since X is a second countable metric space and X is not complete, X is not **P**-dominated by Theorem 28. Thus X is not hemicompact by Lemma 29. □

Proposition 31. *If $X = \mathbb{Q}$, then X is a* **1***-like space but $(\mathcal{C}(X), V^+)$ is not a nc-***1***-like space.*

Proof. As proved in Example 24, the space X is a **1**-like space. By Corollary 27, a metric space Y is hemicompact if and only if $(\mathcal{C}(Y), V^+)$ is a *nc*-**1**-like space. Since X is not hemicompact by Proposition 30, the hyperspace $(\mathcal{C}(X), V^+)$ is not a *nc*-**1**-like space by Corollary 27. □

Thus we have:

Suppose that X is a σ-compact T_2-space. The hyperspace $(\mathcal{C}(X), V^+)$ cannot be a *nc*-**1**-like space. Thus the converse of Theorem 19 does not hold.

In what follows, we study the D-property and covering properties of hyperspaces of a space X.

A *neighborhood assignment* for a space X is a function ϕ from X to the topology of the space X such that $x \in \phi(x)$ for any $x \in X$ [5]. A space X is called a *D-space* if for any neighborhood assignment ϕ for X there exists a closed discrete subspace D of X such that $X = \bigcup\{\phi(d) : d \in D\}$ [5]. In what follows, we study some duality in connection with D-spaces and covering properties.

Theorem 32. *If X is a T_1 D-space, then $(2^X, V^-)$ $((\mathcal{C}(X), V^-))$ is a D-space.*

Proof. Let ϕ be any neighborhood assignment for $(2^X, V^-)$. Since X is a T_1-space, $\{x\} \in 2^X$ for each $x \in X$. Thus there exists an open neighborhood $\psi(x)$ of x in X such that $\psi(x)^- \subset \phi(\{x\})$ for each $x \in X$. Thus $\{\psi(x) : x \in X\}$ is a neighborhood assignment for X. Since X is a D-space, there exists a closed discrete subspace D of X such that $X = \bigcup\{\psi(d) : d \in D\}$. Let $D^* = \{\{d\} : d \in D\}$. Then $D^* \subset 2^X$ and D^* is a closed discrete subspace of $(2^X, V^-)$ by Lemma 11. Thus $D^* \subset 2^X$ and D^* is a closed discrete subspace of $(2^X, V^-)$ such that $2^X = \bigcup\{\phi(p) : p \in D^*\}$. Thus $(2^X, V^-)$ is a D-space.

Similarly, we can show that $(\mathcal{C}(X), V^-)$ is a D-space if X is a T_1 D-space. □

Theorem 33. *Let X be a T_1-space. Then $(\mathcal{C}(X), V^-)$ is a D-space if and only if X is a D-space.*

Proof. If X is a T_1 D-space, then $(\mathcal{C}(X), V^-)$ is a D-space by Theorem 32.

Now assume that $(\mathcal{C}(X), V^-)$ is a D-space. Let ϕ be any neighborhood assignment for X. For each $A \in \mathcal{C}(X)$, A is a compact closed subset of X. Thus there exists a finite subset $D_A \subset A$ such that $A \subset \bigcup\{\phi(d) : d \in D_A\} = O_A$. If $\psi(A) = O_A^-$ for each $A \in \mathcal{C}(X)$, then $\{\psi(A) : A \in \mathcal{C}(X)\}$ is a neighborhood assignment for $(\mathcal{C}(X), V^-)$. Since $(\mathcal{C}(X), V^-)$ is a D-space, there exists a closed discrete subspace $\mathcal{F}$ of $\mathcal{C}(X)$ such that $\mathcal{C}(X) = \bigcup\{\psi(A) : A \in \mathcal{F}\}$. Let $D = \bigcup\{D_A : A \in \mathcal{F}\}$. Since X is a T_1-space, $\{x\} \in \mathcal{C}(X)$ for each $x \in X$. Let $x \in X$. Thus there exists some $A \in \mathcal{F}$ such that $\{x\} \in \psi(A)$. Thus there exists some $d \in D_A$ such that $x \in \phi(d)$. Thus $X = \bigcup\{\phi(d) : d \in D\}$.

In what follows we show that D is a closed discrete subspace of X. Let x be any point of X. If $\{x\} \notin \mathcal{F}$, then there exists an open neighborhood O_x of x in X such that $O_x^- \cap \mathcal{F} = \emptyset$. Thus $O_x \cap D = \emptyset$. Now assume that $\{x\} \in \mathcal{F}$. Thus $x \in D$. Since $\mathcal{F}$ is discrete in $(\mathcal{C}(X), V^-)$, there exists an open neighborhood W_x of x in X such that $W_x^- \cap (\mathcal{F} \setminus \{x\}) = \emptyset$. Thus $W_x \cap (D \setminus \{x\}) = \emptyset$.

So D is a closed discrete subspace of X and $X = \bigcup\{\phi(d) : d \in D\}$. Thus X is a D-space. □

A space X is a *bD-space* if for each open cover $\mathcal{U}$ of X there exist a locally finite subset A of X and a mapping ϕ of A into $\mathcal{U}$ such that $a \in \phi(a)$ for each $a \in A$ and $\{\phi(a) : a \in A\}$ covers X [1]. It was observed by Borges and Wehrly in [3] that all subparacompact spaces are bD-spaces.

Theorem 34. *Let X be a T_1-space. If $(2^X, V^-)$ is a bD-space, then X is a bD-space.*

Proof. Let $\mathcal{U}$ be any open cover of X. Then $\mathcal{U}^- = \{U^- : U \in \mathcal{U}\}$ is an open cover of $(2^X, V^-)$. Since $(2^X, V^-)$ is a bD-space, there exists a locally finite subset $\mathcal{F} \subset 2^X$ and a mapping φ of $\mathcal{F}$ into $\mathcal{U}^-$ such that $F \in \varphi(F)$ for each $F \in \mathcal{F}$ and $2^X = \bigcup\{\varphi(F) : F \in \mathcal{F}\}$. Since X is a T_1-space, $\{x\} \in 2^X$ for each $x \in X$. Since $\mathcal{F}$ is locally finite in $(2^X, V^-)$, for each $x \in X$ there exists an open neighborhood V_x of x in X such that $\{F \in \mathcal{F} : F \in V_x^-\}$ is finite. So $|\{F \in \mathcal{F} : F \cap V_x \neq \emptyset\}| < \omega$. Thus the family $\mathcal{F}$ is locally finite in X. For each $F \in \mathcal{F}$, the set $\varphi(F) = U_F^-$ for some U_F of $\mathcal{U}$ and $U_F \cap F \neq \emptyset$. Let $d_F \in U_F \cap F$ for each $F \in \mathcal{F}$ and let $D = \{d_F : F \in \mathcal{F}\}$. Define a mapping ϕ from D into $\mathcal{U}$ such that $\phi(d_F) = U_F$ for each $F \in \mathcal{F}$. Thus $\{d_F : F \in \mathcal{F}\}$ is a locally finite subset of X and $X = \bigcup\{\phi(d_F) : F \in \mathcal{F}\}$. So X is a bD-space. □

Since every D-space is a bD-space, we have:

Corollary 35. *Let X be a T_1-space. If $(2^X, V^-)$ is a D-space, then X is a bD-space.*

Since every closed subspace of a D-space is a D-space, by Proposition 15 we have:

Theorem 36. *Let X be a Hausdorff space. If $(2^X, V^-)$ is a D-space, then X is a D-space.*

Recall that a space X is *countably compact* if every countable open cover $\mathcal{U}$ of X there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $X = \bigcup \mathcal{V}$.

Proposition 37. *Let X be a T_1-space. If $(2^X, V^-)$ is countably compact, then X is countably compact.*

Since a bD countably compact T_1-space is compact, by Theorem 34, we have:

Corollary 38. *If X is a countably compact T_1-space and $(2^X, V^-)$ is a bD-space, then X is compact.*

Corollary 39. *If $X = \omega_1$ with the order topology, then $(2^X, V^-)$ is not a bD-space.*

In what follows, we discuss some covering properties of $(\mathcal{C}(X), V^-)$ and $(\mathcal{C}(X), V^+)$.

Recall that a space X is *mesocompact* if for every open cover $\mathcal{U}$ of X there exists an open refinement $\mathcal{V}$ such that for every compact subset C of X, the set $\{V \in \mathcal{V} : V \cap C \neq \emptyset\}$ is finite [2].

Theorem 40. *If X is a mesocompact T_1-space, then $(\mathcal{C}(X), V^-)$ is a metacompact space.*

Proof. Let $\mathcal{U}$ be any open cover of $(\mathcal{C}(X), V^-)$. Since X is a T_1-space, $\{x\} \in \mathcal{C}(X)$ for each $x \in X$. For each $x \in X$, there exists some $U_x \in \mathcal{U}$ such that $\{x\} \in U_x$. Thus there exists an open neighborhood V_x of x in X such that $\{x\} \in V_x^- \subset U_x$. So $\{V_x : x \in X\}$ is an open cover of X. Since X is mesocompat, $\{V_x : x \in X\}$ has a compact-finite open refinement $\mathcal{V}$. If $\mathcal{V}^- = \{W^- : W \in \mathcal{V}\}$, then $\mathcal{V}^-$ is point-finite in $\mathcal{C}(X)$ and is an open refinement of $\mathcal{U}$. Thus $(\mathcal{C}(X), V^-)$ is a metacompact space. $\square$

Since every paracompact space is mesocompact, we have:

Corollary 41. *If X is a paracompact space, then $(\mathcal{C}(X), V^-)$ is metacompact.*

Theorem 42. *Let X_n be a T_1-space for each $n \in \mathbb{N}$. If $\prod_{n\in\mathbb{N}} (\mathcal{C}(X_n), V_n^+)$ is a Lindelöf space, then $\prod_{n\in\mathbb{N}} X_n$ is a Lindelöf space.*

Proof. Let $\mathcal{U}$ be any open cover of $\prod_{n\in\omega} X_n$. Let $x^\star$ be any element of $\prod_{n\in\mathbb{N}} \mathcal{C}(X_n)$. Then $x^\star = \prod_{n\in\mathbb{N}} A_n^{x^\star}$, where $A_n^{x^\star}$ is a closed compact subset of X_n for each $n \in \mathbb{N}$. So there exists a finite subfamily $\mathcal{U}_{x^\star}$ of $\mathcal{U}$ such that $x^\star \subset \bigcup \mathcal{U}_{x^\star}$. By [6, Theorem 3.2.10], there exist open sets $U_n^{x^\star} \subset X_n$ such that $U_n^{x^\star} \neq X_n$ for finitely many $n \in \mathbb{N}$ and $\prod_{n\in\mathbb{N}} A_n^{x^\star} \subset \prod_{n\in\mathbb{N}} U_n^{x^\star} \subset \bigcup \mathcal{U}_{x^\star}$. Thus $x^\star \in \prod_{n\in\mathbb{N}} (U_n^{x^\star})^+$. So $\{\prod_{n\in\mathbb{N}} (U_n^{x^\star})^+ : x^\star \in \prod_{n\in\mathbb{N}} \mathcal{C}(X_n)\}$ is an open cover of $\prod_{n\in\mathbb{N}} \mathcal{C}(X_n)$. Since $\prod_{n\in\mathbb{N}} \mathcal{C}(X_n)$ is Lindelöf, there exists some $x_i^\star \in \prod_{n\in\mathbb{N}} \mathcal{C}(X_n)$ for each $i \in \mathbb{N}$ such that $\prod_{n\in\mathbb{N}} \mathcal{C}(X_n) = \bigcup\{\prod_{n\in\mathbb{N}} (U_n^{x_i^\star})^+ : i \in \mathbb{N}\}$. For any $x \in \prod_{n\in\mathbb{N}} X_n$, we let $x = (x_n : n \in \mathbb{N})$. Define $x^\star = \prod_{n\in\mathbb{N}} \{x_n\}$. Thus $x^\star \in \prod_{n\in\mathbb{N}} \mathcal{C}(X_n)$. So there is some $i \in \mathbb{N}$ such that $x^\star \in \prod_{n\in\mathbb{N}} (U_n^{x_i^\star})^+$. Thus $x_n \in U_n^{x_i^\star}$ for each $n \in \mathbb{N}$ and hence $x \in \prod_{n\in\mathbb{N}} U_n^{x_i^\star} \subset \bigcup \mathcal{U}_{x_i^\star}$. If $\mathcal{V} = \bigcup\{\mathcal{U}_{x_i^\star} : i \in \mathbb{N}\}$, then $\mathcal{V} \subset \mathcal{U}$, $|\mathcal{V}| \leq \omega$ and $\prod_{n\in\mathbb{N}} X_n = \bigcup \mathcal{V}$. Thus $\prod_{n\in\mathbb{N}} X_n$ is a Lindelöf space. □

By Proposition 15, we have:

Theorem 43. *Let X_n be a Hausdorff space for each $n \in \mathbb{N}$. If $\prod_{n\in\mathbb{N}} (2^{X_n}, V^-)$ has property $\mathcal{P}$ and $\mathcal{P}$ is hereditary with respects to closed sets, then $\prod_{n\in\mathbb{N}} X_n$ has property $\mathcal{P}$.*

Proposition 44. *If X is a compact space, then $(\mathcal{C}(X), V^+)$ is compact.*

Proof. Let $\mathcal{T}$ be the topology of the space X. Then $\mathcal{B} = \{W^+ : W \in \mathcal{T} \setminus \{\emptyset\}\}$ is a base of $(\mathcal{C}(X), V^+)$. Let $\mathcal{W} \subset \mathcal{B}$ be any subfamily of $\mathcal{B}$ such that $\mathcal{C}(X) = \bigcup \mathcal{W}$. So for each $C \in \mathcal{C}(X)$, there exists some $\mathcal{V}_C \in \mathcal{W}$ such that $C \in \mathcal{V}_C$. Since X is compact, $X \in \mathcal{C}(X)$. So $X \in \mathcal{V}_X$. Since $\mathcal{V}_X \in \mathcal{B}$, there exists some $W \in \mathcal{T}$ such that $\mathcal{V}_X = W^+$. Since $X \in W^+$, the set $W = X$. Thus for each $C \in \mathcal{C}(X)$ the point $C \in W^+ = \mathcal{V}_X$. So $\mathcal{C}(X) = \mathcal{V}_X$. Thus $(\mathcal{C}(X), V^+)$ is compact. □

Corollary 45. *Let X be a T_1-space. If $Y \subset X$ is a closed compact subset of X, then $(\mathcal{C}(Y), V_Y^+) = (\mathcal{C}(Y), V^+|Y)$ is compact.*

Recall that a space X is *locally compact* if every point $x \in X$ has a neighborhood which is compact.

Theorem 46. *If X is a locally compact Hausdorff space, then $(\mathcal{C}(X), V^+)$ is locally compact.*

Proof. Let F be any element of $\mathcal{C}(X)$. Then F is a compact closed subset of X. Since X is locally compact, for each $x \in X$ there exists a compact neighborhood V_x of X. Thus there exist $n_F \in \mathbb{N}$ and $x_i \in F$ for each $i \le n_F$ such that $F \subset \bigcup\{V^\circ_{x_i} : i \le n_F\} = W \subset \bigcup\{V_{x_i} : i \le n_F\} = Y$. If $W^+ = \{C \in \mathcal{C}(X) : C \subset W\}$ and $Y^+ = \{C \in \mathcal{C}(X) : C \subset Y\}$, then $F \in W^+ \subset Y^+$. Since X is a Hausdorff space and Y is compact, the set Y is closed in X. So $Y^+ = \mathcal{C}(Y)$ is compact by Corollary 45. Thus $(\mathcal{C}(X), V^+)$ is locally compact. □

Theorem 47. *Let X be a T_1-space. Then X has a countable base if and only if $(2^X, V^-)$ has a countable base.*

Proof. Assume that X has a countable base $\mathcal{B}$. Let $\mathcal{B}^* = \{\bigcap_{i\le n} B_i^- : B_i \in \mathcal{B}, i \le n, n \in \mathbb{N}\}$. Then $|\mathcal{B}^*| \le \omega$. Let $F \in 2^X$ be any element of 2^X and let $\mathcal{V}$ be any open neighborhood of F in $(2^X, V^-)$.

There exist some $n \in \mathbb{N}$ and an open subset U_i for each $i \le n$ such that $F \in \bigcap_{i\le n} U_i^- \subset \mathcal{V}$. For each $i \le n$ there exists some $x_i \in F \cap U_i$. Since $\mathcal{B}$ is a base of X, for each $i \le n$ there exists some $B_i \in \mathcal{B}$ such that $x_i \in B_i \subset U_i$. Thus $F \in \bigcap_{i\le n} B_i^- \subset \bigcap_{i\le n} U_i^- \subset \mathcal{V}$ and $\bigcap_{i\le n} B_i^- \in \mathcal{B}^*$. So $\mathcal{B}^*$ is a countable base of $(2^X, V^-)$.

The inverse implication follows from Proposition 13. □

Theorem 48. *Let X be a T_1-space. Then X has a countable base if and only if $(\mathcal{C}(X), V^+)$ has a countable base.*

Proof. Let $\mathcal{B}$ be a countable base of X. If $\mathcal{B}^* = \{(\bigcup\mathcal{V})^+ : \mathcal{V} \subset \mathcal{B}$ and $|\mathcal{V}| < \omega\}$, then $\mathcal{B}^*$ is a countable family of open subsets of $(\mathcal{C}(X), V^+)$. Let F be any element of $\mathcal{C}(X)$. If $\mathcal{U}$ is an open neighborhood of F in $(\mathcal{C}(X), V^+)$, then there is an open subset U of X such that $F \in U^+ \subset \mathcal{U}$. Thus there exists a finite family $\mathcal{V} \subset \mathcal{B}$ such that $F \subset \bigcup\mathcal{V} \subset U$. So $F \in (\bigcup\mathcal{V})^+ \subset U^+ \subset \mathcal{U}$ and $(\bigcup\mathcal{V})^+ \in \mathcal{B}^*$. Thus $\mathcal{B}^*$ is a countable base of $(\mathcal{C}(X), V^+)$.

Now we prove the inverse implication. By Proposition 14, the space X is homeomorphic to a subspace of $(\mathcal{C}(X), V^+)$. Thus X has a countable base if $(C(X), V^+)$ has a countable base. □

Similar to Theorem 47, we have:

Proposition 49. *Let X be a T_1-space. Then X has a countable network if and only if $(2^X, V^-)$ has a countable network.*

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