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TOPOLOGY PROCEEDINGS

Volume 53, 2019

Pages 255-268

http://topology.nipissingu.ca/tp/

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by

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Electronically published on January 23, 2019

Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 53 (2019)
Pages 255-268

E-Published on January 23, 2019

ON H-CLOSED AND MINIMAL HAUSDORFF SPACES AND THE BOOLEAN PRIME IDEAL THEOREM

ELEFTHERIOS TACHTSIS

Abstract. In ZF (i.e., Zermelo–Fraenkel set theory without the Axiom of Choice (AC)), we establish that the *Boolean Prime Ideal Theorem* (BPI) is equivalent to each one of the following statements:

(1) A Hausdorff space is H-closed if and only if every open ultrafilter on the space converges;

(2) Products of H-closed Hausdorff spaces are H-closed;

(3) Products of minimal Hausdorff spaces are minimal;

(4) For every Hausdorff space X, the Katětov space κX is an H-closed extension of X;

(5) Every Hausdorff space has a (unique up to homeomorphism) projectively largest Katětov H-closed extension.

We also establish the following implications: BPI $\Rightarrow$ "products of non-empty H-closed Hausdorff spaces are non-empty" $\Rightarrow$ "products of non-empty minimal Hausdorff spaces are non-empty" $\Rightarrow$ $\mathsf{AC_{fin}}$ (i.e., "every family of non-empty finite sets has a choice function").

1. Introduction

An *extension* of a topological space X is a space which contains X as a dense subspace. The construction of extensions such as compactifications, realcompactifications and H-closed extensions has been an area of intense research in general topology for a long time. For a systematic and deep study of extensions (and absolutes) of Hausdorff spaces the reader is referred to the book of Porter and Woods [20].

2010 *Mathematics Subject Classification.* Primary 03E25; Secondary 54B10, 54D10, 54D30, 54D35.

Key words and phrases. Axiom of Choice, Boolean Prime Ideal Theorem, compact Hausdorff space, minimal Hausdorff space, H-closed Hausdorff space, extension of a Hausdorff space, product space, open filter, open ultrafilter, adherence of an open filter, Katětov's H-closed extension.

Let X be a Hausdorff space. X is called *Hausdorff-closed* or simply H-*closed* if X is closed in every Hausdorff space in which X can be embedded. The notion of H-closed space was introduced in 1924 by Alexandroff and Urysohn [1]. In [1], it was shown that a Hausdorff space is H-closed iff every open cover has a finite subfamily with dense union; an H-closed Hausdorff space is compact iff it is regular; there exists (in ZF) an H-closed, Urysohn space (i.e., any two distinct elements of the underlying set have respectively two open neighborhoods whose closures are disjoint) which is not compact.

A Hausdorff space $(X, \mathcal{T})$ is called *minimal* if there is no Hausdorff topology $\mathcal{T}'$ on X such that $\mathcal{T}' \subsetneq \mathcal{T}$; in other words, X is minimal if $\mathcal{T}$ is a minimal element of the lattice $\mathrm{Haus}(X)$ of all Hausdorff topologies on X partially ordered by $\subseteq$. (Note that if X is finite, then $\mathrm{Haus}(X)$ is a singleton, and thus $\mathrm{Haus}(X)$ has a minimum element; hence the notion of minimal Hausdorff space becomes interesting when X is infinite. Furthermore, since compact Hausdorff spaces are minimal, and there are compact Hausdorff topologies on any set X, it follows that $\mathrm{Haus}(X)$ always has minimal elements.) It was a question of Alexandroff and Urysohn in [1] that triggered off the study of minimal Hausdorff spaces, namely whether a Hausdorff space has an H-closed extension. The answer to the above question was shown to be in the affirmative by Stone in 1937, Katětov in 1940, Fomin in 1941, A. D. Alexandroff in 1942 and Šanin in 1943 (detailed historical notes and references can be found by the interested reader in [3] and [20]). It should be noted here that it was Tychonoff [23] in 1930 who first gave a partial answer to Alexandroff and Urysohn's inquiry, namely that every Hausdorff space can be embedded in an H-closed space. In 1992, Porter [18] proved that Tychonoff was close in providing a complete answer; in particular, Porter established that the closure of Tychonoff's embedding is H-closed.

Katětov [13] showed that there is a strong connection between minimal Hausdorff spaces and H-closed Hausdorff spaces. In particular, he showed that a minimal Hausdorff space is H-closed, and that a Hausdorff space is minimal iff it is H-closed and semiregular (where a Hausdorff space $(X, \mathcal{T})$ is semiregular if the complete Boolean algebra of the regular open sets of X is a base for $\mathcal{T}$).

Katětov's H-closed extension space of a given Hausdorff space has properties similar to the *Stone–Čech compactification* of a Tychonoff space, i.e., T_1 and completely regular space (we will explicitly mention those properties in the sequel – see Proposition 3.7 and Theorem 3.8 of Section 3). Furthermore, we recall that BPI (the Boolean Prime Ideal Theorem) is equivalent to the Stone-Čech compactification theorem for a Tychonoff space (see Form [14L] in [10]), and is also equivalent to *the Tychonoff* *product theorem restricted to compact Hausdorff spaces* (i.e., "Products of compact Hausdorff spaces are compact", and Form [14J] in [10]) – the latter two results were established by Rubin and Scott [22] in 1954, whereas the above restricted product theorem was in fact proved earlier by Łoś and Ryll-Nardzewski [17] in 1951; see also Howard and Rubin [10] for a list of equivalents of BPI. In the realm of H-closed and minimal Hausdorff spaces, it is known that *products of H-closed (minimal) Hausdorff spaces are H-closed (resp. minimal)*; see [6] (resp. [11]). *The above facts provide a first implicit suggestion that* BPI *may possess a prominent role in the theory of H-closed and minimal Hausdorff spaces.*

A second fact which strongly points at this direction is, on one hand, the (ZFC, i.e., ZF + AC) result by Bourbaki [5] in 1961 which characterizes H-closed Hausdorff spaces, namely that a Hausdorff space is H-closed iff every open ultrafilter on the space converges, and on the other hand, the fairly recent result by Rhineghost [21] in 2002 that BPI is equivalent to each one of "*For each non-empty topological space, its lattice of open sets contains an ultrafilter*" and "*For topological spaces, each open filter can be extended to an open ultrafilter*". The latter two equivalences have been established independently by Keremedis and Tachtsis [16], and Zisis [24]. (We would like to point out here that each one of "For each non-empty topological space, its lattice of closed sets contains an ultrafilter" and "For topological spaces, each closed filter can be extended to a closed ultrafilter" is *equivalent to the full* AC *in* ZFA set theory, i.e., in ZF with the Axiom of Extensionality modified in order to allow the existence of atoms. The first equivalence was established by Herrlich [8] and the second one by Keremedis and Tachtsis [15]. The question of whether the above principle restricted to the class of T_1 spaces or Hausdorff spaces is equivalent to AC is still an *open problem.*)

It is exactly the *chief purpose* of this note to justify the above implicit suggestions and in particular to prove that BPI is equivalent to some of the most fundamental results in the area of H-closed and minimal Hausdorff spaces. We would like to stress the fact that the axiomatic system in which most of the theory of H-closed and minimal Hausdorff spaces has been developed is ZFC, and to the best of our knowledge, *there is no published work in the literature which discusses the set-theoretic strength of results of this area in comparison with* AC *or weak forms of* AC. (We also note that there is no mention of results related to H-closed and minimal Hausdorff spaces in the list of forms (i.e., of consequences of AC) of the encyclopedic book of Howard and Rubin [10].) We thus consider it *important* that the relative strength of fundamental results of this area of general topology be clarified since this will provide a deeper insight and understanding of the theory as well as a comprehension of its possible limitations.

The study in this note is motivated by the above considerations and one of its goals is to initiate the investigation in this important part of general topology from the set-theoretic viewpoint.

2. Terminology and known results in ZF

All topological spaces considered in this paper are assumed to be Hausdorff unless it is explicitly stated otherwise. Therefore, the word "space" shall henceforth mean "Hausdorff topological space". In only a couple of cases, we shall remind the reader of this assumption for emphasis.

Definition 2.1. Let $(X, \mathcal{T})$ be a space. (Often, the short term "X" shall be used for $(X, \mathcal{T})$.)

(1) A space Y is called an *extension* of X if X is a dense subspace of Y. (Equivalently, Y is an extension of X if X can be densely embedded in Y.)

(2) X is called *compact* if every open cover of X has a finite subcover.

(3) X is called H-*closed* if X is closed in every Hausdorff space in which X can be embedded.

(4) X is called *minimal* if there is no Hausdorff topology $\mathcal{T}'$ on X such that $\mathcal{T}' \subsetneq \mathcal{T}$.

(5) X is called *semiregular* if the regular open sets of X form a base for the topology $\mathcal{T}$ on X.

Definition 2.2. Let $(X, \mathcal{T})$ be any (not necessarily Hausdorff) space.

(1) A subset $\mathcal{F} \subseteq \mathcal{T}$ is called an *open filter* on X if $\mathcal{F}$ satisfies:

(a) $\mathcal{F} \neq \emptyset$, and for all $F \in \mathcal{F}$, $F \neq \emptyset$;

(b) if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$;

(c) if $F_1 \in \mathcal{F}$ and $F_1 \subseteq F_2 \in \mathcal{T}$, then $F_2 \in \mathcal{F}$.

(2) An open filter $\mathcal{F}$ on X is called an *open ultrafilter* on X if $\mathcal{F}$ is a maximal element in the set of all open filters on X when partially ordered by inclusion.

(3) Let $\mathcal{F}$ be an open filter on X. The set $\bigcap\{\mathrm{cl}_X(F) : F \in \mathcal{F}\}$ (where $\mathrm{cl}_X(F)$ is the closure of F in X) is called the *adherence* of $\mathcal{F}$ and it is denoted by $a(\mathcal{F})$. Every element of $a(\mathcal{F})$ is called an *adherent point* (or a *cluster point*) of $\mathcal{F}$. If $a(\mathcal{F}) \neq \emptyset$, then the open filter $\mathcal{F}$ is called *fixed*; otherwise, $\mathcal{F}$ is called *free*. The open filter $\mathcal{F}$ is said to *converge* to a point $x \in X$ if every open neighborhood of x belongs to $\mathcal{F}$.

(4) An open filter $\mathcal{F}$ on X is called *prime* if whenever $A \in \mathcal{T}$ and $B \in \mathcal{T}$ are such that $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

(5) If X is a discrete space, i.e., $\mathcal{T} = \wp(X)$ (the power set of X), then an open filter on X is called a *filter* on X and an open ultrafilter on X is

called an *ultrafilter* on X. Note that in this case (that is, $\mathcal{T} = \wp(X)$), a filter on X is an ultrafilter iff it is prime.

Definition 2.3. The *Boolean Prime Ideal Theorem* (BPI, and Form 14 in [10]) is the principle: Every non-trivial Boolean algebra has a prime ideal.

$\mathsf{AC}_{\mathsf{fin}}$ (Form 62 in [10]) is the principle: Every family of non-empty finite sets has a choice function.

The following result is well-known (see for example Herrlich [9, Lemma 4.35]).

Proposition 2.4. (ZF) *Let $(X, \mathcal{T})$ be any (not necessarily Hausdorff) space and let $\mathcal{F}$ be an open filter on X. Then $\mathcal{F}$ is an open ultrafilter on X iff for every $O \in \mathcal{T}$, either $O \in \mathcal{F}$ or $\mathrm{int}_X(X \setminus O) \in \mathcal{F}$.*

(For a subset $Y \subseteq X$, $\mathrm{int}_X(Y)$ denotes the interior of Y in X.)

For clarity and the reader's convenience and complete understanding, we shall list some known central results whose proofs are *choice free*, and most of which will play a key role in the establishment of the main results. Their ZF-proofs can be found by the interested reader in Porter and Woods [20]. Some bibliographic references shall be attached to the subsequent results, but the totality of references can be located in [3] and [20].

Theorem 2.5. (ZF) *A compact space is minimal, and a minimal space is H-closed. None of the converses is in general true.*

Theorem 2.6. (ZF) *Let X be a space. Then the following are equivalent:*

(1) X is H-closed;

(2) ([1]) Every open cover of X has a finite subfamily whose union is dense in X;

(3) ([4], [6]) Every open filter on X has non-empty adherence.

The subsequent result follows easily from Theorem 2.6 and the observation that *in* ZF*, every compact space is normal* (and recall that by "space" we mean a Hausdorff space), and thus regular (see [10] for the latter observation).

Theorem 2.7. (ZF) *The following hold:*

(1) An H-closed space is compact iff it is regular.

(2) A continuous image of an H-closed space is H-closed.

(3) Every open ultrafilter on an H-closed space converges.

(4) If X is H-closed and $W \subseteq X$ is open, then $\mathrm{cl}_X(W)$ is H-closed.

(5) If a product of spaces is H-closed, then each coordinate space is H-closed.

Theorem 2.8. (ZF) *The following hold:*

(1) ([4]) A space is minimal iff every open filter on the space with a unique adherent point converges to this point.

(2) ([13]) A space is minimal iff it is H-closed and semiregular.

(3) ([2]) If a product of spaces is minimal, then each coordinate space is minimal.

Theorem 2.9. *([9], [12], [16], [17], [21], [22], [24]) In* ZF*, the following are equivalent:*

(1) BPI*;*

(2) Every filter on an infinite set can be extended to an ultrafilter;

(3) Products of compact spaces are compact;

(4) Every non-empty (and not necessarily Hausdorff) space has an open ultrafilter;

(5) Every open filter on a space can be extended to an open ultrafilter.

Proof. (*Sketch of* (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).)

(2) $\Rightarrow$ (4) Let $(X, \mathcal{T})$ be a non-empty (not necessarily Hausdorff) space. Let $\mathcal{F} = \{U \in \mathcal{T} : \mathrm{cl}_X(U) = X\}$; then $\mathcal{F}$ is an open filter on X. It follows that $\mathcal{G} = \{A : A \in \wp(X) \text{ and } A \supseteq F \text{ for some } F \in \mathcal{F}\}$ is a filter on X. By (2), there exists an ultrafilter $\mathcal{H}$ on X which extends $\mathcal{F}$. It is easy to see that $\mathcal{U} = \mathcal{H} \cap \mathcal{T}$ is a prime open filter on X which extends $\mathcal{F}$. Since for any $W \in \mathcal{T}$, we have $W \cup \mathrm{int}_X(X \setminus W)$ is dense in X, it follows that $W \cup \mathrm{int}_X(X \setminus W)$ belongs to $\mathcal{F}$, and hence to $\mathcal{U}$. Since $\mathcal{U}$ is prime, we conclude that for every $W \in \mathcal{T}$, either $W \in \mathcal{U}$ or $\mathrm{int}_X(X \setminus W) \in \mathcal{U}$. Thus, $\mathcal{U}$ is an open ultrafilter on X.

(4) $\Rightarrow$ (5) Let $(X, \mathcal{T})$ be a space and also let $\mathcal{F}$ be an open filter on X. Let $Y = \{\mathcal{G} : \mathcal{G} \text{ is an open filter on } X \text{ which extends } \mathcal{F}\}$. Let $\mathcal{T}' = \{W : W \in \wp(Y) \text{ such that for every } \mathcal{U} \in W \text{ and for every } \mathcal{V} \in Y, \text{ if } \mathcal{U} \subseteq \mathcal{V}, \text{ then } \mathcal{V} \in W\}$. Then $\mathcal{T}'$ is a topology on Y; hence by (4), there exists an open ultrafilter $\mathcal{M}$ on Y. For every $O \in \mathcal{T}$, let $U(O) = \{\mathcal{W} \in Y : O \in \mathcal{W}\}$. Let $\mathcal{H} = \{O : O \in \mathcal{T}, U(O) \in \mathcal{M}\}$. Then $\mathcal{H}$ is an open ultrafilter on X, which extends $\mathcal{F}$.

(5) $\Rightarrow$ (2) Apply (5) to any infinite discrete space. □

A well-known ZF-result, which shall be useful for the proof of the forthcoming Theorem 3.1 of Section 3, is the product invariance of semiregularity.

Proposition 2.10. (ZF) *Products of semiregular spaces are semiregular.*

Proof. Let X be a product of semiregular spaces X_i ($i \in I$), and let $V = \bigcap_{j \in J} \pi_j^{-1}[V_j]$ be a basic neighborhood of a point $x \in X$, with V_j regular open neighborhoods of x_j in X_j ($j \in J \subseteq I$, J finite). Then V is regular open, too, by the rules for interior and closure operators. Hence, X is semiregular. □

Concluding this section, we would like to draw the reader's attention to a related recent paper by Erné [7] (though the results therein are not needed explicitly in the present conclusions) on the equivalence of certain lattice-theoretical and topological statements to BPI; furthermore, the rich source of references in [7] is an excellent guide to the interested reader towards a deep study of the role of BPI and other choice principles in set theory, topology, lattice theory, and category theory.

3. Main results

Our first result in this section is the equivalence of BPI to each of "a space is H-closed iff every open ultrafilter on the space converges", "products of H-closed spaces are H-closed", and "products of minimal spaces are minimal".

The statement "products of H-closed spaces are H-closed" was established in 1941 by Chevalley and Frink [6], who initially presented another proof of the Tychonoff product theorem for arbitrary (not necessarily Hausdorff) compact spaces (which is equivalent to the full AC in ZFA as shown by Kelley [14]) – which is the one that is commonly used – and then applied the same method to prove their result on products of H-closed spaces by simply replacing *sets* by *open sets* throughout their proof. It should be pointed out that in [6], the full power of AC is used in the disguise of Zorn's Lemma (or equivalently of the Teichmüller–Tukey Lemma—for the equivalence of AC to each of those lemmas, see for example Herrlich [9, Theorem 2.2]).

Furthermore, the authors in [6] establish that a space is H-closed iff every open filter on the space has non-empty adherence (Theorem 2.6 of Section 2). However, their proof uses Zorn's Lemma; in fact, by Theorem 2.9, the proof employs BPI in its equivalent form "every open filter on a space can be extended to an open ultrafilter". The use of AC, or of BPI, in the particular proof is *redundant*, as the reader may easily verify. (We also note that, in [6], the term "absolutely closed" is used for H-closed.)

Theorem 3.1. *The following are equivalent:*

(1) BPI*;*
(2) A space is H-closed iff every open ultrafilter on the space converges;
(3) Products of H-closed spaces are H-closed;
(4) Products of minimal spaces are minimal;
(5) Products of compact spaces are compact.

Proof. (1) $\Rightarrow$ (2) By Theorem 2.7(3), every open ultrafilter on an H-closed space converges (without invoking any choice principle). Conversely, let X be a space on which every open ultrafilter converges. By BPI, any open filter $\mathcal{F}$ on X extends to an open ultrafilter (see Theorem 2.9), which converges to some point x, whence $x \in a(\mathcal{F})$. Thus, X is H-closed (see Theorem 2.6).

(2) ⇒ (3) Let X be a product of H-closed spaces $(X_i, \mathcal{T}_i)$ $(i \in I)$, and let $\mathcal{F}$ be an open ultrafilter on X. Then for each $i \in I$, $\mathcal{F}_i = \{\pi_i[F] : F \in \mathcal{F}\}$ (where π_i is the canonical projection of X onto X_i) is an open ultrafilter on X_i, which converges to a unique point x_i. (Since $\mathcal{F}$ is an open filter, and for each $i \in I$, π_i is open, continuous and onto, it easily follows that for each $i \in I$, $\mathcal{F}_i$ is an open filter on X_i and $\mathcal{F}_i = \{F : F \in \mathcal{T}_i, \pi_i^{-1}[F] \in \mathcal{F}\}$. Now, for any $i \in I$, let $O \in \mathcal{T}_i$ such that $\mathcal{F}_i \cup \{O\}$ has the finite intersection property; then $\pi_i^{-1}[O] \cap F \neq \emptyset$ for all $F \in \mathcal{F}$, and thus $\pi_i^{-1}[O] \in \mathcal{F}$ since $\mathcal{F}$ is an open ultrafilter. Hence, $O = \pi_i[\pi_i^{-1}[O]] \in \mathcal{F}_i$, and thus $\mathcal{F}_i$ is an open ultrafilter on X_i.)

Then $\mathcal{F}$ converges to $x = (x_i)_{i \in I}$. (Let $V = \pi_{i_1}^{-1}[V_{i_1}] \cap \pi_{i_2}^{-1}[V_{i_2}] \cap \cdots \cap \pi_{i_n}^{-1}[V_{i_n}]$ be a basic neighborhood of x, with $V_{i_j} \in \mathcal{T}_{i_j}$ for all $j \in \{1, \ldots, n\}$. Since for each $j \in \{1, \ldots, n\}$, $\mathcal{F}_{i_j}$ converges to x_{i_j}, we have $V_{i_j} \in \mathcal{F}_{i_j}$ for all $j \in \{1, \ldots, n\}$; hence $\pi_{i_j}^{-1}[V_{i_j}] \in \mathcal{F}$ for all $j \in \{1, \ldots, n\}$, and thus $V \in \mathcal{F}$ since $\mathcal{F}$ is an open filter.) By (2), this entails that X is H-closed.

(3) ⇒ (4) This follows from Theorem 2.8(2) and Proposition 2.10.

(4) ⇒ (5) This follows from Theorems 2.5, 2.7(1), and the product invariance of regularity.

(5) ⇒ (1) This was shown by Łoś and Ryll-Nardzewski [17], and by Rubin and Scott [22]. □

We recall here that the statement "products of minimal spaces are minimal" has been established by Ikenaga [11]. Our corresponding proof above is *considerably simpler* than the original one in [11]. The proof in [11] is based on the ZF-result that a space is minimal iff every open filter on the space with a unique adherent point converges to this point (see Theorem 2.8(1)), but *inevitably* uses BPI (as established in Theorem 3.1) in its equivalent form "products of H-closed spaces are H-closed"; we refer the interested reader to [11] in order to verify this.

Two statements which are relevant to the context of this paper, are the following ones:

(a) products of non-empty H-closed spaces are non-empty, and

(b) products of non-empty minimal spaces are non-empty.

(Note that, by Theorem 2.5 (or by Theorem 2.8(2)), (a) ⇒ (b).) We show next that (a) and (b) lie in strength between BPI and $\mathsf{AC}_{\mathsf{fin}}$.

Theorem 3.2. *Each of the following statements implies the one beneath it:*

(1) BPI*;*
(2) Products of non-empty H-closed spaces are non-empty;
(3) Products of non-empty minimal spaces are non-empty;

(4) Products of non-empty compact spaces are non-empty;
(5) $\mathsf{AC}_{\mathsf{fin}}$.

(We note that (4) is Form 343 in [10], and that (4) does not imply BPI in ZF, see [10]; it is an *open problem* whether or not (5) implies (4).)

Proof. (1) $\Rightarrow$ (2) Let $(X_i)_{i\in I}$ be a family of non-empty H-closed spaces. For each $i \in I$, add a new element ∞ to X_i, and consider

$$Y_i = X_i \cup \{\infty\}$$

as the disjoint union topological space. Then for each $i \in I$, Y_i is Hausdorff (since X_i is Hausdorff and ∞ is an isolated point of Y_i) and H-closed. Indeed, fix an $i \in I$, and let $\mathcal{F}$ be an open filter on Y_i. We will show that $a(\mathcal{F}) \neq \emptyset$. If $\infty \in a(\mathcal{F})$, then the conclusion is straightforward, so we assume without loss of generality that $\infty \notin a(\mathcal{F})$; then $\{\infty\} \notin \mathcal{F}$. Let $\mathcal{G} = \{F \cap X_i : F \in \mathcal{F}\}$. Note that $G \neq \emptyset$ for all $G \in \mathcal{G}$. Furthermore, $\mathcal{G}$ is an open filter on X_i, and thus has an adherent point $x \in X_i$. Clearly x is an adherent point of $\mathcal{F}$ in Y_i, and thus Y_i is H-closed (Theorem 2.6).

By Theorem 3.1, the product space $Y = \prod_{i\in I} Y_i$ is H-closed. Let $\mathcal{G}$ be the open filter on Y, which is generated by the family $\mathcal{H} = \{\pi_i^{-1}[X_i] : i \in I\}$ (which consists of non-empty open sets in Y, and has the finite intersection property—since $X_i \neq \emptyset$ for all $i \in I$), i.e., $\mathcal{G} = \{G : G$ is open in Y and $G \supseteq \bigcap \mathcal{Q}$ for some non-empty finite subset $\mathcal{Q}$ of $\mathcal{H}\}$. Since Y is H-closed, it follows (by Theorem 2.6) that $\mathcal{G}$ has an adherent point $y \in Y$. Then $y(i) \in X_i$ for all $i \in I$. (If for some $i_0 \in I$, $y(i_0) = \infty$, then the open neighborhood $\pi_{i_0}^{-1}[\{\infty\}]$ of y does not meet $\pi_{i_0}^{-1}[X_{i_0}]$, which belongs to $\mathcal{H} \subseteq \mathcal{G}$; this contradicts the fact that $y \in a(\mathcal{G})$.) Therefore, $\prod_{i\in I} X_i \neq \emptyset$ as required.

(2) $\Rightarrow$ (3) This follows from Theorem 2.5 (or Theorem 2.8(2)).

(3) $\Rightarrow$ (4) This follows from Theorem 2.5.

(4) $\Rightarrow$ (5) This is well-known (see [9, Exercise E 9, p. 94], or [10]). $\square$

Except for "BPI $\Rightarrow$ products of non-empty compact spaces are non-empty", which is known to be non-reversible in ZF (see [10]), we *do not know* whether any of the other implications of Theorem 3.2 is reversible in ZF, or in ZFA.

Now we turn our attention to the Katětov H-closed extension of a space. Firstly, we need the following definition due to Katětov [13].

Definition 3.3. Let $(X, \mathcal{T})$ be a space and let

$$\kappa X = X \cup \{\mathcal{F} : \mathcal{F} \text{ is a free open ultrafilter on } X\}.$$

(We note that $X \cup \{\mathcal{F} : \mathcal{F}$ is a free open ultrafilter on $X\}$ is a *disjoint union*. Also, we recall that an open filter $\mathcal{F}$ is free if $a(\mathcal{F}) = \emptyset$, i.e., if $\mathcal{F}$ has no adherent points.) We let

$$\mathcal{B}_X = \mathcal{T} \cup \{W \cup \{\mathcal{F}\} : W \in \mathcal{F}, \mathcal{F} \in \kappa X \setminus X\}.$$

Proposition 3.4. (ZF) *Let* $(X, \mathcal{T}), \kappa X$, *and* $\mathcal{B}_X$ *be as in Definition 3.3. Then* $\mathcal{B}_X$ *is a base for a Hausdorff topology* $\mathcal{Q}_X$ *on* κX, *and* X *is open in* κX.

Proof. We only show that $(\kappa X, \mathcal{Q}_X)$ is Hausdorff. Let $x, y \in \kappa X$ with $x \neq y$. If $x, y \in X$, then since X is Hausdorff and $\mathcal{T} \subseteq \mathcal{Q}_X$, there exist two disjoint $\mathcal{Q}_X$-open neighborhoods of x and y, respectively.

If $x \in X$ and $y = \mathcal{F} \in \kappa X \setminus X$, then since $a(\mathcal{F}) = \emptyset$, it follows that x is not an adherent point of $\mathcal{F}$; hence there exist $R, S \in \mathcal{T}$ such that $x \in R$, $S \in \mathcal{F}$ and $R \cap S = \emptyset$. Then R and $S \cup \{\mathcal{F}\}$ are disjoint $\mathcal{Q}_X$-open neighborhoods of x and y, respectively. Similarly, one finds disjoint $\mathcal{Q}_X$-open neighborhoods of x and y, in case $x \in \kappa X \setminus X$ and $y \in X$.

If $x = \mathcal{U} \in \kappa X \setminus X$ and $y = \mathcal{V} \in \kappa X \setminus X$, then since $\mathcal{U} \neq \mathcal{V}$ and $\mathcal{U}, \mathcal{V}$ are open ultrafilters on X, it follows that $\mathcal{U} \nsubseteq \mathcal{V}$, and thus we may let $U \in \mathcal{U} \setminus \mathcal{V}$. Then there exists an element $V \in \mathcal{V}$ such that $U \cap V = \emptyset$. (Otherwise, $\mathcal{V} \cup \{U\}$ has the finite intersection property, and thus $\mathcal{V}$ could be properly extended to an open filter on X, contrary to $\mathcal{V}$'s being an open ultrafilter on X.) It follows that $U \cup \{\mathcal{U}\}$ and $V \cup \{\mathcal{V}\}$ are disjoint $\mathcal{Q}_X$-open neighborhoods of x and y, respectively. Hence, κX is Hausdorff as required. □

Given a space X, the space $(\kappa X, \mathcal{Q}_X)$ is called *Katětov extension* of X. Theorem 3.5 below justifies the term "extension" (this does not require any choice form); furthermore, κX is H-closed – assuming that BPI is true. The result about κX was established by Katětov [13], with the full power of AC (in the disguise of Zorn's Lemma) being used in the proof (for the existence of free open ultrafilters on a given non-H-closed space X); see also 1.16 ("Every space can be densely embedded in an H-closed space") of Porter and Stephenson [19].

Theorem 3.5. *The following are equivalent:*

(1) BPI;

(2) For every non-empty space X, κX *is an H-closed extension of* X.

Proof. (1) $\Rightarrow$ (2) Let $(X, \mathcal{T})$ be a space (with $X \neq \emptyset$), and also let $(\kappa X, \mathcal{Q}_X)$ be the space (by virtue of Proposition 3.4) associated with X. If X is H-closed, then $\kappa X = X$ (since, by Theorem 2.6, every open filter on X has non-empty adherence), and thus κX is H-closed. So we assume that X is not H-closed, which means that there exists an open filter $\mathcal{F}$ on X such that $a(\mathcal{F}) = \emptyset$. By BPI, there is an open ultrafilter $\mathcal{G}$ on X such that $\mathcal{F} \subseteq \mathcal{G}$ (recall Theorem 2.9). Since $a(\mathcal{F}) = \emptyset$, it follows that $a(\mathcal{G}) = \emptyset$; hence $\kappa X \setminus X \neq \emptyset$.

X is dense in κX. Indeed, let $O \in \mathcal{B}_X \setminus \{\emptyset\}$. If $O \in \mathcal{T}$, then $O \cap X = O \neq \emptyset$. Assume that $O = W \cup \{\mathcal{F}\}$ for some $\mathcal{F} \in \kappa X \setminus X$ and $W \in \mathcal{F}$. Then $X \cap (W \cup \{\mathcal{F}\}) = W \neq \emptyset$ (since $\mathcal{F}$ is an open filter, it follows that none of its elements are empty).

κX is H-closed. By Theorem 3.1, it suffices to show that every open ultrafilter on κX converges. Let $\mathcal{H}$ be an open ultrafilter on κX. By way of contradiction, assume that $\mathcal{H}$ is not convergent in κX; hence $\mathcal{H}$ is free. Since $X \in \mathcal{Q}_X$ and X is dense in κX, it follows that X meets every element of $\mathcal{H}$ non-trivially, and since $\mathcal{H}$ is an open ultrafilter, we have $X \in \mathcal{H}$. Let $\mathcal{R}$ be the trace of $\mathcal{H}$ on X, that is, $\mathcal{R} = \{Z \cap X : Z \in \mathcal{H}\}$. It is fairly straightforward to see that $\mathcal{R}$ is an open ultrafilter on X and $\mathcal{R} \subseteq \mathcal{H}$. Furthermore, since $\mathcal{H}$ is free, so is $\mathcal{R}$; hence $\mathcal{R} \in \kappa X \setminus X$. Now for every $R \in \mathcal{R}$, we have $R \subseteq R \cup \{\mathcal{R}\}$, and thus $R \cup \{\mathcal{R}\} \in \mathcal{H}$, for all $R \in \mathcal{R}$. Hence $\mathcal{H}$ converges to $\mathcal{R}$, contrary to $\mathcal{H}$'s being free. Consequently, κX is H-closed.

(2) $\Rightarrow$ (1) By Theorem 3.1, it suffices to show that if every open ultrafilter on a space X converges, then X is H-closed. (Recall that "H-closed $\Rightarrow$ every open ultrafilter converges" is provable in ZF.) To this end, let X be a space such that every open ultrafilter on X converges. Then $\kappa X = X$, and hence (by (2)) X is H-closed. $\square$

Definition 3.6. Let X be a space. Two extensions Y_1 and Y_2 of X are called *equivalent* if there is a topological homeomorphism $f : Y_1 \to Y_2$ such that $f \upharpoonright X = \mathrm{id}_X$ (the identity function on X). In this case, we write $Y_1 \equiv_X Y_2$. (It is easy to see that the binary relation $\equiv_X$ on the proper class of extensions of X is an equivalence relation.)

If Y, Z are two extensions of X, then Y is said to be *projectively larger than* Z, denoted by $Y \geq Z$ (or $Z \leq Y$), if there is a continuous function $f : Y \to Z$ such that $f \upharpoonright X = \mathrm{id}_X$. (The binary relation $\leq$ is a partial order on the quotient class $E(X) = \{[Y]_{\equiv_X} : Y \text{ is an extension of } X\}$; see Proposition 4.1(g) of [20].)

Let $\mathcal{Q}$ be a non-empty set of extensions of X. An extension Y of X is a *projective maximum* in $\mathcal{Q}$ if $Y \in \mathcal{Q}$ and $Y \geq Z$ for all $Z \in \mathcal{Q}$.

For the proof of the subsequent proposition, the reader is referred to [20, Theorem 4.8(n), Proposition 4.8(p)]. We note that the use of BPI in that proof is exactly as in the proof of '(1) $\Rightarrow$ (2)' of Theorem 3.5, i.e., in the assumption that $\kappa X \setminus X \neq \emptyset$ if X is not H-closed, and in this case, κX is H-closed.

Proposition 3.7. *Assume that* BPI *is true. Let X be a space. Then the following hold:*

(1) If Y is an H-closed extension of X, then there is a unique continuous function $f : \kappa X \to Y$ such that $f \restriction X = \mathrm{id}_X$, i.e., $\kappa X \geq Y$.

(2) If Z is an H-closed extension of X and $Z \geq Y$ for all H-closed extensions Y of X, then $\kappa X \equiv_X Z$; in particular, κX is a projective maximum in the set of all H-closed extensions of X, which is unique up to homeomorphism.

(3) If K is a compact space and $f : X \to K$ is a continuous function, then there exists a unique continuous function $F : \kappa X \to K$ such that $F \restriction X = f$.

Theorem 3.8. *The following are equivalent:*

(1) BPI*;*

(2) Every space X has a (unique up to homeomorphism) projectively largest H-closed extension κX which satisfies (3) of Proposition 3.7.

Proof. (1) $\Rightarrow$ (2) This follows from Theorem 3.5 and Proposition 3.7.

(2) $\Rightarrow$ (1) We will prove that products of compact spaces are compact (see Theorem 3.1). To this end, let X be a product of compact spaces X_i ($i \in I$). We assume that $X_i \neq \emptyset$ for all $i \in I$, and also that $X \neq \emptyset$ (otherwise the conclusion is straightforward). Clearly, X is regular (being a product of compact (Hausdorff), and hence regular, spaces).

By (2), X has an H-closed extension κX, which satisfies (3) of Proposition 3.7. For each $i \in I$, let π_i be the canonical projection of X onto X_i. By Proposition 3.7(3), let $E_{\pi_i} : \kappa X \to X_i$ be the unique continuous extension of (the continuous function) π_i. We define a function $\Phi : \kappa X \to X$ by requiring

$$\Phi(x)(i) = E_{\pi_i}(x), \text{ for all } x \in \kappa X \text{ and } i \in I.$$

Φ is continuous since for each $i \in I$, the coordinate function E_{π_i} is continuous. Furthermore, Φ is onto. Indeed, let $x = (x_i)_{i \in I} \in X$ (recall that $X \neq \emptyset$). Since $X \subseteq \kappa X$, we have $x \in \kappa X$, so for each $i \in I$, we have

$$\Phi(x)(i) = E_{\pi_i}(x) = \pi_i(x) = x_i,$$

and thus $\Phi(x) = x$. Hence X is a continuous image of the H-closed space κX; thus, by Theorem 2.7(2), X is H-closed. Since X is regular and H-closed, we conclude (by Theorem 2.7(1)) that X is compact as required. $\square$

Acknowledgement. We are deeply grateful to the anonymous referee for the very thorough review work, whose valuable comments and suggestions enhanced both the mathematical quality and the overall exposition of our paper. We are especially thankful to the referee for his/her suggestions on the original proof of Theorem 3.1, which considerably simplified it and elevated the quality of exposition.

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