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by

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INSERTION THEOREMS FOR SOME SPACES BY MAPS TO ORDERED TOPOLOGICAL VECTOR SPACES

ER-GUANG YANG

Abstract. In this paper, we present characterizations of some spaces such as stratifiable spaces, perfectly normal spaces, k-semi-stratifiable spaces with maps to ordered topological vector spaces. The results obtained generalize real valued functions in some known results to maps to ordered topological vector spaces. Some of them improve the corresponding results in [9].

1. Introduction

Real-valued functions are closely related to the characterizations of topological spaces. It turned out that many classes of topological spaces such as normal spaces, monotonically normal spaces, stratifiable spaces, monotonically countably paracompact spaces can be characterized with real-valued functions satisfying certain conditions. For example.

Theorem 1.1. *For a space X, the following are equivalent.*

(a) *X is stratifiable.*

(b) [11] *There exists an operator ϕ assigning to each lower semi-continuous function $h : X \to [0, \infty)$, a continuous function $\phi(h) : X \to [0, \infty)$ with $\phi(h) \leq h$ such that $0 < \phi(h)(x) < h(x)$ whenever $h(x) > 0$ and $\phi(h) \leq \phi(h')$ whenever $h \leq h'$.*

(c) [18] *There exist operators ψ, ϕ assigning to each lower semi-continuous function $h : X \to [0, \infty)$, a lower semi-continuous function $\psi(h) : X \to [0, \infty)$ and an upper semi-continuous function $\phi(h) : X \to [0, \infty)$ with $\psi(h) \leq \phi(h) \leq h$ such that $0 < \psi(h)(x) \leq \phi(h)(x) < h(x)$ whenever $h(x) > 0$ and $\psi(h) \leq \psi(h')$, $\phi(h) \leq \phi(h')$ whenever $h \leq h'$.*

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Theorem 1.2. [12] *A space X is k-semi-stratifiable if and only if there exists an operator ϕ assigning to each lower semi-continuous function $h : X \to [0, \infty)$, an upper semi-continuous function $\phi(h) : X \to [0, \infty)$ such that* (1) *$\phi(h) \leq h$,* (2) *$0 < \phi(h)(x) < h(x)$ whenever $h(x) > 0$,* (3) *$\phi(h) \leq \phi(h')$ whenever $h \leq h'$,* (4) *for each compact subset K of X, if $h(x) > 0$ for each $x \in K$, then $\inf\{\phi(h)(x) : x \in K\} > 0$.*

Theorem 1.3. [6, 19] *For a space X, the following are equivalent.*

(a) *X is monotonically countably paracompact.*

(b) *There exist operators ϕ, ψ assigning to each upper semi-continuous function $h : X \to [0, \infty)$, a lower semi-continuous function $\phi(h)$ and an upper semi-continuous function $\psi(h)$ with $h \leq \phi(h) \leq \psi(h)$ such that $\phi(h) \leq \phi(h')$, $\psi(h) \leq \psi(h')$ whenever $h \leq h'$.*

(c) *There exist operators ψ, ϕ assigning to each lower semi-continuous function $h : X \to (0, \infty)$, a lower semi-continuous function $\psi(h) : X \to (0, \infty)$ and an upper semi-continuous function $\phi(h) : X \to (0, \infty)$ with $\psi(h) \leq \phi(h) \leq h$ such that $\psi(h) \leq \psi(h')$, $\phi(h) \leq \phi(h')$ whenever $h \leq h'$.*

In [20], Yamazaki generalized real-valued functions in (b) of Theorem 1.3 and some other insertion theorems to maps with values into ordered topological vector spaces. Characterizations of some spaces, such as monotonically countably paracompact spaces, monotonically countably metacompact spaces and *cb*-spaces were obtained. In [24], we obtained some preliminary results for stratifiable spaces and k-semi-stratifiable spaces. For example.

Proposition 1.4. [24] *Let X be a topological space and Y an ordered topological vector space with a positive interior point e of Y^+. Then X is stratifiable if and only if for each open subset U of X, there exists a continuous map $f_U : X \to [\mathbf{0}, e]$ such that $X \setminus U = f_U^{-1}(\mathbf{0})$ and $f_U \leq f_V$ whenever $U \subset V$.*

In [21], Yamazaki presented some insertion theorems for some related spaces such as normal and countably paracompact spaces, perfectly normal spaces with maps to extended ordered topological vector spaces. As applications, some characterizations of perfectly normal spaces with maps to ordered topological vector spaces were presented. Along the same line, Jin et al [9] considered insertion of maps to ordered topological vector spaces for some other spaces such as stratifiable spaces, semi-stratifiable spaces, monotonically countably paracompact spaces. For example, the following characterization of stratifiable spaces was obtained which extends (c) of Theorem 1.1.

Theorem 1.5. [9] *Let X be a topological space and Y an ordered topological vector space with an interior point e of Y^+. Then X is stratifiable if and only if there exist operators Ψ, Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, two maps $\Psi(h) \in L(X,Y)$ and $\Phi(h) \in U(X,Y)$ with $\Psi(h) \leq \Phi(h) \leq h$ such that $\mathbf{0} \ll \Psi(h)(x) \leq \Phi(h)(x) < h(x)$ whenever $h(x) \neq \mathbf{0}$ and $\Psi(h) \leq \Psi(h')$, $\Phi(h) \leq \Phi(h')$ whenever $h \leq h'$.*

Recall Theorem 1.1 and Theorem 1.2, a natural question is that whether the real-valued functions in (b) of Theorem 1.1 and in Theorem 1.2 can also be generalized to maps to ordered topological vector spaces. In this paper, we shall give a positive answer to this question. The results obtained improve some corresponding results in [9]. Moreover, we present characterizations of some other spaces such as perfectly normal spaces, k-perfect spaces with maps to ordered topological vector spaces.

2. Preliminaries

Throughout, all spaces are assumed to be T_1 topological spaces. A vector space always means a real vector space. The origin of a vector space is denoted by $\mathbf{0}$. The set of all positive integers is denoted by $\mathbb{N}$. τ and τ^c denote the topology of X and the family of all closed subsets of X, respectively. For a space X and $A \subset X$, we use $intA$ and $\overline{A}$ to denote the interior and the closure of A in X, respectively. Also, we use χ_A to denote the characteristic function of A.

A vector space Y with a partial ordered $\leq$ is called an *ordered vector space* if

(1) for each $x, y, z \in Y$, if $x \leq y$ then $x + z \leq y + z$,

(2) for each $x, y \in Y$ and $r \geq 0$, if $x \leq y$ then $rx \leq ry$.

Let $(Y, \leq)$ be an ordered vector space. For $y_1, y_2 \in Y$, $y_1 \leq y_2$ will be sometimes written as $y_2 \geq y_1$. $y_1 < y_2$ means that $y_1 \leq y_2$ and $y_1 \neq y_2$. The set $\{y \in Y : y \geq \mathbf{0}\}$ is denoted by Y^+ and is called the *positive cone* of Y.

A topological vector space Y is called an *ordered topological vector space* if Y is an ordered vector space and the positive cone Y^+ is closed in Y. In this paper, an ordered topological vector space is always non-trivial, that is, $Y \neq \{\mathbf{0}\}$.

Let Y be an ordered topological vector space and $e \in Y^+$. Then e is called an *interior point* of Y^+ if $e \in int_Y Y^+$. e is called an *order unit* if for each $y \in Y$, there exists $\lambda > 0$ such that $y \leq \lambda e$. It is clear that (see [20]) that every interior point of Y^+ is an order unit. It is also clear that if e is an interior point of Y^+ then for each $r > 0$, both $-re + Y^+$ and $re - Y^+$ are $\mathbf{0}$-neighborhoods. Note that if Y is non-trivial and e is an order unit of Y or an interior point of Y^+, then $e > \mathbf{0}$.

The following notation was introduced in [21]. For an ordered topological vector space Y and $y_1, y_2 \in Y$, we write $y_1 \ll y_2$ if $y_2 - y_1$ is an interior point of Y^+. As a generalization, we introduce the following notation. We write $y_1 \lessdot y_2$ if $y_2 - y_1$ is an order unit. $y_1 \ll y_2$ ($y_1 \lessdot y_2$) will be sometimes written as $y_2 \gg y_1$ ($y_2 \gtrdot y_1$). Note that $y \gg \mathbf{0}$ ($y \gtrdot \mathbf{0}$) simply implies that y is an interior point of Y^+ (an order unit).

Recall that a real-valued function f on a space X is called *lower* (*upper*) *semi-continuous* if for any real number r, the set $\{x \in X : f(x) > r\}$ ($\{x \in X : f(x) < r\}$) is open. We write $L(X)$ ($U(X)$) for the set of all lower (upper) semi-continuous functions from X into the unit interval $[0, 1]$. $C(X)$ is the set of all continuous functions from X into $[0, 1]$.

Borwein and Théra [3] generalized the notion of real-valued semi-continuous functions to semi-continuous maps with values into ordered topological vector spaces as follows. Let X be a topological space and Y an ordered topological vector space. A map $f : X \to Y$ is called *lower semi-continuous* if the set-valued mapping $\varphi : X \to 2^Y$, defined by letting $\varphi(x) = f(x) - Y^+$ for each $x \in X$, is lower semi-continuous. f is *upper semi-continuous* if $-f$ is lower semi-continuous. For a topological space X and an ordered topological vector space Y, we use $L(X, Y)$ ($U(X, Y)$) to denote the set of all lower (upper) semi-continuous maps from X to Y. $C(X, Y)$ is the set of all continuous maps from X to Y.

Definition 2.1. [2] A space X is called *stratifiable* if there is a map $\rho : \mathbb{N} \times \tau \to \tau$ such that

(1) $U = \bigcup_{n\in\mathbb{N}} \rho(n, U) = \bigcup_{n\in\mathbb{N}} \overline{\rho(n, U)}$ for each $U \in \tau$,

(2) if $U, V \in \tau$ and $U \subset V$, then $\rho(n, U) \subset \rho(n, V)$ for all $n \in \mathbb{N}$.

Definition 2.2. A space X is called *k-semi-stratifiable* [13] if there is a map $\rho : \mathbb{N} \times \tau \to \tau^c$, such that

(1) $U = \bigcup_{n\in\mathbb{N}} \rho(n, U)$ for each $U \in \tau$,

(2) if $U, V \in \tau$ and $U \subset V$, then $\rho(n, U) \subset \rho(n, V)$ for each $n \in \mathbb{N}$.

(3) for each compact subset K of X and $U \in \tau$ with $K \subset U$, there is $n \in \mathbb{N}$ such that $K \subset \rho(n, U)$.

Note that, without loss of generality, we may assume that the map ρ in the above definitions is increasing with respect to n.

As the end of this section, we list some preliminary lemmas which will be used frequently in the sequel.

Lemma 2.3. [20] *Let X be a topological space and Y an ordered topological vector space. For a map $f : X \to Y$,* (1)*,* (2) *are equivalent and* (1) *implies* (3)*.*

(1) *f is lower (upper) semi-continuous.*

(2) *For each $x \in X$ and each $\mathbf{0}$-neighborhood V, there exists a neighborhood O_x of x such that $f(O_x) \subset f(x) + V + Y^+$ ($f(O_x) \subset f(x) + V - Y^+$).*

(3) *$f^{-1}(y - Y^+)$ ($f^{-1}(y + Y^+)$) is closed in X for each $y \in Y$.*

Lemma 2.4. [23] *Let X be a topological space and Y an ordered topological vector space and f a real-valued function on X. If f is continuous, then for each $y \in Y$, the map $g : X \to Y$ defined by letting $g(x) = f(x)y$ for each $x \in X$ is continuous. If f is lower (upper) semi-continuous, then for each $y \in Y^+$, the map $g : X \to Y$ defined by letting $g(x) = f(x)y$ for each $x \in X$ is lower (upper) semi-continuous.*

Lemma 2.5. *Let Y be an ordered topological vector space. If $y \geq e \gg \mathbf{0}$, then $y \gg \mathbf{0}$.*

Proof. Since $e \gg 0$, we have that $e \in int_Y Y^+$. From $y \geq e$ it follows that $y \in e + Y^+ \subset int_Y Y^+ + Y^+ \subset Y^+$. $int_Y Y^+ + Y^+ = int_Y Y^+$ is open in Y and thus $y \gg 0$. $\square$

Remark 2.6. By Lemma 2.5, if a non-trivial ordered topological vector space Y has an interior point of Y^+, then for each $y \in Y$, $y \gg \mathbf{0}$ if and only if $y \gtrdot \mathbf{0}$. Indeed, let e be an interior point of Y^+. If $y \gtrdot \mathbf{0}$, then there exists $r > 0$ such that $y \geq re$. Since $re \gg \mathbf{0}$, it follows from Lemma 2.5 that $y \gg \mathbf{0}$.

In this paper, for an ordered topological vector space having an interior point of Y^+ and $y \in Y$, we shall always use the notation $y \gtrdot \mathbf{0}$ instead of $y \gg \mathbf{0}$ although they are equivalent by Remark 2.6.

To abbreviate the expressions of our results, we use the following notation.

(H) X is a topological space and Y is an ordered topological vector space with an interior point e of Y^+.

For undefined terminologies, we refer the readers to [4, 16].

3. Stratifiable spaces

In this section, we generalize real-valued functions in (b) of Theorems 1.1 to maps with values into ordered topological vector spaces. Moreover, with similar methods, we present some characterizations of perfectly normal spaces in terms of maps to ordered topological vector spaces. The results improve some corresponding results in [9]. To begin, we need the following Lemma.

Lemma 3.1. [10] *A space X is monotonically normal if and only if there exists an operator Λ assigning to each pair (f, g) of real-valued functions with f upper semi-continuous, g lower semi-continuous and $f \leq g$, a continuous function $\Lambda(f, g)$ such that $f \leq \Lambda(f, g) \leq g$ and $\Lambda(f, g) \leq \Lambda(f', g')$ whenever $f \leq f'$, $g \leq g'$.*

Theorem 3.2. *Assume (H). Then the following are equivalent.*

(a) *X is stratifiable.*

(*b*) *There exists an operator* Φ *assigning to each* $h \in L(X,Y)$ *satisfying* $h(x) \gg \mathbf{0}$ *whenever* $h(x) \neq \mathbf{0}$, *a map* $\Phi(h) \in C(X,Y)$ *with* $\Phi(h) \leq h$ *such that* $\mathbf{0} \ll \Phi(h)(x) < h(x)$ *whenever* $h(x) \neq \mathbf{0}$ *and* $\Phi(h) \leq \Phi(h')$ *whenever* $h \leq h'$.

(*c*) *There exists an operator* Φ *assigning to each* $h \in L(X,Y)$ *satisfying* $h(x) \gg \mathbf{0}$ *whenever* $h(x) \neq \mathbf{0}$, *a map* $\Phi(h) \in C(X,Y)$ *with* $\Phi(h) \leq h$ *such that* $\Phi(h)(x) \gg \mathbf{0}$ *whenever* $h(x) \neq \mathbf{0}$ *and* $\Phi(h) \leq \Phi(h')$ *whenever* $h \leq h'$.

Proof. (a) $\Rightarrow$ (b) Let ρ be the map in Definition 2.1. Let $h \in L(X,Y)$ be such that $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$. For each $n \in \mathbb{N}$, let $U(n,h) = int(h^{-1}(\frac{1}{2^n}e + Y^+))$. Then $U(n,h)$ is an open subset of X. Let

$$\alpha(n,h) = \sum_{i=1}^{\infty} \frac{1}{2^i} \chi_{\overline{\rho(i,U(n,h))}}, \quad \beta(n,h) = \chi_{U(n,h)}.$$

Then $\alpha(n,h) \in U(X)$, $\beta(n,h) \in L(X)$ and $\alpha(n,h) \leq \beta(n,h)$ for each $n \in \mathbb{N}$. Since X is stratifiable, it is monotonically normal [2]. Let Λ be the operator in Lemma 3.1 and $\varphi_n(h) = \Lambda(\alpha(n,h), \beta(n,h))$ for each $n \in \mathbb{N}$. Then let

$$\Phi(h) = (\sum_{n=1}^{\infty} \frac{1}{2^{n+2}} \varphi_n(h))e.$$

By Lemma 2.4, $\Phi(h) \in C(X,Y)$.

If $h \leq h'$, then $U(n,h) \subset U(n,h')$ for each $n \in \mathbb{N}$. It follows that $\alpha(n,h) \leq \alpha(n,h')$, $\beta(n,h) \leq \beta(n,h')$ and so $\varphi_n(h) \leq \varphi_n(h')$ for each $n \in \mathbb{N}$. Thus $\Phi(h) \leq \Phi(h')$.

Let $x \in X$. If $h(x) = \mathbf{0}$, then $x \notin U(n,h)$ for all $n \in \mathbb{N}$. Thus $\beta(n,h)(x) = 0$ from which it follows that $\varphi_n(h)(x) = 0$ and so $\Phi(h)(x) = \mathbf{0}$. If $h(x) \neq \mathbf{0}$, then $h(x) \gg \mathbf{0}$ which implies that $h(x)$ is an order unit. Then there exists $m \in \mathbb{N}$ such that $h(x) \geq \frac{1}{2^{m-1}}e$. Since h is lower semi-continuous and $-\frac{1}{2^m}e + Y^+$ is a $\mathbf{0}$-neighborhood, by Lemma 2.3, there exists a neighborhood O_x of x such that

$$h(O_x) \subset h(x) - \frac{1}{2^m}e + Y^+ + Y^+ \subset \frac{1}{2^m}e + Y^+.$$

This implies that $x \in int(h^{-1}(\frac{1}{2^m}e + Y^+)) = U(m,h)$. Then there exists $i \geq m$ such that $x \in \rho(i, U(m,h))$. It follows that $\varphi_m(h)(x) \geq \alpha(m,h)(x) \geq \frac{1}{2^i}$. Thus $\Phi(h)(x) \geq \frac{1}{2^{m+i+2}}e \gg \mathbf{0}$. By Lemma 2.5, $\Phi(h)(x) \gg \mathbf{0}$. Now, let $k = \min\{n \in \mathbb{N} : x \in U(n,h)\}$. Then $x \in U(k,h)$ and $x \notin U(n,h)$ for each $n < k$. Thus $\varphi_n(h)(x) = 0$ for each $n < k$ and so

$$\Phi(h)(x) = (\sum_{n=k}^{\infty} \frac{1}{2^{n+2}} \varphi_n(h)(x))e \leq (\sum_{n=k}^{\infty} \frac{1}{2^{n+2}})e = \frac{1}{2^{k+1}}e < \frac{1}{2^k}e \leq h(x).$$

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a) For each open subset U of X, let $h_U = \chi_U e$. Then $h_U \in L(X,Y)$ by Lemma 2.4 and if $h_U(x) \neq \mathbf{0}$ then $h_U(x) = e \gg \mathbf{0}$. For each $n \in \mathbb{N}$, let

$$\rho(n,U) = int(\Phi(h_U)^{-1}(\frac{1}{n}e + Y^+)),\ F(n,U) = \Phi(h_U)^{-1}(\frac{1}{n}e + Y^+).$$

It is easy to see that if U, V are open subsets of X and $U \subset V$, then $\rho(n,U) \subset \rho(n,V)$ for each $n \in \mathbb{N}$. Since $\rho(n,U) \subset F(n,U)$ and $F(n,U)$ is a closed subset of X, we have that $\overline{\rho(n,U)} \subset F(n,U)$.

For each $n \in \mathbb{N}$, if $x \in \overline{\rho(n,U)} \subset F(n,U)$, then $\frac{1}{n}e \leq \Phi(h_U)(x) \leq h_U(x)$ from which it follows that $\chi_U(x) \geq \frac{1}{n}$ and thus $x \in U$. This implies that $\bigcup_{n\in\mathbb{N}} \overline{\rho(n,U)} \subset U$. Now, if $x \in U$, then $h_U(x) = e \neq \mathbf{0}$ and so $\Phi(h_U)(x) \gg \mathbf{0}$ which implies that $\Phi(h_U)(x)$ is an order unit. Then there exists $m \in \mathbb{N}$ such that $\Phi(h_U)(x) \geq \frac{2}{m}e$. Since $\Phi(h_U)$ is continuous and $-\frac{1}{m}e + Y^+$ is a $\mathbf{0}$-neighborhood, there exists an open neighborhood O_x of x such that

$$\Phi(h_U)(O_x) \subset \Phi(h_U)(x) - \frac{1}{m}e + Y^+ \subset \frac{1}{m}e + Y^+.$$

This implies that $x \in int(\Phi(h_U)^{-1}(\frac{1}{m}e + Y^+)) = \rho(m,U)$. Thus $U \subset \bigcup_{n\in\mathbb{N}} \rho(n,U)$. By Definition 2.1, X is a stratifiable space. $\square$

Corollary 3.3. *Assume* (H). *Then the following are equivalent.*

(a) *X is stratifiable.*

(b) [9] *There exist operators Ψ, Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, two maps $\Psi(h) \in L(X,Y)$ and $\Phi(h) \in U(X,Y)$ with $\Psi(h) \leq \Phi(h) \leq h$ such that $\mathbf{0} \ll \Psi(h)(x) \leq \Phi(h)(x) < h(x)$ whenever $h(x) \neq \mathbf{0}$ and $\Psi(h) \leq \Psi(h')$, $\Phi(h) \leq \Phi(h')$ whenever $h \leq h'$.*

(c) *There exist operators Ψ, Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, two maps $\Psi(h) \in L(X,Y)$ and $\Phi(h) \in U(X,Y)$ with $\Psi(h) \leq \Phi(h) \leq h$ such that $\Psi(h)(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$ and $\Psi(h) \leq \Psi(h')$ whenever $h \leq h'$.*

Proof. (a) $\Rightarrow$ (b) follows directly from (a) $\Rightarrow$ (b) of Theorem 3.2, since a continuous map is both lower semi-continuous and upper semi-continuous.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a) For each open subset U of X, let $h_U = \chi_U e$. Then for each $n \in \mathbb{N}$, let

$$\rho(n,U) = int(\Psi(h_U)^{-1}(\frac{1}{n}e + Y^+)),\ F(n,U) = \Phi(h_U)^{-1}(\frac{1}{n}e + Y^+).$$

If U, V are open subsets of X and $U \subset V$, then $\rho(n,U) \subset \rho(n,V)$ for each $n \in \mathbb{N}$.

From $\Psi(h_U) \leq \Phi(h_U)$ it follows that $\rho(n,U) \subset F(n,U)$. Since $\Phi(h_U)$ is upper semi-continuous, by Lemma 2.3, $F(n,U)$ is a closed subset of X and so $\overline{\rho(n,U)} \subset F(n,U)$.

Analogous to the proof of (c) $\Rightarrow$ (a) of Theorem 3.2, we can show that $\bigcup_{n\in\mathbb{N}} \overline{\rho(n,U)} \subset U$. Now, if $x \in U$, then $h_U(x) = e$ and so $\Psi(h_U)(x) \gg \mathbf{0}$. Then $\Psi(h_U)(x) \geq \frac{2}{m}e$ for some $m \in \mathbb{N}$. Since $\Psi(h_U)$ is lower semi-continuous and $-\frac{1}{m}e + Y^+$ is a $\mathbf{0}$-neighborhood, by Lemma 2.3, there exists a neighborhood O_x of x such that

$$\Psi(h_U)(O_x) \subset \Psi(h_U)(x) - \frac{1}{m}e + Y^+ + Y^+ \subset \frac{1}{m}e + Y^+.$$

This implies that $x \in \rho(m,U)$. Thus $U \subset \bigcup_{n\in\mathbb{N}} \rho(n,U)$. By Definition 2.1, X is a stratifiable space. $\square$

Corollary 3.4. *Assume* (H). *Then the following are equivalent.*

(a) *X is stratifiable.*

(b) *There exists an operator Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, a map $\Phi(h) \in C(X,Y)$ with $\Phi(h) \leq h$ such that $\Phi(h) \leq \Phi(h')$ whenever $h \leq h'$, and if $h(x) \neq \mathbf{0}$ then there is an open neighborhood O_x of x and $r > 0$ such that $\Phi(h)(O_x) \subset re + Y^+$.*

(c) *There exist operators Ψ, Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, two maps $\Psi(h) \in L(X,Y)$ and $\Phi(h) \in U(X,Y)$ with $\Psi(h) \leq \Phi(h) \leq h$ such that $\Psi(h) \leq \Psi(h')$, $\Phi(h) \leq \Phi(h')$ whenever $h \leq h'$, and if $h(x) \neq \mathbf{0}$ then there is an open neighborhood O_x of x and $r > 0$ such that $\Psi(h)(O_x) \subset re + Y^+$.*

(d) *There exists an operator Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, a map $\Phi(h) \in U(X,Y)$ with $\Phi(h) \leq h$ such that $\Phi(h) \leq \Phi(h')$ whenever $h \leq h'$, and if $h(x) \neq \mathbf{0}$ then there is an open neighborhood O_x of x and $r > 0$ such that $\Phi(h)(O_x) \subset re + Y^+$.*

Proof. (a) $\Rightarrow$ (b) Let $\Phi(h) \in C(X,Y)$ be the map in (b) of Theorem 3.2. If $h(x) \neq \mathbf{0}$, then $\Phi(h)(x) \gg \mathbf{0}$ which implies that $\Phi(h)(x) \in int_Y Y^+$. Then $\Phi(h)(x) - int_Y Y^+$ is an open $\mathbf{0}$-neighborhood and thus there exists $m \in \mathbb{N}$ such that $\frac{1}{m}e \in \Phi(h)(x) - int_Y Y^+$. It follows that $\Phi(h)(x) \in \frac{1}{m}e + int_Y Y^+ = int_Y(\frac{1}{m}e + Y^+)$. Since $\Phi(h)$ is continuous, we have that $x \in \Phi(h)^{-1}(int_Y(\frac{1}{m}e + Y^+)) \subset int(\Phi(h)^{-1}(\frac{1}{m}e + Y^+))$. Set $O_x = int(\Phi(h)^{-1}(\frac{1}{m}e + Y^+))$. Then O_x is an open neighborhood of x and $\Phi(h)(O_x) \subset \frac{1}{m}e + Y^+$.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are clear.

(d) $\Rightarrow$ (a) For each open subset U of X, let $h_U = \chi_U e$. Then for each $n \in \mathbb{N}$, let

$$\rho(n,U) = int(\Phi(h_U)^{-1}(\frac{1}{n}e + Y^+)),\ F(n,U) = \Phi(h_U)^{-1}(\frac{1}{n}e + Y^+).$$

Since $\rho(n,U) \subset F(n,U)$ and $F(n,U)$ is a closed subset of X, we have that $\overline{\rho(n,U)} \subset F(n,U)$. By the proof of (c) $\Rightarrow$ (a) of Theorem 3.2, we only need show that $U \subset \bigcup_{n\in\mathbb{N}} \rho(n,U)$. Let $x \in U$. Then $h_U(x) = e \neq \mathbf{0}$. By (d), there exists an open neighborhood O_x of x and $m \in \mathbb{N}$ such that $\Phi(h_U)(O_x) \subset \frac{1}{m}e + Y^+$. This implies that $x \in int(\Phi(h_U)^{-1}(\frac{1}{m}e + Y^+)) = \rho(m,U)$. Thus $U \subset \bigcup_{n\in\mathbb{N}} \rho(n,U)$. By Definition 2.1, X is a stratifiable space. □

Lemma 3.5. *A space X is perfectly normal if and only if there is a map $\rho : \mathbb{N}\times\tau \to \tau$ such that $U = \bigcup_{n\in\mathbb{N}} \rho(n,U) = \bigcup_{n\in\mathbb{N}} \overline{\rho(n,U)}$ for each $U \in \tau$.*

Lemma 3.5 is easy to be shown and we believe that it is known, but we can not find a reference.

Lemma 3.6. [17] *A space X is normal if and only if there exists an operator Λ assigning to each pair (f,g) of functions with f upper semi-continuous, g lower semi-continuous and $f \leq g$, a continuous function $\Lambda(f,g)$ such that $f \leq \Lambda(f,g) \leq g$.*

With a similar argument to that in the proof of Theorem 3.2, we can prove the following analogous result for perfectly normal spaces.

Theorem 3.7. *Assume (H). Then the following are equivalent.*

(a) *X is perfectly normal.*

(b) *There exists an operator Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, a map $\Phi(h) \in C(X,Y)$ with $\Phi(h) \leq h$ such that $\mathbf{0} \ll \Phi(h)(x) < h(x)$ whenever $h(x) \neq \mathbf{0}$.*

(c) *There exists an operator Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, a map $\Phi(h) \in C(X,Y)$ with $\Phi(h) \leq h$ such that $\Phi(h)(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$.*

Proof. (a) $\Rightarrow$ (b) is similar to the proof of (a) $\Rightarrow$ (b) of Theorem 3.2 by using Lemma 3.5 and Lemma 3.6.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a) is similar to the proof of (c) $\Rightarrow$ (a) of Theorem 3.2 by using Lemma 3.5. □

Corollary 3.8. *Assume (H). Then the following are equivalent.*

(a) *X is perfectly normal.*

(b) [9] *There exist operators Ψ, Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, two maps $\Psi(h) \in L(X,Y)$ and $\Phi(h) \in U(X,Y)$ with $\Psi(h) \leq \Phi(h) \leq h$ such that $\mathbf{0} \ll \Psi(h)(x) \leq \Phi(h)(x) < h(x)$ whenever $h(x) \neq \mathbf{0}$.*

(c) *There exist operators Ψ, Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, two maps $\Psi(h) \in L(X,Y)$ and $\Phi(h) \in U(X,Y)$ with $\Psi(h) \leq \Phi(h) \leq h$ such that $\Psi(h)(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$.*

Proof. (a) $\Rightarrow$ (b) follows from (a) $\Rightarrow$ (b) of Theorem 3.7.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a) is similar to the proof of (c) $\Rightarrow$ (a) of Corollary 3.3. $\square$

Analogous to Corollary 3.4, we have the following.

Corollary 3.9. *Assume* (H). *Then the following are equivalent.*

(a) *X is perfectly normal.*

(b) *There exists an operator Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, a map $\Phi(h) \in C(X,Y)$ with $\Phi(h) \leq h$ such that if $h(x) \neq \mathbf{0}$ then there is an open neighborhood O_x of x and $r > 0$ such that $\Phi(h)(O_x) \subset re + Y^+$.*

(c) *There exist operators Φ, Ψ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, two maps $\Phi(h) \in U(X,Y)$ and $\Psi(h) \in L(X,Y)$ with $\Psi(h) \leq \Phi(h) \leq h$ such that if $h(x) \neq \mathbf{0}$ then there is an open neighborhood O_x of x and $r > 0$ such that $\Psi(h)(O_x) \subset re + Y^+$.*

(d) *There exists an operator Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, a map $\Phi(h) \in U(X,Y)$ with $\Phi(h) \leq h$ such that if $h(x) \neq \mathbf{0}$ then there is an open neighborhood O_x of x and $r > 0$ such that $\Phi(h)(O_x) \subset re + Y^+$.*

4. k-semi-stratifiable spaces

In this section, we generalize real-valued functions in Theorems 1.2 to maps to ordered topological vector spaces. Moreover, we give a similar result for k-perfect spaces.

Theorem 4.1. *Assume* (H). *Then the following are equivalent.*

(a) *X is k-semi-stratifiable.*

(b) *There exists an operator Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, a map $\Phi(h) \in U(X,Y)$ such that* (1) *$\Phi(h) \leq h$,* (2) *$\mathbf{0} \ll \Phi(h)(x) < h(x)$ whenever $h(x) \neq \mathbf{0}$,* (3) *$\Phi(h) \leq \Phi(h')$ whenever $h \leq h'$,* (4) *for each compact subset K of X, if $h(x) \neq \mathbf{0}$ for each $x \in K$, then $\Phi(h)(K) \subset re + Y^+$ for some $r > 0$.*

(c) *There exists an operator Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, a map $\Phi(h) \in U(X,Y)$ such that* (1) *$\Phi(h) \leq h$,* (2) *$\Phi(h) \leq \Phi(h')$ whenever $h \leq h'$,* (3) *for each compact subset K of X, if $h(x) \neq \mathbf{0}$ for each $x \in K$, then $\Phi(h)(K) \subset re + Y^+$ for some $r > 0$.*

(d) *There exists an operator Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, a map $\Phi(h) \in U(X,Y)$ such that* (1) *$\Phi(h) \leq h$,* (2) *$\Phi(h) \leq \Phi(h')$ whenever $h \leq h'$,* (3) *for each compact subset K of X, if $h(K) \subset re + Y^+$ for some $r > 0$, then $\Phi(h)(K) \subset se + Y^+$ for some $s > 0$.*

Proof. (a) $\Rightarrow$ (b) Let ρ be the map in Definition 2.2 which is increasing with respect to n. For each $n \in \mathbb{N}$ and each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, let $U(n,h) = int(h^{-1}(\frac{1}{2^n}e + Y^+))$ and

$$\Phi(h) = (\sum_{n=1}^{\infty} \frac{1}{2^{n+2}} \chi_{\rho(n,U(n,h))})e.$$

Then $\Phi(h) \in U(X,Y)$. It is clear that if $h \leq h'$ then $\Phi(h) \leq \Phi(h')$.

Let $x \in X$. If $h(x) = \mathbf{0}$, then $x \notin U(n,h)$ and so $x \notin \rho(n,U(n,h))$ for all $n \in \mathbb{N}$ from which it follows that $\Phi(h)(x) = \mathbf{0}$. If $h(x) \neq \mathbf{0}$, then $x \in U(n,h)$ for some $n \in \mathbb{N}$. Let $k = \min\{n \in \mathbb{N} : x \in U(n,h)\}$. Then $x \in U(k,h)$ and $x \notin U(n,h) \supset \rho(n,U(n,h))$ for each $n < k$. Thus

$$\Phi(h)(x) = (\sum_{n=k}^{\infty} \frac{1}{2^{n+2}} \chi_{\rho(n,U(n,h))}(x))e \leq (\sum_{n=k}^{\infty} \frac{1}{2^{n+2}})e = \frac{1}{2^{k+1}}e < \frac{1}{2^k}e \leq h(x).$$

To prove (4), let K be a compact subset of X and $h(x) \neq \mathbf{0}$ for each $x \in K$. Then for each $x \in K$, there exists $n_x \in \mathbb{N}$ such that $x \in U(n_x,h)$. Since $\{U(n_x,h) : x \in K\}$ is an open cover of K, there exists a finite subset A of K such that $\{U(n_x,h) : x \in A\}$ covers K. Let $n = \max\{n_x : x \in A\}$. Then $K \subset U(n,h)$. By (3) of Definition 2.2, $K \subset \rho(m,U(n,h))$ for some $m \in \mathbb{N}$. Let $i = \max\{m,n\}$. Then $K \subset \rho(i,U(i,h))$. Thus for each $x \in K$, $\Phi(h)(x) \geq \frac{1}{2^{i+2}}e$.

From (4) it follows that $\Phi(h)(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are clear.

(d) $\Rightarrow$ (a) For each open subset U of X, let $h_U = \chi_U e$. Then for each $n \in \mathbb{N}$, let $\rho(n,U) = \Phi(h_U)^{-1}(\frac{1}{n}e + Y^+)$. Since $\Phi(h_U)$ is upper semi-continuous, by Lemma 2.3, $\rho(n,U)$ is closed in X. It is easy to verify that if U,V are open subsets of X and $U \subset V$, then $\rho(n,U) \subset \rho(n,V)$ for each $n \in \mathbb{N}$.

If $x \in \rho(n,U)$ for some $n \in \mathbb{N}$, then $\frac{1}{n}e \leq \Phi(h_U)(x) \leq h_U(x)$ from which it follows that $\chi_U(x) \geq \frac{1}{n}$ and thus $x \in U$. This implies that $\bigcup_{n\in\mathbb{N}} \rho(n,U) \subset U$.

Now, let K be a compact subset of X and $U \in \tau$ with $K \subset U$. Then $h_U(x) = e$ for each $x \in K$. By (3) of (d), there exists $n \in \mathbb{N}$ such that $K \subset \Phi(h_U)^{-1}(\frac{1}{n}e + Y^+) = \rho(n,U)$. By Definition 2.2, X is a k-semi-stratifiable space. $\square$

Definition 4.2. [22] A space X is called *k-perfect* if there is a map $\rho : \mathbb{N} \times \tau \to \tau^c$, such that (1) $U = \bigcup_{n\in\mathbb{N}} \rho(n,U)$ for each $U \in \tau$, (2) for each compact subset K of X and $U \in \tau$ with $K \subset U$, there is $n \in \mathbb{N}$ such that $K \subset \rho(n,U)$.

With a similar argument to that in the proof of Theorem 4.1, we can prove the following result for k-perfect spaces.

Proposition 4.3. *Assume (H). Then the following are equivalent.*

(a) *X is k-perfect.*

(b) *There exists an operator Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, a map $\Phi(h) \in U(X,Y)$ such that* (1) *$\Phi(h) \leq h$,* (2) *$\mathbf{0} \ll \Phi(h)(x) < h(x)$ whenever $h(x) \neq \mathbf{0}$,* (3) *for each compact subset K of X, if $h(x) \neq \mathbf{0}$ for each $x \in K$, then $\Phi(h)(K) \subset re + Y^+$ for some $r > 0$.*

(c) *There exists an operator Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, a map $\Phi(h) \in U(X,Y)$ such that* (1) *$\Phi(h) \leq h$,* (2) *for each compact subset K of X, if $h(x) \neq \mathbf{0}$ for each $x \in K$, then $\Phi(h)(K) \subset re + Y^+$ for some $r > 0$.*

(d) *There exists an operator Φ assigning to each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, a map $\Phi(h) \in U(X,Y)$ such that* (1) *$\Phi(h) \leq h$,* (2) *for each compact subset K of X, if $h(K) \subset re + Y^+$ for some $r > 0$, then $\Phi(h)(K) \subset se + Y^+$ for some $s > 0$.*

Proof. (a) $\Rightarrow$ (b) Let ρ be the map in Definition 4.2. For each $n \in \mathbb{N}$ and each $h \in L(X,Y)$ satisfying $h(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$, let $U(n,h) = int(h^{-1}(\frac{1}{2^n}e + Y^+))$. Then let

$$\Phi(h) = (\sum_{n=1}^{\infty} \frac{1}{2^{n+2}} \sum_{i=1}^{\infty} \frac{1}{2^i} \chi_{\rho(i,U(n,h))})e.$$

Then $\Phi(h) \in U(X,Y)$.

Let $x \in X$. If $h(x) = \mathbf{0}$, then $x \notin U(n,h)$ for all $n \in \mathbb{N}$. Thus $x \notin \rho(i, U(n,h))$ for all $i, n \in \mathbb{N}$ and so $\Phi(h)(x) = \mathbf{0}$. If $h(x) \neq \mathbf{0}$, then $x \in U(m,h)$ for some $m \in \mathbb{N}$. Let $k = \min\{n \in \mathbb{N} : x \in U(n,h)\}$. Then $x \in U(k,h)$ and $x \notin U(n,h)$ for each $n < k$. Hence

$$\begin{aligned}\Phi(h)(x) &= (\sum_{n=k}^{\infty} \tfrac{1}{2^{n+2}} \sum_{i=1}^{\infty} \tfrac{1}{2^i} \chi_{\rho(i,U(n,h))}(x))e \\ &\leq (\sum_{n=k}^{\infty} \tfrac{1}{2^{n+2}})e = \tfrac{1}{2^{k+1}}e < \tfrac{1}{2^k}e \leq h(x).\end{aligned}$$

Now, let K be a compact subset of X and $h(x) \neq \mathbf{0}$ for each $x \in K$. With a similar argument as that in the proof of (a) $\Rightarrow$ (b) of Theorem 4.1, we can show that $K \subset U(m,h)$ for some $m \in \mathbb{N}$. By (2) of Definition 4.2, $K \subset \rho(i, U(m,h))$ for some $i \in \mathbb{N}$. Thus for each $x \in K$, $\Phi(h)(x) \geq \frac{1}{2^{m+i+2}}e$.

From (3) it follows that $\Phi(h)(x) \gg \mathbf{0}$ whenever $h(x) \neq \mathbf{0}$.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (d) are clear.

(d) $\Rightarrow$ (a) is similar to the proof of (d) $\Rightarrow$ (a) of Theorem 4.1. $\square$

5. Some other spaces

For two sequences $\langle A_j \rangle$ and $\langle B_j \rangle$ of subsets of a space X, we write $\langle A_j \rangle \preceq \langle B_j \rangle$ if $A_n \subset B_n$ for each $n \in \mathbb{N}$.

Definition 5.1. [5] A space X is called *monotonically countably paracompact* (abbr., MCP) if there is an operator O assigning to each decreasing sequence $\langle F_j \rangle$ of closed subsets of X with empty intersection, a sequence of open sets $\{O(n, \langle F_j \rangle) : n \in \mathbb{N}\}$ such that

(1) $F_n \subset O(n, \langle F_j \rangle)$ for each $n \in \mathbb{N}$,
(2) if $\langle F_j \rangle \preceq \langle G_j \rangle$, then $O(n, \langle F_j \rangle) \subset O(n, \langle G_j \rangle)$ for all $n \in \mathbb{N}$,
(3) $\bigcap_{n \in \mathbb{N}} \overline{O(n, \langle F_j \rangle)} = \emptyset$.

Note that, without loss of generality, we may assume that $\{O(n, \langle F_j \rangle) : n \in \mathbb{N}\}$ is decreasing with respect to n in the above definition.

In [9], the following characterization of MCP spaces was obtained which extends (c) of Theorem 1.3.

Theorem 5.2. [9] *Assume (H). Then X is an MCP space if and only if there exist operators Ψ, Φ assigning to each $h \in L(X, Y)$ with $h \gtrdot \mathbf{0}$, two maps $\Psi(h) \in L(X, Y)$ and $\Phi(h) \in U(X, Y)$ with $\mathbf{0} \ll \Psi(h) \leq \Phi(h) < h$ $(\leq h)$ such that $\Psi(h) \leq \Psi(h')$, $\Phi(h) \leq \Phi(h')$ whenever $h \leq h'$.*

As a corollary, we have the following.

Corollary 5.3. *Assume (H). Then X is an MCP space if and only if there exists an operator Φ assigning to each $h \in L(X, Y)$ with $h \gtrdot \mathbf{0}$, a map $\Phi(h) \in U(X, Y)$ with $\Phi(h) < h$ $(\leq h)$ such that $\Phi(h) \leq \Phi(h')$ whenever $h \leq h'$ and for each $x \in X$ there exists an open neighborhood O_x of x and $r > 0$ such that $\Phi(h)(O_x) \subset re + Y^+$.*

Proof. Let $\Psi(h), \Phi(h)$ be the maps in Theorem 5.2. We shall show that for each $x \in X$ there exists an open neighborhood O_x of x and $r > 0$ such that $\Phi(h)(O_x) \subset re + Y^+$. For each $x \in X$, since $\Psi(h)(x) \gg \mathbf{0}$, there exists $m \in \mathbb{N}$ such that $\Psi(h)(x) \geq \frac{2}{m}e$. Since $-\frac{1}{m}e + Y^+$ is a $\mathbf{0}$-neighborhood and $\Psi(h)$ is lower semi-continuous, there exists an open neighborhood O_x of x such that $\Psi(h)(O_x) \subset \frac{2}{m}e + Y^+ - \frac{1}{m}e + Y^+ + Y^+ = \frac{1}{m}e + Y^+$. For each $x' \in O_x$, $\Phi(h)(x') \geq \Psi(h)(x') \geq \frac{1}{m}e + Y^+$ which implies that $\Phi(h)(O_x) \subset \frac{1}{m}e + Y^+$.

Conversely, assume the condition. Let $h \in L(X, Y)$ be such that $h \gtrdot \mathbf{0}$ and $x \in X$. Then there exists an open neighborhood O_x of x and $m \in \mathbb{N}$ such that $\Phi(h)(O_x) \subset \frac{1}{m}e + Y^+$. It follows that $x \in int(\Phi(h)^{-1}(\frac{1}{m}e + Y^+))$. Let $n_x(h) = \min\{n \in \mathbb{N} : x \in int(\Phi(h)^{-1}(\frac{1}{n}e + Y^+))\}$ and $\Psi(h)(x) = \frac{1}{n_x(h)}e$. Then $\Psi(h)(x) \gg \mathbf{0}$ and $\Phi(h)(x) \geq \frac{1}{n_x(h)}e = \Psi(h)(x)$. This implies that $\mathbf{0} \ll \Psi(h) \leq \Phi(h)$.

To show that $\Psi(h)$ is lower semi-continuous. Let $x \in X$ and V a $\mathbf{0}$-neighborhood. Let $O_x = int(\Phi(h)^{-1}(\frac{1}{n_x(h)}e + Y^+))$. Then O_x is an open neighborhood of x. For each $x' \in O_x$, we have that $n_{x'}(h) \leq n_x(h)$ and thus $\Psi(h)(x') \geq \Psi(h)(x)$. It follows that $\Psi(h)(x') \in \Psi(h)(x) + Y^+ \subset \Psi(h)(x) + V + Y^+$. This implies that $\Psi(h)$ is lower semi-continuous.

Now, suppose that $h \leq h'$. Then $\Phi(h) \leq \Phi(h')$ and thus $\Phi(h)^{-1}(\frac{1}{n}e + Y^+) \subset \Phi(h')^{-1}(\frac{1}{n}e + Y^+)$ for each $n \in \mathbb{N}$. Hence for each $x \in X$, $x \in int(\Phi(h)^{-1}(\frac{1}{n_x(h)}e + Y^+)) \subset int(\Phi(h')^{-1}(\frac{1}{n_x(h)}e + Y^+))$. It follows that $n_x(h') \leq n_x(h)$ and thus $\Psi(h)(x) \leq \Psi(h')(x)$. This implies that $\Psi(h) \leq \Psi(h')$.

By Theorem 5.2, X is an MCP space. □

Recall that a space X is called a *cb*-space [8] if for each locally bounded function f on X there is a continuous function g on X such that $|f| \leq g$. In [20], Yamazaki showed that X is a *cb*-space if and only if for each upper semi-continuous map $f : X \to Y$,there exists a continuous map $\Phi(f) : X \to Y$ such that $f \leq \Phi(f)$. In [14], it was shown that X is a *cb*-space if and only if there exists an operator ϕ assigning to each lower semi-continuous function $f : X \to (0, \infty)$, a continuous function $\phi(f) : X \to (0, \infty)$ such that $\phi(f) < f$. In view of Theorem 5.2, a natural question is that whether there is a similar result for *cb*-spaces. We have the following partial answer.

Proposition 5.4. *Assume* (H). *If* X *is a cb-space, then there exists an operator* Φ *assigning to each* $h \in L(X, Y)$ *with* $h \gg \mathbf{0}$, *a map* $\Phi(h) \in C(X, Y)$ *such that* $\mathbf{0} \ll \Phi(h) < h$.

To prove Proposition 5.4, we need the following lemma.

Lemma 5.5. [15] *A space* X *is a cb-space if and only if for every decreasing sequence* $\{f_n \in U(X) : n \in \mathbb{N}\}$ *of functions such that* $f_n \to 0$, *there is a sequence* $\{g_n \in C(X) : n \in \mathbb{N}\}$ *of functions such that* $f_n \leq g_n$ *for each* $n \in \mathbb{N}$ *and* $g_n \to 0$.

Proof of Proposition 5.4. For each $n \in \mathbb{N}$ and each $h \in L(X, Y)$ with $h \gg \mathbf{0}$, let $F(n, h) = X \setminus int(h^{-1}(\frac{1}{2^{n-1}}e + Y^+))$. Then $\{F(n, h) : n \in \mathbb{N}\}$ is a decreasing sequence of closed subsets of X with empty intersection.

For each $n \in \mathbb{N}$, let $f(n, h) = \chi_{F(n,h)}$, then $\{f(n, h) \in U(X) : n \in \mathbb{N}\}$ is decreasing and $f(n, h) \to 0$. By Lemma 5.5, there exists a sequence $\{g(n, h) \in C(X) : n \in \mathbb{N}\}$ of functions such that $f(n, h) \leq g(n, h)$ for each $n \in \mathbb{N}$ and $g(n, h) \to 0$. Let

$$\Phi(h) = (\frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} g(n, h))e.$$

Then $\Phi(h) \in C(X, Y)$. For each $x \in X$, since $g(n, h)(x) \to 0$, there is $m \in \mathbb{N}$ such that $g(m, h)(x) < 1$. Hence

$$\frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} g(n, h)(x) > 0$$

from which it follows that $\Phi(h)(x) \gg \mathbf{0}$. Since $\bigcap_{n\in\mathbb{N}} F(n, h) = \emptyset$, $x \notin F(n, h)$ for some $n \in \mathbb{N}$. Let $k = \min\{n \in \mathbb{N} : x \notin F(n, h)\}$, then $x \notin F(n, h)$ for all $n \geq k$ while $x \in F(n, h)$ for each $n < k$. Thus

$$\Phi(h)(x) = (\frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} g(n, h)(x))e \leq (\frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{2^{n}} f(n, h)(x))e$$

$$= (\frac{1}{2} - \sum_{n=1}^{k-1} \frac{1}{2^{n+1}})e = \frac{1}{2^k} e < \frac{1}{2^{k-1}} e \leq h(x).$$

Question 5.6. Is the condition in Proposition 5.4 also sufficient?

The referee reminded the author that in a recent paper [K. Yamazaki, A method of returning vector-valued maps to real-valued functions on mono-tone operators, Top. Appl., **246** (2018), 69-82], Yamazaki gave a characterization of cb-spaces which answers positively Question 5.6.

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