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HOMOTOPY GROUPS OF INFINITE WEDGE

LEONARD MDZINARISHVILI

Abstract. In *Homotopy Theory* (Pure and Applied Mathematics, Vol. VIII, Academic Press, New York–London, 1959), Sze-tsen Hu proved for $X \vee Y$, the wedge sum of pointed spaces (X, x_0), and (Y, y_0) that for $n \geq 2$ there is an isomorphism

(1) $\pi_n(X \vee Y, u_0) \approx \pi_n(X, x_0) \oplus \pi_n(Y, y_0) \oplus \pi_{n+1}(X \times Y, X \vee Y, u_0)$,

where $u_0 = (x_0, y_0)$.

This result was not generalized for an infinite wedge $\vee Y_\omega$, $\omega \in \Omega$, of pointed spaces (Y_ω, y_ω^0) in view of the fact that an infinite wedge $\vee Y_\omega$ is not a subspace of the direct product $\prod Y_\omega$, $\omega \in \Omega$.

In the present work we prove that for $n \geq 2$ there is an isomorphism

$$\pi_n(\vee Y_\omega, y^0) \approx \sum_{\omega \in \Omega} \pi_n(Y_\omega, y_\omega^0) \oplus \pi_{n+1}(LY_\omega, \vee Y_\omega, y^0),$$

where LY_ω is the weak product of pointed topological spaces (Y_ω, y_ω^0), $\omega \in \Omega$ (see C. J. Knight, *Weak products of spaces and complexes*, Fund. Math. **53** (1963), 1–12.)

Let Top_* be the category of pointed topological spaces and continuous maps preserving base point [4].

If $\Omega = \{\omega\}$ is an infinite set and $\{(Y_\omega, y_\omega^0)\}_{\omega \in \Omega}$ is a family of objects from Top_* indexed by Ω, their infinite wedge is denoted by $\vee Y_\omega$ and is defined by $\bigcup\limits_{\omega \in \Omega} Y_\omega / \bigcup y_\omega^0$ the quotient space of $\bigcup Y_\omega$ obtained by identifying all of $\bigcup y_\omega^0$ to a single point u^0. We define a topology by declaring a subset $U \subset \bigcup\limits_{\omega \in \Omega} Y_\omega$ to be open if and only if the intersection $U \cap Y_\omega$ is open in Y_ω for all $\omega \in \Omega$ [1, Definition 2.2.8].

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In [3], C. J. Knight defined the weak product LY_ω of pointed topological spaces (Y_ω, y^0_ω), $\omega \in \Omega$, to consist of all elements $y \in \prod_{\omega\in\Omega} Y_\omega$ of the product such that all but a finite number of coordinates y_ω of y are the base points. However, its topology is not the relative topology, but the topology of the union of the finite products $\prod_{i=1}^{k} Y_i \subset \prod_{\omega\in\Omega} Y_\omega$.

Consider in the weak product LY_ω the subset $\bigcup_{\omega\in\Omega} L_\omega$ where $L_\omega = \prod_{\alpha\in\Omega} X_\alpha$, $X_\omega = Y_\omega$ if $\alpha = \omega$, and $X_\alpha = \{y^0_\alpha\}$ if $\alpha \neq \omega$. We define a topology by declaring a subset $V \subset \bigcup_{\omega\in\Omega} L_\omega$ to be open if and only if the intersection $U \cap L_\omega$ is open in L_ω for all $\omega \in \Omega$ and denote it by ML_ω.

Lemma 1. *The infinite wedge $\vee Y_\omega$ and the space ML_ω are homeomorphic.*

Proof. The space $ML_\omega = \bigcup_{\omega\in\Omega} L_\omega$ has only one point $y^0 = \{y^0_\omega\}_{\omega\in\Omega}$ such that $y^0 \in L_\omega$ for all $\omega \in \Omega$. For each $\omega \in \Omega$ the space L_ω is homeomorphic to Y_ω. The maps $i : \vee Y_\omega \to ML_\omega$ and $p : ML_\omega \to \vee Y_\omega$, defined by

$$i(u) = \begin{cases} i_\omega(u) & \text{if } u \in Y_\omega, \\ y^0 & \text{if } u = u^0, \end{cases} \qquad p(y) = \begin{cases} p_\omega(y) & \text{if } y \in L_\omega, \\ u^0 & \text{if } y = y^0, \end{cases}$$

are continuous maps and $pi = 1$ and $ip = 1$. □

Lemma 2. *The space ML_ω has the subspace topology inherited from LY_ω.*

Proof. Let $V \subset LY_\omega$ be an open subset. Then for each $\alpha \in \Omega$, the intersection $V \cap L_\alpha$ is an open subset of L_α and $V \cap ML_\omega$ is an open subset in ML_ω, $(V \cap ML_\omega) \cap L_\alpha = V \cap L_\alpha$. We show that for any open subset $U \subset ML_\omega$, there is an open subset $V \subset LY_\omega$ such that $V \cap ML_\omega = U$. The system $\mathcal{U} = \{U\}$ of all open sets $U \subset ML_\omega$ is the union of two subsystems $\mathcal{U}'$ and $\mathcal{U}''$ such that $\mathcal{U}' \cap \mathcal{U}'' = \varnothing$, where $\mathcal{U}' = \{U \in \mathcal{U} : y^0 \notin U\}$ and $\mathcal{U}'' = \{U \in \mathcal{U} : y^0 \in U\}$.

Let $U \in \mathcal{U}'$ and notice that $U = \bigcup_{\omega\in\Omega} (U \cap L_\omega)$. Let $U^*_\omega = U \cap L_\omega$ and $\rho_\omega U^*_\omega = U_\omega \subset Y_\omega$. For each $\alpha \in \Omega$, define an open set $W_\alpha \subset \prod Y_\omega$ such that $y = \{y_\omega\} \in W_\alpha$ if and only if $y_\alpha \in U_\alpha$. The union $\bigcup_{\omega\in\Omega} W_\omega$ is an open subset of $\prod Y_\omega$ under the product topology.

Since the topology of LY_ω is finer than the topology induced as a subset of the product $\prod Y_\omega$, the set $\bigcup_{\omega\in\Omega} W_\omega \cap LY_\omega = V$ is open in LY_ω. For each $\alpha \neq \omega$, the intersection $W_\alpha \cap L_\omega = \varnothing$ since any element $y \in W_\alpha$ does not have α-coordinate $y_\alpha = y^0_\alpha$, but all elements $y \in L_\omega$ have α-coordinate $y_\alpha = y^0_\alpha$. We have the equality $W_\omega \cap L_\omega = U^*_\omega$. Hence,

$V \cap L_\omega = \Big(\bigcup_{\alpha\in\Omega} W_\alpha\Big) \cap L_\omega = \bigcup_{\alpha\in\Omega}(W_\alpha \cap L_\omega) = W_\omega \cap L_\omega = U^*_\omega$, and therefore,

$$V \cap ML_\omega = V \cap \Big(\bigcup_{\omega\in\Omega} L_\omega\Big) = \bigcup_{\omega\in\Omega}(V \cap L_\omega) = \bigcup U^*_\omega = U.$$

Let $U \in \mathcal{U}''$ and $U = \bigcup_{\omega\in\Omega}(U \cap L_\omega) = \bigcup U^*_\omega$. For each $\alpha \in \Omega$, we define the open set $W_\alpha \subset \prod Y_\omega$ as in the previous case. Considering the intersection $\bigcap_{\omega\in\Omega} W_\omega = W$, we show that $W \cap LY_\omega = V$ is an open subset in LY_ω. It is clear that $V = W \cap LY_\omega = \Big(\bigcap_{\omega\in\Omega} W_\omega\Big) \cap LY_\omega = \bigcap_{\omega\in\Omega}(W_\omega \cap LY_\omega) = \bigcap_{\omega\in\Omega} V_\omega$ is an intersection of open sets V_ω in LY_ω.

We show that $V \cap L^*_N$ is an open set in L^*_N for all L^*_N (where $L^*_N = \{y^*\}$ is the subspace of $\prod Y_\omega$ such that, if $y^* = \{y_\omega\} \in L^*_N$, then $y_\omega = y^0_\omega$ for all $\omega \notin N$, $N \in \Omega f$, Ωf is the system of all finite subsets N of Ω). There is $V \cap L^*_N = [\cap(W_\omega \cap LY_\omega)] \cap L^*_N = \bigcap_{\omega\in\Omega}(W_\omega \cap L^*_N)$. We have $W_\omega \cap L^*_N = L^*_N$ if $\omega \notin N$, and if $\omega \in N$, we have $W_\omega \cap L^*_N = \{\{y\} \in L^*_N : y_\omega \in U_\omega\}$. Since $N \in \Omega f$, $V \cap L^*_N$ will be a finite intersection of open sets in L^*_N, and therefore open. Hence, V is an open subset in LY_ω.

Now we show that $V \cap ML_\omega = U$. Since $V \cap L_\omega = \bigcap_{\alpha\in\Omega}(W_\alpha \cap L_\omega) = U^*_\omega$, $\omega \in \Omega$, there is

$$V \cap ML_\omega = V \cap \Big(\bigcup_{\omega\in\Omega} L_\omega\Big) = \bigcup_{\omega\in\Omega}(V \cap L_\omega) = \bigcup_{\omega\in\Omega} U^*_\omega = U. \qquad \square$$

Theorem. *If Y_ω is a T_1 space for each $\omega \in \Omega$, then, for $n \geq 2$, there is an isomorphism*

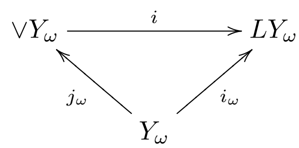
$$\pi_n(\vee Y_\omega, u^0) \approx \sum \pi_n(Y_\omega, y^0_\omega) \oplus \pi_{n+1}(LY_\omega, \vee Y_\omega, y^0).$$

Proof. Using lemmas 1 and 2, we have an exact homotopy sequence

$$\cdots \xrightarrow{\partial} \pi_{n+1}(\vee Y_\omega, u^0) \xrightarrow{i_*} \pi_{n+1}(LY_\omega, y^0) \longrightarrow \pi_{n+1}(LY_\omega, \vee Y_\omega, y^0)$$
$$\xrightarrow{\partial} \pi_n(\vee Y_\omega, u^0) \xrightarrow{i_*} \pi_n(LY_\omega, y^0) \longrightarrow \cdots . \tag{2}$$

Define a homomorphism $j_* : \sum \pi_n(Y_\omega, y^0_\omega) \to \pi_n(\vee Y_\omega, u^0)$ by $j_*(h) = \sum_{\omega\in\Omega} j^*_\omega(h_\omega)$, where $j^*_\omega : \pi_n(Y_\omega, y^0_\omega) \to \pi_n(\vee Y_\omega, u^0)$ is induced by inclusion.

We show that $\xi_* i_* j_* = 1$, where $\xi_* : \pi_n(LY_\omega, y^0) \to \sum\limits_{\omega\in\Omega} \pi_n(Y_\omega, y^0_\omega)$. Since we have a commutative diagram



and the isomorphism $\xi_* : \pi_n(LY_\omega, y^0) \approx \sum\limits_{\omega\in\Omega} \pi_n(Y_\omega, y^0_\omega)$ [3], there is $\xi_* i_* j_* = \sum\limits_{\omega\in\Omega} \xi_* i_* j_{\omega,*} = \sum\limits_{\omega\in\Omega} \xi_* i_{\omega,*} = 1$. Hence, j_* is a monomorphism and i_* is an epimorphism.

As is known [4], π_n is a covariant functor from the category of pairs of pointed spaces to the category of abelian groups if $n \geq 3$, the category of groups if $n = 2$, and the category of pointed sets if $n = 1$. In particular, π_n is a covariant functor from the category Top_* to the category of abelian groups if $n \geq 2$, the category of groups if $n = 1$, and the category of pointed sets if $n = 0$. Hence, $\pi_n(\vee Y_\omega, u^0) \approx \operatorname{Im} j_* + \operatorname{Ker} i_*$, $n \geq 2$.

Since j_* is a monomorphism, there is $\operatorname{Im} j_* = \sum \pi_n(Y_\omega, y^0_\omega)$. Since i_* is an epimorphism, using the exact sequence (2), there is ∂ – a monomorphism and

$$\operatorname{Ker} i_* = \operatorname{Im} \partial = \pi_{n+1}(LY_\omega, \vee Y_\omega, y^0). \qquad \square$$

Since sequence (2) is exact, $i_* j_* = \sum i_{\omega,*}$ and $\sum i_{\omega,*}$ is an isomorphism for $n \geq 1$ [3, Theorem 2]; the homomorphism i_* is an epimorphism; and there is an exact sequence

$$0 \longrightarrow \pi_2(LY_\omega, \vee Y_\omega, y^0) \longrightarrow \pi_1(\vee Y_\omega, u^0) \longrightarrow \pi_1(LY_\omega, y^0) \longrightarrow 0.$$

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