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ON PERIODIC DATA OF DIFFEOMORPHISMS WITH ONE SADDLE ORBIT

T. MEDVEDEV, E. NOZDRINOVA, AND O. POCHINKA

Abstract. In this paper we find all possible periodic data for orientation preserving gradient-like diffeomorphisms of orientable surfaces with one saddle orbit. We also construct a system of this class for every admissible collection of periodic data.

1. Introduction

In the study of discrete dynamical systems, i.e., the study of orbits of self-maps f defined on a given compact manifold, the periodic behavior plays an important role. In the last forty years there was a growing number of results showing that certain simple hypotheses force qualitative and quantitative properties (like the set of periods) of a system. One of the best-known results is the title of the paper "Period three implies chaos for the interval continuous self-maps" [11]. The effect described there was discovered by O. M. Šarkovs'kiĭ in [14]. The most useful tools for proving the existence of fixed points or, more generally, of periodic points for a continuous self-map f of a compact manifold is the Lefschetz fixed point theorem and its improvements (see, for instance [3] and [4]). The Lefschetz zeta-function simplifies the study of the periodic points of f. This is a generating function for all the Lefschetz numbers of all iterates of f.

The periodic data of diffeomorphisms with regular dynamics on surfaces were studied by means zeta-function in a series of already classical works by such authors as Paul R. Blanchard, John M. Franks, Rufus Bowen, Steve Batterson, John Smillie, William H. Jaco, Peter B. Shalen, Carolyn C. Narasimhan, and others. A description of periodic data of gradient-like diffeomorphisms of surfaces was given in [1] by means of classification of periodic surface transformations obtained by Jakob Nielsen [12].

In [8], the authors show that the study of periodic data of arbitrary Morse–Smale diffeomorphisms on surfaces is reduced by filtration to the problem of computing

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periodic data of diffeomorphisms with a unique saddle periodic orbit. In the present paper we solve this problem.

2. Definitions, Notation, and Main Results

Let S_g be a closed orientable surface of a genus $g \geq 0$ with a metric d and let $f : S_g \to S_g$ be an orientation preserving diffeomorphism with the finite hyperbolic non-wandering set Ω_f.

A point $x \in S_g$ is said to be *wandering* for f if there is an open neighborhood U_x of x such that $f^n(U_x) \cap U_x = \emptyset$ for all $n \in \mathbb{N}$. Otherwise the point x is called *non-wandering*. The set of non-wandering points of f is called the *non-wandering set* denoted by Ω_f.

If Ω_f is finite, then every point $p \in \Omega_f$ is periodic, its period being $m_p \in \mathbb{N}$. A point $p \in \Omega_f$ is *hyperbolic* if the absolute values of all the eigenvalues of the Jacobian matrix $\left(\frac{\partial f^{m_p}}{\partial x}\right)|_p$ are not 1. If the absolute values of all the eigenvalues are less (greater) than 1, then p is called a *sink point* (a *source point*). Sink and source points are called *nodes*. If a hyperbolic periodic point is not a node it a *saddle point*.

The hyperbolic structure of a periodic point p implies the existence of the *stable* W^s_p and the *unstable* W^u_p manifolds defined as follows:

$$W^s_p = \{x \in S_g : \lim_{k\to+\infty} d(f^{k\cdot per(p)}(x), p) = 0\},$$

$$W^u_p = \{x \in S_g : \lim_{k\to+\infty} d(f^{-k\cdot per(p)}(x), p) = 0\}.$$

Stable and unstable manifolds are called *invariant manifolds*. A connected component of the set $W^u_p \setminus p$ ($W^s_p \setminus p$) is called an *unstable (stable) separatrix*.

The *periodic data* of the periodic orbit $\mathcal{O}_p$ of a periodic point p is the collection of numbers (m_p, q_p, ν_p) where m_p is the period of p, $q_p = \dim\ W^u_p$, and ν_p is the orientation type of p: $p = +1$ ($p = -1$) if $f^{m_p}|_{W^u_p}$ preserves (changes) the orientation. For orientation preserving diffeomorphisms, the orientation type of all nodes is $+1$, while that of saddle points may equal either $+1$ or -1.

Denote by G the set of diffeomorphisms $f : S_g \to S_g$ having a unique saddle periodic orbit $\mathcal{O}_\sigma$. Let G_1 and G_2 ($G = G_1 \cup G_2$) be the sets of diffeomorphisms whose orientation type of the saddle orbit is -1 and $+1$, respectively.

Theorem 2.1. (1) *The non-wandering set of every diffeomorphism $f \in G_1$ consists of an unique saddle orbit, one sink orbit, and one source orbit.*

(2) *The non-wandering set of every diffeomorphism $f \in G_2$ consists of an unique saddle orbit and three node orbits (one sink and two source orbits or ne source and two sink orbits).*

Let $\mathcal{O}_\omega$ and $\mathcal{O}_\alpha$ denote the sink and the source orbits of a diffeomorphism $f \in G_1$. If a diffeomorphism $f \in G_2$, we assume that it has a unique sink orbit $\mathcal{O}_\omega$ and two source orbits $\mathcal{O}_{\alpha_1}$ and $\mathcal{O}_{\alpha_2}$ (otherwise, one considers the inverse diffeomorphism f^{-1}).

Let $f \in G$. It is well known that the Euler characteristic for an orientable surface is given by the formula: $\chi(S_g) = 2 - 2g$ where g is the number of handles (see, for example, [7] and [2]). On the other hand, by [15], the manifold S_g is a two-dimensional cell complex

$$S_g = \bigcup_{p\in\Omega_f} W^u_p,$$

where the number c_2 of the two-dimensional cells equals the number of the source points; the number c_1 of the one-dimensional cells equals the number of the saddle points; and the number c_0 of the zero-dimensional cells equals the number of the sink points. Thus,

$$(*) \qquad c_2 - c_1 + c_0 = 2 - 2g.$$

Then, for $f \in G_1$, we have

$$(*_1) \qquad m_\alpha - m_\sigma + m_\omega = 2 - 2g,$$

and for $f \in G_2$,

$$(*_2) \qquad m_{\alpha_1} + m_{\alpha_2} - m_\sigma + m_\omega = 2 - 2g.$$

Let (a, b) denote the greatest common divisor of the natural numbers a and b, and we assume $(0, b) = b$.

Theorem 2.2. (1) *Every diffeomorphism $f \in G_1$ has one of the following collections of periodic data:*

$$(**_1) \qquad \begin{array}{llll} \bullet\ m_\omega = 1, & m_\sigma = 2g, & m_\alpha = 1, & g > 0, \\ \bullet\ m_\omega = 1, & m_\sigma = 2g+1, & m_\alpha = 2, & g \geq 0, \\ \bullet\ m_\omega = 2, & m_\sigma = 2g+1, & m_\alpha = 1, & g \geq 0. \end{array}$$

(2) *Every diffeomorphism $f \in G_2$ has the following periodic data,*

$$(**_2) \qquad \begin{aligned} &m_\omega = m, \quad m_\sigma = km, \\ &m_{\alpha_1} = (k, j+1)\left(\tfrac{k}{(k,j+1)}, m\right), \\ &m_{\alpha_2} = (k, j)\left(\tfrac{k}{(k,j)}, m\right), \end{aligned}$$

for some natural numbers $m \in \mathbb{N}$, $k \in \mathbb{N}$, and $j \in \{0, \dots, k-1\}$.

Corollary 2.3. *Every orientable surface of genus g admits a diffeomorphism from the class G, for example, with periodic data $m_\omega = 2$, $m_\sigma = 2g+1$, and $m_\alpha = 1$ in the class G_1 and with periodic data $m_\sigma = 2g+1$ and $m_\omega = m_{\alpha_1} = m_{\alpha_2} = 1$ in the class G_2.*

Corollary 2.4. *Every diffeomorphism of G on the 2-sphere has one of the following periodic data:*

(1) $m_\omega = 1, m_\sigma = 1, m_\alpha = 2$;
(2) $m_\omega = 2, m_\sigma = 1, m_\alpha = 1$;
(3) $m_\omega = 1, m_\sigma = m_{\alpha_1} = k, m_{\alpha_2} = 1, k \in \mathbb{N}$;
(4) $m_\omega = m, m_\sigma = m, m_{\alpha_1} = m_{\alpha_2} = 1, m \in \mathbb{N}$.

Theorem 2.5. (1) *Every collection of the type $(**_1)$ can be realized by a diffeomorphism $f \in G_1$ with the corresponding periodic data on a surface of genus g.*

(2) *Every triplet of natural numbers $k \in \mathbb{N}$, $m \in \mathbb{N}$, and $j \in \{0, \dots, k-1\}$ can be realized by a diffeomorphism $f \in G_2$ with periodic data of the form $(**_2)$ on a surface of genus*

$$g = 1 + \frac{1}{2}\left((k-1)m - (k, j+1)\left(\frac{k}{(k, j+1)}, m\right) - (k, j)\left(\frac{k}{(k, j)}, m\right)\right).$$

3. Illustrations

3.1. Illustrations for Theorem 2.2.

1. $f \in G_1$

- Consider the octagon shown in Figure 1. Identify the pairs of its sides that have center points with the same notation and get the diffeomorphism $f \in G_1$ on the surface of genus two S_2 with the periodic data of the first type from $(**_1)$ for $g = 2$.

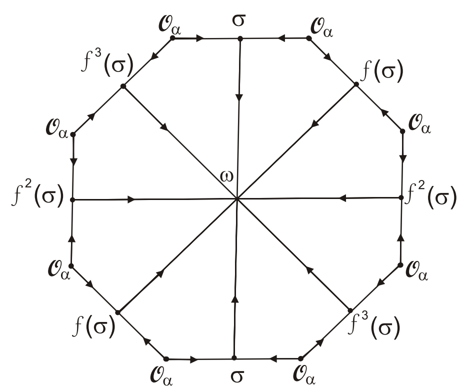

Figure 1. Diffeomorphisms from G_1 with $m_\omega = 1$, $m_\sigma = 4$, $m_\alpha = 1$, and $g = 2$

- Consider the hexagon shown in Figure 2. Identify the pairs of its sides that have center points with the same notation and get the diffeomorphism $f \in G_1$ on the torus S_1 with the periodic data of the second type from $(**_1)$ for $g = 1$.

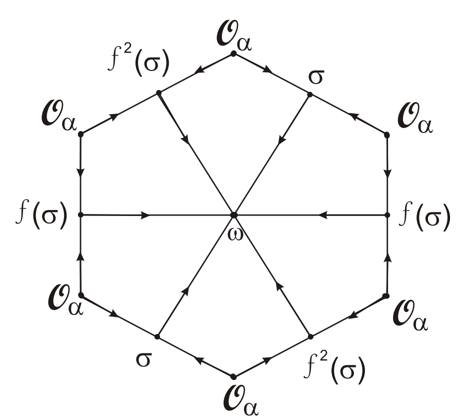

Figure 2. Diffeomorphisms from G_1 with $m_\omega = 1$, $m_\sigma = 3$, $m_\alpha = 2$, and $g = 1$

- Consider the two pentagons shown in Figure 3. Identify the pairs of its sides that have center points with the same notation and get the diffeomorphism

$f \in G_1$ on the surface of genus two S_2 with the periodic data of the third type from $(**_1)$ for $g = 2$.

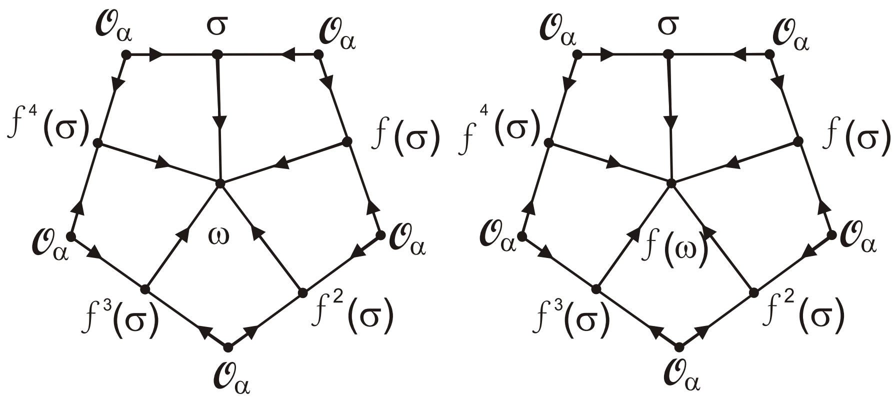

FIGURE 3. Diffeomorphisms from G_1 with $m_\omega = 2$, $m_\sigma = 5$, $m_\alpha = 1$, and $g = 2$

2. $f \in G_2$
   - Consider the hexagon shown in Figure 4. Identify the pairs of its sides that have center points with the same notation and get the diffeomorphism $f \in G_2$ on the sphere with the periodic data $(**_2)$.

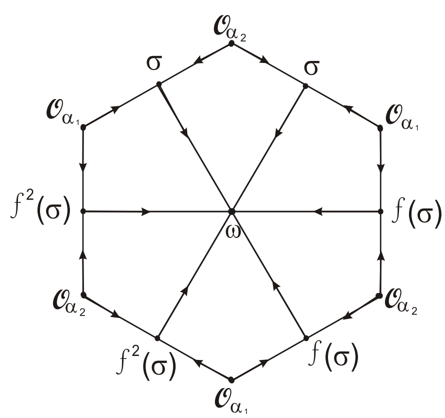

FIGURE 4. Diffeomorphisms from G_2 with $m_\omega = 1$, $m_\sigma = 3$, $m_{\alpha_1} = 1$, $m_{\alpha_2} = 3$, and $g = 0$

3.2. Illustrations for Corollary 2.3.

- Consider the two polygons shown in Figure 3. Identify the pairs of its sides that have center points with the same notation and get the closed surface and the diffeomorphism $f \in G_1$.

- Consider the polygon shown in Figure 5. Identify the pairs of its sides that have center points with the same notation and get the closed surface and the diffeomorphism $f \in G_2$.

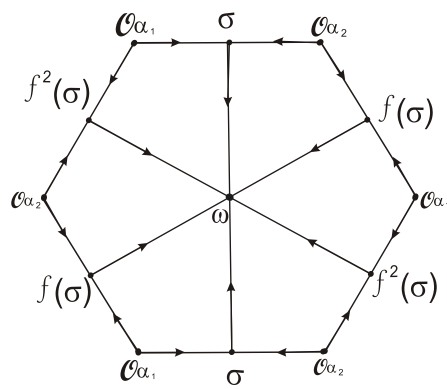

FIGURE 5. Diffeomorphisms from G_2 with $m_\sigma = 3$, $m_\omega = m_{\alpha_1} = m_{\alpha_2} = 1$, and $g = 1$

3.3. Illustrations for Corollary 2.4.

(1) Figure 6(A) shows a diffeomorphism $f \in G_1$ on the sphere with periodic data $m_\omega = 1, m_\sigma = 1, m_\alpha = 2$;
(2) Figure 6(B) shows a diffeomorphism $f \in G_1$ on the sphere with periodic data $m_\omega = 2, m_\sigma = 1, m_\alpha = 1$;

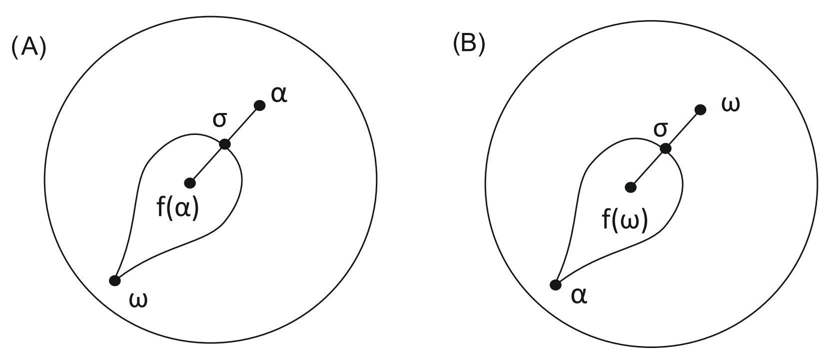

FIGURE 6. Diffeomorphism from G_1

(3) Figure 7(A) shows a diffeomorphism $f \in G_2$ on the sphere with periodic data $m_\omega = 1, m_\sigma = 3, m_{\alpha_1} = 3, m_{\alpha_2} = 1$;
(4) Figure 7(B) shows a diffeomorphism $f \in G_2$ on the sphere with periodic data $m_\omega = 2, m_\sigma = 2, m_{\alpha_1} = 1, m_{\alpha_2} = 1$.

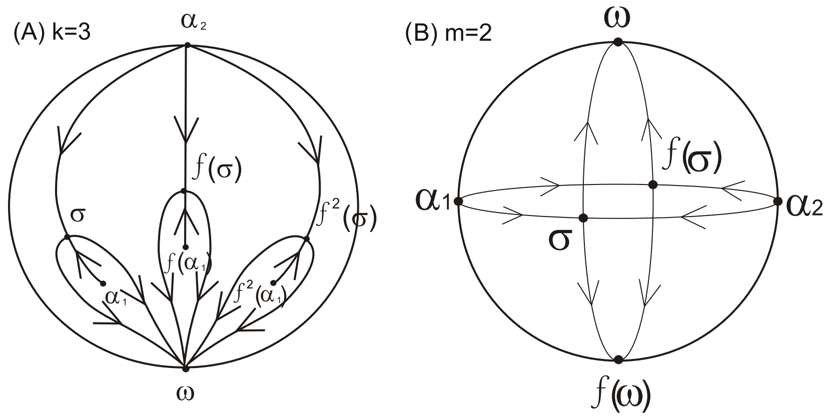

FIGURE 7. Diffeomorphism from G_2 (A) k=3 and (B) m=2

4. The Structure of the Non-Wandering Set

In this section we prove Theorem 2.1. The detailed proofs of the auxiliary statements can be found in [7].

Proof of Theorem 2.1. We are going to show that the non-wandering set of every diffeomorphism $f \in G$ consists of a unique saddle orbit and three node orbits (one sink and two sources or one source and two sinks).

Let m_ℓ denote the *period* of the saddle separatrix ℓ, i.e., the smallest natural number μ such that $f^\mu(\ell) = \ell$, and denote by $\mathcal{O}_\ell$ the orbit of the separatrix ℓ. By [7, Proposition 2.3] the closure of each unstable saddle separatrix contains a unique sink, while the closure of each stable saddle separatrix contains a unique source. Let an unstable separatrix ℓ^u of a saddle point σ contain the sink ω in its closure. Let m be the period of ω. According to [13, Theorem 5.5], f^m at the point ω is locally conjugate with the linear diffeomorphism of the space $\mathbb{R}^2$ defined by

$$L(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right).$$

Denote by $\mathcal{O}_\omega$ the orbit of the point ω and let $V_\omega = W^s_{\mathcal{O}_\omega} \setminus \mathcal{O}_\omega$. Denote by $\hat{V}_\omega = V_\omega/f$ the orbits space of the action of the group $F = \{f^k, k \in \mathbb{Z}\}$ on V_ω, and denote by $p_{_\omega} : V_\omega \to \hat{V}_\omega$ the natural projection.

Due to [7, Proposition 2.5], the space $\hat{V}_\omega$ is diffeomorphic to the 2-torus; the natural projection $p_{_\omega} : V_\omega \to \hat{V}_\omega$ is a cover that induces the epimorphism $\eta_{_\omega}$: $\pi_1(\hat{V}_\omega) \to m_\omega\mathbb{Z}$ (here, $m_\omega\mathbb{Z}$ is the group of integers which are multiples to m_ω) defined by the following rule. Let $[\hat{c}] \in \pi_1(\hat{V}_\omega)$. Any lift c of the loop $\hat{c}$ starting at $x \in V_\omega$ has the end point in $f^n(x)$ where $n \in \mathbb{Z}$ does not depend on the choice of the lift. Therefore, $\eta_{_\omega}([\hat{c}]) = n$.

Define the diffeomorphisms $a_i : \mathbb{R}^2 \to \mathbb{R}^2, i = 1, 2$ by

$$a_1(x, y) = \left(-\frac{x}{2}, -2y\right) \text{ and } a_2(x, y) = \left(\frac{x}{2}, 2y\right).$$

Both $a_i : \mathbb{R}^2 \to \mathbb{R}^2 (i = 1, 2)$ have the unique fixed saddle point at the origin O with the stable manifold $W^s_O = Ox$ and the unstable manifold $W^u_O = Oy$; α_1 reverses the orientation while α_1 preserves it. The diffeomorphism f^{m_σ} in some neighborhood of the point σ is topologically conjugate to the diffeomorphism a_i in a neighborhood of the point O (see, for example, [13, Theorem 5.5]).

Let $\hat{\ell}^u = p_{_\omega}(\ell^u)$ and let $j_{\hat{\ell}^u} : \hat{\ell}^u \to \hat{V}_\omega$ be the inclusion map. Due to [7, Proposition 2.5], the set $\hat{\ell}^u$ is a circle smoothly embedded in $\hat{V}_\omega$ and such that $\eta_{_\omega}(j_{\hat{\ell}^u *}(\pi_1(\hat{\ell}^u))) = m_{\ell^u}\mathbb{Z}$. Notice that $p_{_\omega}(\mathcal{O}_{\ell^u}) = \hat{\ell}^u$.

Figure 8 shows the torus $\hat{V}_\omega$ with the projection $\hat{\ell}^u$ of the separatrix ℓ^u such that $\frac{m_{\ell^u}}{m_\omega} = 3$.

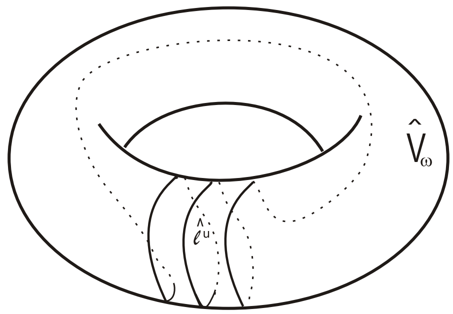

FIGURE 8. The projection of the saddle separatrix to the orbits space of the sink basin homeomorphic to the torus

Let $\mathcal{N} = \{(x,y) \in \mathbb{R}^2 : |xy| \leq 1\}$. Notice that the set $\mathcal{N}$ is invariant with respect to the diffeomorphism a_i. The neighborhood of N_σ of the point σ of a diffeomorphism $g \in G_i$ is called *linearizing* if there exists a homeomorphism $\mu_\sigma : N_\sigma \to \mathcal{N}$ conjugating the diffeomorphism $f^{m_\sigma}|_{N_\sigma}$ with the diffeomorphism $a_i|_{\mathcal{N}}$.

The neighborhood $N_{\mathcal{O}_\sigma} = \bigcup\limits_{j=0}^{m_\sigma - 1} f^j(N_\sigma)$ of the orbit $\mathcal{O}_\sigma = \bigcup\limits_{j=0}^{m_\sigma - 1} f^j(\sigma)$ equipped with the map $\mu_{\mathcal{O}_\sigma}$ which is composed of the homeomorphisms $\mu_\sigma f^{-j} : f^j(N_\sigma) \to \mathcal{N}, j = 0, \dots, m_\sigma - 1$ is called the *linearizing neighborhood* of the orbit $\mathcal{O}_\sigma$.

Due to [7, Theorem 2.2], the saddle point (orbit) of the diffeomorphism f has a linearizing neighborhood.

Let $\mathcal{N}^u = \mathcal{N} \setminus Ox$, and let $\hat{\mathcal{N}}_i^u = \mathcal{N}^u / a_i, (i = 1, 2)$ denote the orbit space of the action of the group $\{a_i^n, n \in \mathbb{Z}, i = 1, 2\}$ on $\mathcal{N}^u$. Denote by $p_{_{\hat{\mathcal{N}}_i^u}} : \mathcal{N}^u \to \hat{\mathcal{N}}_i^u$ the natural projection and by $\eta_{_{\hat{\mathcal{N}}_i^u}}$ the map composed of non-trivial homomorphisms from the fundamental groups of connected components of the space $\hat{\mathcal{N}}_i^u$ to the group $\mathbb{Z}$.

The fundamental domain of the action of the group $\{a_1^n, n \in \mathbb{Z}\}$ on $\mathcal{N}_1^u$ is one curvilinear trapezoid with equivalent points lying on the horizontal segments of the boundary (Figure 9(a)). By identifying the horizontal boundaries of the trapezoid by the diffeomorphism a_1, we get the manifold $\hat{\mathcal{N}}_1^u$. Thus, the space $\hat{\mathcal{N}}_1^u$ is homeomorphic to the 2-annulus K and $\eta_{_{\hat{\mathcal{N}}_1^u}}(K) = 2\mathbb{Z}$.

The fundamental domain of the action of the group $\{a_2^n, n \in \mathbb{Z}\}$ on $\mathcal{N}_2^u$ consists of two disjoint curvilinear trapezoids, each of which has equivalent points lying on the horizontal segments of the boundary. In Figure 9(b) these trapezoids are shaded. By identifying their horizontal boundaries by the diffeomorphism a_2, we obtain the manifold $\hat{\mathcal{N}}_2^u$. Thus, the space $\hat{\mathcal{N}}_2^u$ is homeomorphic to a pair of the 2-annuli K_1 and K_2, and $\eta_{\hat{\mathcal{N}}_2^u}(K_1) = \eta_{\hat{\mathcal{N}}_2^u}(K_2) = \mathbb{Z}$.

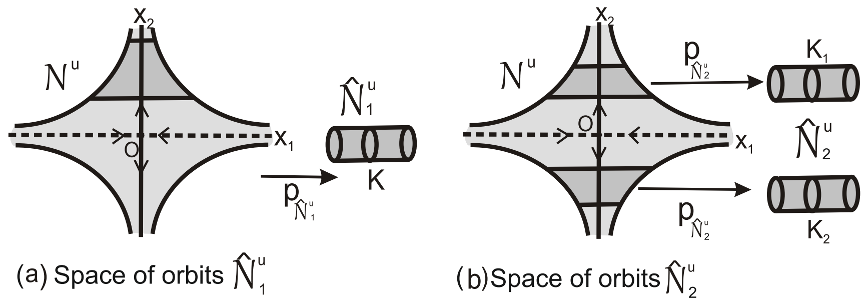

FIGURE 9. The orbit space $\hat{\mathcal{N}}_1^u$

Let $N_\sigma^u = N_\sigma \backslash W_\sigma^s$ and $N_\sigma^s = N_\sigma \backslash W_\sigma^u$. Denote by N_{ℓ^u} the connected component of the set N_σ^u containing the unstable separatrix ℓ^u. Let $\hat{N}_{\ell^u} = p_{_\omega}(N_{\ell^u})$ and let $j_{N_{\hat{\ell}^u}} : N_{\hat{\ell}^u} \to \hat{V}_\omega$ be the inclusion map. The set $\hat{N}_{\ell^u}$ is the smoothly embedded annulus in $\hat{V}_\omega$ such that $\eta_{_\omega}(j_{\hat{N}_{\ell^u}*}(\pi_1(\hat{N}_{\ell^u}])) = m_{\ell^u}\mathbb{Z}$.

Denote by A the union of the sink points of the diffeomorphism $f \in G$. Let $V_A = W_A^s \backslash A$ and $\hat{V}_A = V_A/f$, and let $p_A : V_A \to \hat{V}_A$ be the natural projection. Similar to the arguments above, the orbit space in the sink basin is homeomorphic to the torus and that implies that each connected component of the set $\hat{V}_A$ is homeomorphic to the 2-torus and the number of the connected components coincides with the number of the sink orbits.

Let $N_{\mathcal{O}_\sigma}^u = N_{\mathcal{O}_\sigma} \setminus W_{\mathcal{O}_\sigma}^s$, $N_{\mathcal{O}_\sigma}^s = N_{\mathcal{O}_\sigma} \setminus W_{\mathcal{O}_\sigma}^u$, and $\hat{N}_{\mathcal{O}_\sigma}^u = N_{\mathcal{O}_\sigma}^u/f$. Due to [7, Theorem 2.4], if $f \in G_1$, then the set $\hat{N}_{\mathcal{O}_\sigma}^u$ is the annulus smoothly embedded into $\hat{V}_A$, while if $f \in G_2$, then $\hat{N}_{\mathcal{O}_\sigma}^u$ is two smoothly embedded into $\hat{V}_A$. According to [7, Corollary 2.1], the set $\hat{V}_A$ is not empty, and due to [7, Corollary 2.2], each torus in $\hat{V}_A$ contains at least one annulus of $\hat{N}_{\mathcal{O}_\sigma}^u$.

Similar statements can be formulated for the source point α and for the stable separatrix ℓ^s of the saddle point σ such that $\ell^s \subset W_\alpha^u$.

Denote by R the union of the source points of the diffeomorphism $f \in G$. Let $V_R = W_R^u \setminus R$ and $\hat{V}_R = V_R/f$, and let $p_{_R} : V_R \to \hat{V}_R$ be the natural projection. Similar to the previous arguments, the orbit space in the source basin is homeomorphic to the torus and that implies that each connected component of the set $\hat{V}_R$ is homeomorphic to the 2-torus and the number of connected components coincides with the number of the source orbits. On the other hand, it follows from $V_R = V_A \setminus N_{\mathcal{O}_\sigma}^u \cup N_{\mathcal{O}_\sigma}^s$ (see, for instance, [7, Theorem 2.1]) that

$$\hat{V}_A = \hat{V}_R \setminus \hat{N}_{\mathcal{O}_\sigma}^u \cup \hat{N}_{\mathcal{O}_\sigma}^s.$$

Thus, to get the space $\hat{V}_R$, we have to delete $\hat{N}_{\mathcal{O}_\sigma}^u$ from the torus $\hat{V}_A$ and glue the set $\hat{N}_{\mathcal{O}_\sigma}^s$ to the boundary of the resulting set.

If $\nu_\sigma = -1$, then each of the sets $\hat{N}_{\mathcal{O}_\sigma}^u$ and $\hat{N}_{\mathcal{O}_\sigma}^s$ consists of one annulus and $\hat{N}_{\mathcal{O}_\sigma}^u$ is homotopically non-trivially embedded into the torus $\hat{V}_\omega$. Then $\hat{V}_\omega \setminus \hat{N}_{\mathcal{O}_\sigma}^u$ is an annulus. Therefore, having glued its boundary to that of the annulus $\hat{N}^s_{\mathcal{O}_\sigma}$, we again get one 2-torus. But this means that $f \in G_1$ has exactly one source orbit.

If $\nu_\sigma = +1$, then each of the sets $\hat{N}^u_{\mathcal{O}_\sigma}$ and $\hat{N}^s_{\mathcal{O}_\sigma}$ consists of two annuli. Moreover, the annuli $\hat{N}^u_{\mathcal{O}_\sigma}$ are homotopically non-trivially embedded in the torus $\hat{V}_A$. If we assume that $\hat{V}_A$ consists of a unique connected component, then $\hat{V}_A \setminus \hat{N}^u_{\mathcal{O}_\sigma}$ consists of two annuli, and gluing $\hat{N}^s_{\mathcal{O}_\sigma}$ to their boundaries produces two 2-tori. (See Figure 10 where the transition from the sink basins to the sources basins is illustrated for a diffeomorphism of the 2-sphere. For convenience, in these basins, the fundamental regions are selected after identification of their boundary circles.) This means that there are exactly two source orbits for the diffeomorphism $f \in G_2$; that is, $R = \mathcal{O}_{\alpha_1} \cup \mathcal{O}_{\alpha_2}$ for some periodic sources α_1 and α_2.

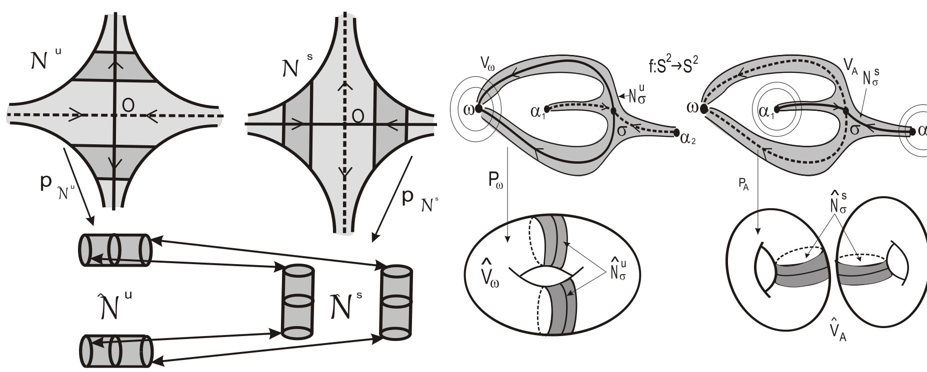

FIGURE 10. Regluing along annuli

If we assume that $\hat{V}_A$ consists of two connected components, then the similar cutting and gluing operation implies the existence of the unique source orbit in this case. $\square$

5. Periodic Data of a Diffeomorphism $f \in G_1$

In this section we prove the first statement of Theorem 2.2; that is, we show that a diffeomorphism $f \in G_1$ has one of the following collections of the periodic data $(**_1)$:

- $m_\omega = 1, \quad m_\sigma = 2g, \quad m_\alpha = 1, \quad g > 0,$
- $m_\omega = 1, \quad m_\sigma = 2g+1, \quad m_\alpha = 2, \quad g \geq 0,$
- $m_\omega = 2, \quad m_\sigma = 2g+1, \quad m_\alpha = 1, \quad g \geq 0.$

Proof. We treat the cases $m_\omega = 1$ and $m_\omega \neq 1$ separately.

Case 1. $m_\omega = 1$. All the unstable separatrices of saddles (their number equals $2m_\sigma$) lie in the basin W^s_ω. Let $V = \mathbb{S}^1 \times \mathbb{R}^+$, and let $L_\beta = \bigcup\limits_{j=0}^{\beta-1} e^{i(\frac{\pi}{2} - \frac{2\pi j}{\beta})} \times \mathbb{R}^+$ for $\beta \in \mathbb{N}$. If $\beta = 1$, then let $\mu = 0$; otherwise, let $\mu \in \{1, \dots, \beta-1\}$ be such that $(\beta, \mu) = 1$. Define the diffeomorphism $\phi_{\beta,\mu} : V \to V$ by

$$\phi_{\beta,\mu}(z, r) = \left(e^{-\frac{2\pi\mu}{\beta} i} z, \frac{r}{2}\right).$$

Let $\hat{V}_{\beta,\mu} = V/\phi_{\beta,\mu}$ and let $p_{\beta,\mu} : V \to \hat{V}_{\beta,\mu}$ be the natural projection. By construction, the set $\hat{V}_{\beta,\mu}$ is the 2-torus. Therefore, there exists a natural μ such that $(2m_\sigma, \mu) = 1$, and there exists a diffeomorphism $\hat{h}_\omega : \hat{V}_\omega \to \hat{V}_{m_\sigma,\mu}$ that sends the circle $\hat{\ell}^u_\sigma$ to the circle $p_{2m_\sigma,\mu}(L_{2m_\sigma})$. Then there is a lifting $h_\omega : V_\omega \to V$ of the diffeomorphism $\hat{h}_\omega$ that sends the separatrices $W^u_{\mathcal{O}_\sigma} \setminus \mathcal{O}_\sigma$ to the collection of lines L_{2m_σ} and conjugates the diffeomorphism $f|_{V_\omega}$ with the diffeomorphism $\phi_{2m_\sigma,\mu}$ (see, for example, [7] and [10]). From now on we identify the conjugated objects.

Since the period of the saddle point equals m_σ and since the map $\phi_{2m_\sigma,\mu}$ is the rotation through the angle $\frac{\pi\mu}{m_\sigma}$, the separatrices of the same saddle of f are diametrically opposite. The stable manifolds of the saddles, as well as the sources, lie in the closure of W^s_ω; the surface S_g is obtained from $2m_\sigma$-gon Π by identification of the diametrically opposite sides (see Figure 11 and Figure 12), and the diffeomorphism $f : S_g \to S_g$ is induced by the diffeomorphism $\phi_{2m_\sigma,\mu}$. According to Lemma 2.1, the diffeomorphism f has a unique source orbit $\mathcal{O}_\alpha$. In order to get its period, we construct the 3-colored graph for f in the following way (for details, see [5] and [6]).

We say the stable (unstable) separatrices are the s (u)-sides (shown in red (blue) in the electronic version), and we say the segments connecting the vertices of the polygon with its center are the t-sides (green in the electronic version). Thus, the sides divide the polygon Π into uniform triangles. We number these triangles in counterclockwise order.

Construct the 3-colored graph T_f for the diffeomorphism f in the following way:
(1) the vertices T_f correspond to the triangular domains in a one-to-one way;
(2) two vertices of the graph are incident to an edge of color s, t, or u if the triangular domains corresponding to these vertices have a common s, t, or u side.

Denote by B_f the set of the vertices of T_f, denote by Δ_f the set of the triangles in the division of Π, and denote by $\pi_f : \Delta_f \to B_f$ the one-to-one correspondence between Δ_f and T_f. The diffeomorphism f induces an automorphism $P_f = \pi_f f \pi_f^{-1}$ on the set of vertices and edges of T_f. Moreover,

- the correspondence between the set of the sinks of f and the set of the tu-cycles of T_f is one-to-one;
- the correspondence between the set of the saddles of f and the set of the su-cycles of T_f is one-to-one;
- the correspondence between the set of the sources of f and the set of the ts-cycles of T_f is one-to-one.

Thus, in order to get the period of the point α, one has to calculate the length of an arbitrary st-cycle (all st-cycles are of the same length). Denote this length by 2λ since it is even, then

$$m_\alpha = \frac{2m_\sigma}{\lambda}.$$

The diametrically opposite sides of Π are identified; therefore, if one considers the st-cycle starting from s-edge $(0_1, (m_\sigma)_2)$, then one comes to

$$(m_\sigma + 1)\lambda = 2\gamma m_\sigma$$

for some natural γ such that $(\lambda, \gamma) = 1$. If m_σ is even, then $m_\sigma + 1$ and $2m_\sigma$ are mutually prime; therefore, $\lambda = 2m_\sigma$ and $m_\alpha = 1$. If m_σ is odd, then $\frac{m_\sigma+1}{2}$ and m_σ are mutually prime; therefore, $\lambda = m_\sigma$ and $m_\alpha = 2$. Thus, we get the following periodic data:

- $m_\omega = 1, \quad m_\sigma = 2q, \qquad m_\alpha = 1, \quad q \in \mathbb{N};$
- $m_\omega = 1, \quad m_\sigma = 2q+1, \quad m_\alpha = 2, \quad q \geq 0.$

It follows from $(**_1)$ that $q = g$, and we get periodic data of the first two types.

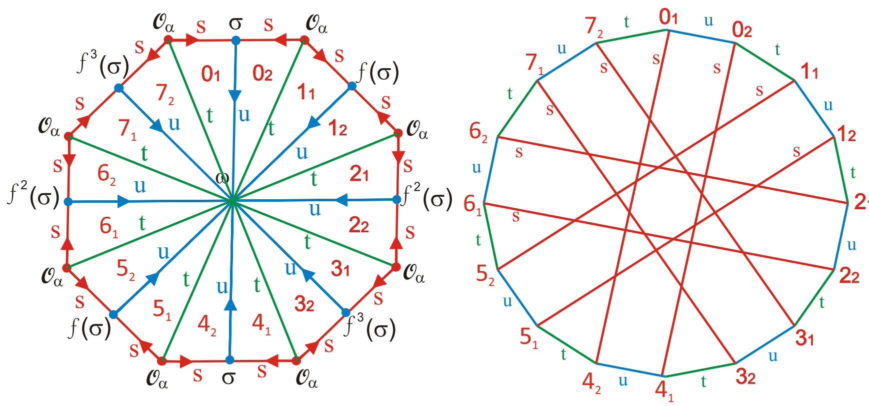

FIGURE 11. The polygonal Π and its 3-colored graph

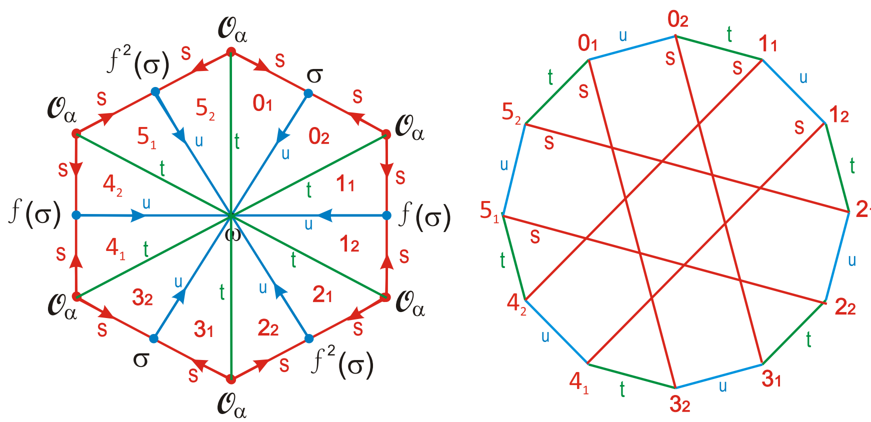

FIGURE 12. The hexagonal Π and its 3-colored graph

Case 2. $m_\omega \neq 1$. In this case $b > 1$ separatrices lie in each basin of the sink.

Let $V_m = V \times \mathbb{Z}_m$ for $m > 1$ and let $L_{\beta,m} = L_\beta \times \mathbb{Z}_m$ for $\beta \in \mathbb{N}$. If $\beta = 1$, then let $\mu = 0$; otherwise let $\mu \in \{1, \ldots, \beta - 1\}$ be such that $(\beta, m\mu) = 1$. Define the diffeomorphism $\phi_{m,\beta,\mu} : V_m \to V_m$ by

$$\phi_{m,\beta,\mu}(z, r, w) = (\phi_{\beta,\mu}(z, r), w + 1 \sim mod\ m).$$

Let $\hat{V}_{m,\beta,\mu} = V_m/\phi_{m,\beta,\mu}$ and let $p_{m,\beta,\mu} : V_m \to \hat{V}_{m,\beta,\mu}$ be the natural projection. The set $\hat{V}^l_{\beta,\mu}$ is the 2-torus by construction. There exists a natural μ such that $(b, m_\omega\mu) = 1$ and there exists a diffeomorphism $\hat{h}_\omega : \hat{V}_\omega \to \hat{V}_{m_\omega,b,\mu}$ sending the circle $\hat{\ell}^u_\sigma$ to the circle $p_{m_\omega,b,\mu}(L^{m_\omega}_b)$. Then there exists a lift $h_\omega : V_\omega \to V$ of $\hat{h}_\omega$ which sends the separatrices $W^u_{\mathcal{O}_\sigma} \setminus \mathcal{O}_\sigma$ to a collection of lines L_{b,m_ω} and which conjugates the diffeomorphism $f|_{V_\omega}$ and the diffeomorphism $\phi_{m_\omega,b,\mu}$ (see, for example, [7] and [10]). From now on we identify the conjugated objects.

Since the period of the saddle point equals $\frac{bm_\omega}{2}$ and since the map $\phi_{m_\omega,b,\mu}$ is the rotation through the angle $\frac{2\pi\mu}{b}$, the separatrices of the same saddle point of f lie in the basins with numbers w and $w + \frac{bm_\omega}{2}$ for $w \in \{0, \dots, m_\omega - 1\}$. Since the surface S_g is connected, we have $\left(\frac{bm_\omega}{2}, m_\omega\right) = 1$. Therefore, b is odd and $m_\omega = 2$. Then the surface S_g is obtained from two b-gons Π_0 and Π_1 by identification of the corresponding sides (see Figure 13) and the diffeomorphism $f : S_g \to S_g$ is induced by the diffeomorphism $\phi_{2,b,\mu}$.

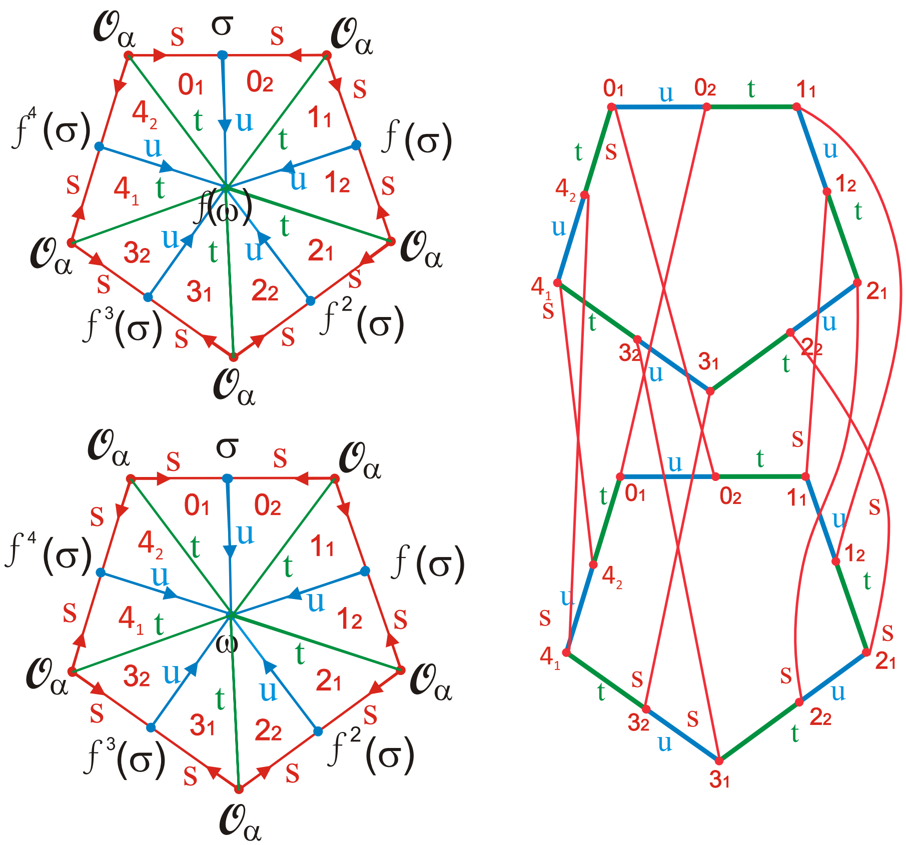

FIGURE 13. Two pentagons Π_1 and Π_2 and their 3-colored graphs

Thus, in order to get the period of the point α, one has to calculate the length of an arbitrary st-cycle (all st-cycles are of the same length). Denote this length by λ, then

$$m_\alpha = \frac{4b}{\lambda}.$$

The corresponding sides of Π_0 and Π_1 are identified; therefore, if one considers the st-cycle starting from s-edge $(0_1, 0_2)$, then one comes to $\lambda = 4\gamma b$ for some natural γ such that $(\lambda, \gamma) = 1$. Therefore, $\lambda = 4b$ and $m_\alpha = 1$.

The Euler characteristic $\chi(S_g) = 2 - (b) + 1 = 2 - 2g$; thus, $b = 2g + 1$, and we have periodic data of the third type

- $m_\omega = 2, \qquad m_\sigma = 2g + 1, \qquad m_\alpha = 1, \qquad g \geq 0.$ □

6. Periodic Data of a Diffeomorphism $f \in G_2$

In this section we prove Theorem 2.2.

Proof. Now we are going to show that every diffeomorphism $f \in G_2$ has the following periodic data $(**_2)$:

$$\begin{aligned} &m_\omega = m, \quad m_\sigma = km, \\ &m_{\alpha_1} = (k, j+1)\left(\tfrac{k}{(k,j+1)}, m\right), \\ &m_{\alpha_2} = (k, j)\left(\tfrac{k}{(k,j)}, m\right), \end{aligned}$$

where $m \in \mathbb{N}, k \in \mathbb{N}, j \in \{0, \ldots, k-1\}$ are natural numbers.

Consider the following abstract model of dynamics in the basin of a periodic sink of period m. Let $m \geq 1$ be an integer and $V_m = \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{Z}_m$. Thus, V_m is a model for the basin of a periodic sink of period m. Let $k \in \mathbb{N}$, $\tau \in \{0, \ldots, k-1\}$, and

$$\gamma_1^\tau = \bigcup_{\tau=0}^{k-1} e^{i\pi\left(\frac{1}{2} - \frac{2\tau}{k}\right)} \times \mathbb{R}^+, \ \ \gamma_2^\tau = \bigcup_{\tau=0}^{k-1} e^{i\pi\left(\frac{1}{2} - \frac{2\tau+1}{k}\right)} \times \mathbb{R}^+,$$

$$\gamma_1 = \bigcup_{\tau=0}^{k-1} \gamma_1^\tau \times \mathbb{Z}_m, \ \ \gamma_2 = \bigcup_{\tau=0}^{k-1} \gamma_2^\tau \times \mathbb{Z}_m.$$

Here, $\gamma_1 \cup \gamma_2$ models the saddle unstable separatrices.

Let $n \geq 0$ be an integer such that if $k = 1$, then $n = 0$; otherwise, let $n \in \{1, \ldots, k-1\}$ be such that nm and k are co-prime. Here, mk models the period of periodic unstable separatrices in V_m, and $\frac{n}{k}$ represents their "rotation number," i.e., how the diffeomorphism permutes these separatrices. As a local model for the diffeomorphism on the basin, consider the contraction $\phi_{m,k,n} : V_m \to V_m$ given by the formula

$$\phi_{m,k,n}(z, r, w) = (e^{-\frac{2\pi n}{mk} i} z, \frac{r}{2^m}, w + 1 \sim \mathit{mod}\ m).$$

Notice that $\hat{V}_{m,k,n} = V_m / \phi_{m,k,n}$ is a torus. Denote by $p_{m,k,n} : V_m \to \hat{V}_{m,k,n}$ the natural projection. The set $\hat{\gamma}_i = p_{m,k,n}(\gamma_i), i = 1, 2$ is a knot on $\hat{V}_{m,k,n}$.

Consider the diffeomorphism $f \in G_2$. For the sink orbit $\mathcal{O}_\omega$, let $V_\omega = W^s_{\mathcal{O}_\omega} \setminus \mathcal{O}_\omega$. Denote by $\hat{V}_\omega = V_\omega / f$ the orbit space of the action of the group $F = \{f^i, i \in \mathbb{Z}\}$ on V_ω and denote by $p_{_\omega} : V_\omega \to \hat{V}_\omega$ the natural projection. The unstable separatrices ℓ_1^u and ℓ_2^u of the saddle point σ are of period m_σ and they lie in the basin V_ω. Since the group F acts transitively on the connected components of V_ω (the number of such components is m) and on the orbit of each unstable separatrix (the number of the connected components of this orbit is m_σ) in each connected component of the set V_ω, there is the same number of separatrices from this orbit. Hence, the period m_σ is a multiple of the period m.

Therefore, each connected component of V_ω contains $k = \frac{m_\sigma}{m}$ separatrices from the orbit of the separatrix ℓ_i^u. Let $\hat{\ell}_1^u = p_{_\omega}(\ell_1^u)$ and $\hat{\ell}_2^u = p_{_\omega}(\ell_2^u)$. Then there is a number n and a diffeomorphism $\hat{h}_\omega : \hat{V}_\omega \to \hat{V}_{m,k,n}$ transforming the knots $\hat{\ell}_1^u$ and $\hat{\ell}_2^u$ to the knots $\hat{\gamma}_1$ and $\hat{\gamma}_2$. Thus, there is a lift $h_\omega : V_\omega \to V_m$ of $\hat{h}_\omega$ which sends the separatrices $W^u_{\mathcal{O}_\sigma} \setminus \mathcal{O}_\sigma$ to the frame of the rays $\gamma_1 \cup \gamma_2$ and conjugates the diffeomorphism $f|_{V_\omega}$ with the diffeomorphism $\phi_{m,k,n}$ (see, for example, [7, Statement 10.35]). From now on we identify the conjugated objects.

The diffeomorphism f uniquely defines the parameters $j \in \{0, \dots, k-1\}$ and $\rho \in \{0, \dots, m-1\}$ so that both the ray $\gamma_1^0 \times \{0\}$ and the ray $\gamma_2^j \times \{\rho\}$ belong to the unstable manifold of the same saddle point. Moreover, due to connectivity of the ambient surface S_g,

$$(\rho, m) = 1.$$

Since h_ω conjugates f to $\phi_{m,k,n}$, the parameters j and ρ correspond to the division of the separatrices into pairs: Two separatrices form a pair if they belong to the unstable manifold of the same saddle; that is, the ray $\gamma_1^\tau \times \{w\}$ has the paired ray $\gamma_2^{(\tau+j) \sim mod\ k} \times \{(w+\rho) \sim mod\ m\}$.

Indeed, j and ρ are uniquely determined by f, but the order of the separatrices depends on h_ω. The numbers j' and ρ' for the reverse order satisfy $j + j' = k$ and $\rho + \rho' = m$; therefore,

$$(k, j') = (k, j) \text{ and } (\rho', m) = (\rho, m),$$

which shows that the periods m_{α_1} and m_{α_2} are independent of the order of the separatrices.

By Theorem 2.1, the non-wandering set of the diffeomorphism f contains exactly two source orbits $\mathcal{O}_{\alpha_1}$ and $\mathcal{O}_{\alpha_2}$ such that $cl(\ell_1^u) = \ell_1^u \cup \alpha_1$ and $cl(\ell_2^u) = \ell_2^u \cup \alpha_2$. Thus,

$$W^s_{\mathcal{O}_\omega} = S_g \setminus cl(W^s_{\mathcal{O}_\sigma}).$$

If we remove from our surface S_g the closures of m_σ stable manifolds, then we get m disks (the basins of the sinks). Since each stable manifold locally separates two such discs on the supporting surface, each stable manifold has two exemplars in the boundaries of the disks after cutting. Thus, the boundary of each disk consists of $\frac{2m_\sigma}{m} = 2k$ stable manifolds and each disk with the boundary is a $2k$-gon (see Figure 14 and Figure 15 on the left).

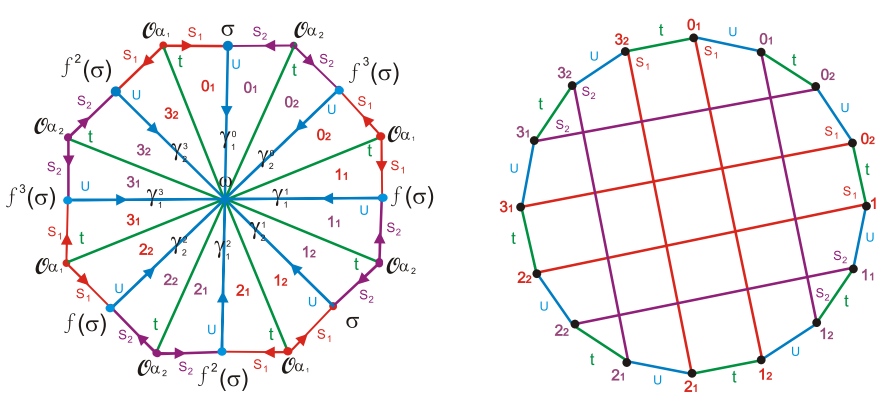

FIGURE 14. The octagon Π which is the closure of the sink basin of the diffeomorphism $f \in G$ (on the left) and the four-color graph T_f constructed for it (on the right). Here $m = 1$, $k = 4$, $n = 1$, $j = 1$, and $\rho = 0$

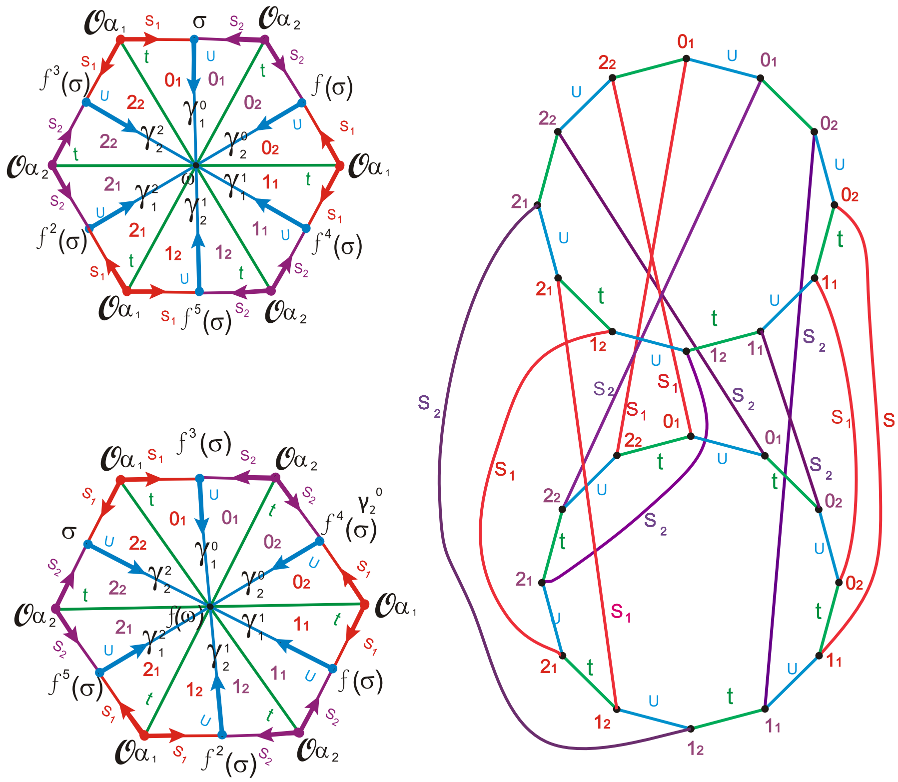

FIGURE 15. The hexagons Π and $f(\Pi)$ which are the closures of the sink basins of the diffeomorphism $f \in G$ (left) and the four-color graph T_f (right) constructed for them. Here $m = 2$, $k = 3$, $n = 1$, $j = 2$ and $\rho = 1$.

The stable separatrices are called s_1- and s_2-curves; the unstable separatrices (they lie on the rays of the frames γ_1 and γ_2) are called u-curves frames; and the segments connecting the vertices of the polygon with its center are called t-curves. These (colored) curves divide each polygon into triangles with s_i-, t-, and u-sides. Enumerate these triangles as shown on Figure 14 and Figure 15 on the left.

The u-sides belonging to the rays $\gamma_1^\tau \times \{w\}$ and $\gamma_2^{(\tau+j)\sim mod\ k} \times \{(w+\rho) \sim mod\ m\}$ are the separatrices of the same saddle point of f. In order to get the surface S_g from the polygons $\Pi_0, \dots, \Pi_{m-1}$, we identify the pairs of those sides of the polygons which are transversal to this pair of the separatrices.

To establish the periods of the source points, we associate a four-color graph with the diffeomorphism f in the following way (see, for example, [5] and [6] for details):

(1) the vertices of the graph T_f are in a one-to-one correspondence with the triangular regions;

(2) two vertices of the graph are incident to the edge of color s_1, s_2, t, or u if the triangular areas corresponding to these vertices have a common s_1, s_2, t, or u side (see Figure 14 and Figure 15 on the right).

Denote by B_f the set of the vertices of the graph T_f and by Δ_f the set of the triangles in the partition of the polygon. Denote by $\pi_f : \Delta_f \to B_f$ a one-to-one correspondence between the set of triangular domains of the diffeomorphism f and the set of vertices of the graph T_f. The diffeomorphism f induces an automorphism $P_f = \pi_f f \pi_f^{-l}$ on the set of vertices and edges of the graph T_f. Additionally,

- the set of the sink points of f is in a one-to-one correspondence with the set of the tu-cycles of the graph T_f;
- the set of the saddle points of f is in a one-to-one correspondence with the set of the su-cycles of T_f;
- the set of the source points of f is in a one-to-one correspondence with the set of the ts-cycles of T_f.

In order to determine the period m_{α_i} of the point α_i, $i = 1, 2$, we have to calculate the number of $s_i t$-cycles. If each such cycle is an image of some other such cycle by f, then all the cycles are of the same period. Hence, the length of each such cycle is some even number (edges s_i and t follow one another) denoted by $2\lambda_i$. Notice that the number of s_i- and t-edges in all the $s_i t$-cycles equals $2km$; therefore,

$$m_{\alpha_i} = \frac{km}{\lambda_i}.$$

Now we calculate the length of the $s_1 t$-cycle starting from the s_1-edge $(0_1, j_2)$. We get the following sequence of the vertices

$$0_1 \to j_2 \to ((j+1) \sim mod\, k)_1 \to ((2j+1) \sim mod\, k)_2 \to$$
$$\to (2(j+1) \sim mod\, k)_1 \to \cdots \to (\lambda_1(j+1) \sim mod\, k)_1.$$

Since the sequence forms the cycle, we have

$$\lambda_1(j+1) \sim mod\, k = 0 \text{ and } \lambda_1\rho \sim mod\, m = 0$$

and, hence,

$$\lambda_1(j+1) = lk \text{ and } \lambda_1\rho = rm$$

for some natural l and r.

Let $A = (k, j+1)$, then $k = pA$ and $j+1 = qA$ where $(p, q) = 1$; hence, $\lambda_1 = \frac{lp}{q} = \frac{rm}{\rho}$. As λ_1 is natural and $(p, q) = 1$ and $(\rho, m) = 1$, we have $l = \mu q$ and $r = \nu\rho$. Therefore, $\lambda_1 = \mu p = \nu m$ and $(\mu, \nu) = 1$ as λ_1 is the minimal number satisfying $\lambda_1 = \tilde{\mu} p = \tilde{\nu} m$ for some natural $\tilde{\mu}$ and $\tilde{\nu}$. Let $B = (p, m)$, then $p = xB$ and $m = yB$ where $(x, y) = 1$. Therefore, $\mu x = \nu y$ and $\mu = y$, $\nu = x$, and $\lambda_1 = yp$.

Thus, $m_{\alpha_1} = \frac{km}{\lambda_1} = \frac{km}{yp} = \frac{pAm}{yp} = AB = (k, j+1)\left(\frac{k}{(k,j+1)}, m\right)$.

The similar construction for α_2 gives $m_{\alpha_2} = (k, j)\left(\frac{k}{(k,j)}, m\right)$. By $(*_2)$, we have $m + (k, j+1)\left(\frac{k}{(k,j+1)}, m\right) + (k, j)\left(\frac{k}{(k,j)}, m\right) - km = 2 - 2g$. □

7. Diffeomorphisms of G Class on the 2-Sphere

In this section we prove Corollary 2.4; that is, we show that on the 2-sphere there exists a diffeomorphism from G with the following periodic data:

(1) $m_\omega = 1, m_\sigma = 1, m_\alpha = 2$;
(2) $m_\omega = 2, m_\sigma = 1, m_\alpha = 1$;
(3) $m_\omega = 1, m_\sigma = m_{\alpha_1} = k, m_{\alpha_2} = 1, k \in \mathbb{N}$;
(4) $m_\omega = m, m_\sigma = m, m_{\alpha_1} = m_{\alpha_2} = 1, m \in \mathbb{N}$.

Proof. Due to [9], every diffeomorphism from G on the sphere has at least one fixed point. Since $(**_1)$ of Theorem 2.2 is proved, we have the complete list of periodic data for diffeomorphisms of G_1:

(1) $m_\omega = 1, m_\sigma = 1, m_\alpha = 2$;
(2) $m_\omega = 2, m_\sigma = 1, m_\alpha = 1$.

If, in $(**_2)$, $m_\omega = 1$ and $m_{\alpha_1} = 1$, then we have

(3) $m_\omega = 1, m_\sigma = k, m_{\alpha_1} = (k, j+1), m_{\alpha_2} = (k, j), k \in \mathbb{N}, j \in \{0, \dots, k-1\}$.

From $(*_2)$, it follows that $(k, j+1) + (k, j) = k+1$, then $(k, j+1) = k, (k, j) = 1$, and therefore $m_\omega = 1, m_\sigma = m_{\alpha_1} = k, m_{\alpha_2} = 1, k \in \mathbb{N}$.

(4) $m_\omega = m, m_\sigma = km, m_{\alpha_1} = 1,\ k, m \in \mathbb{N}$.

From $(*_2)$, it follows that $m + 1 - km + m_{\alpha_2} = 2$ and $m_{\alpha_2} = 1 + m(k-1)$. If $k = 1$, then $m_{\alpha_2} = 1$, and we have periodic data of the fourth type.

The case $k > 1$ is impossible. Indeed, if $k > 1$, since $m > 1$, we have $m_{\alpha_2} > k$. On the other hand, from $(**_2)$, it follows that $m_{\alpha_1} = (k, j+1)\left(\frac{k}{k, j+1}, m\right) = 1$; therefore, $(k, j+1) = 1, (k, m) = 1$, and $m_{\alpha_2} = (k, j)$. So $m_{\alpha_2} \leq k$, and we have a contradiction. $\square$

8. Realization

In this section we prove Theorem 2.5; that is, we construct a diffeomorphism of the class G for the given periodic data. To address this problem, we define on the unit circle D the model vector field by the following system of differential equations in the polar coordinates (r, φ)

$$\begin{cases} \dot{r} = r(r-1), \\ \dot{\varphi} = -(\varphi - \varphi_0)(\varphi - \varphi_1)\dots(\varphi - \varphi_{2b-1}) \end{cases};$$

here, $b \in \mathbb{N}$, $\varphi_0 = \frac{\pi}{2}$, and $\varphi_\nu = \varphi_0 - \nu\frac{\pi}{b}$.

Denote by χ_b^t the flow induced by this vector field and denote by χ_b the diffeomorphism which is the time-1 map of the flow χ_b^t. Denote by Δ_ν the sector of D such that $\varphi_\nu \leq \varphi \leq \varphi_{\nu+1}$ where $\varphi_{2b} = -3\pi/2$. Denote by A_ν the point with the polar coordinates $r = 1$ and $\varphi = \varphi_\nu$ (see Figure 16).

8.1. Realization of diffeomorphisms of the class G_1.

To realize a diffeomorphism with periodic data $m_\omega = 1$, $m_\sigma = 2g$, $m_\alpha = 1$, and $g > 0$, or $m_\omega = 1$, $m_\sigma = 2g+1$, $m_\alpha = 2$, and $g \geq 0$, let $b = 2m_\sigma$. Define the diffeomorphism $\bar{f} : D \to D$ by $\bar{f}(d) = e^{i\frac{\pi}{m_\sigma}} \cdot \chi_{2m_\sigma}(d)$. In order to obtain the surface S_g of genus $g = m_\sigma/2$ from the disk D, we identify the arcs on ∂D by the diffeomorphisms

$$\Psi_\mu(1, \varphi) = (1, \varphi_{2\mu+1} + \varphi_{2\mu-1} - \varphi - \pi),\ \varphi \in [\varphi_{2\mu+1}, \varphi_{2\mu-1}]$$

for $\mu \in \{0, \dots, m_\sigma - 1\}$ where $\varphi_{-1} = \frac{\pi}{2} + \frac{\pi}{2m_\sigma}$. By the construction, the diffeomorphism $\bar{f}$ commutes with the identification; therefore, it induces the homeomorphism $f : S_g \to S_g$ which is smooth except at the source point. Using the technique of [5], it is possible to introduce such a local chart in a neighborhood of the source that the surface becomes smooth and f induces the desired Morse–Smale diffeomorphism on it.

The diffeomorphism f^{-1} for $m_\sigma = 2g+1$ realizes the periodic data $m_\omega = 2, m_\sigma = 2g+1, m_\alpha = 1,\ g \geq 0$.

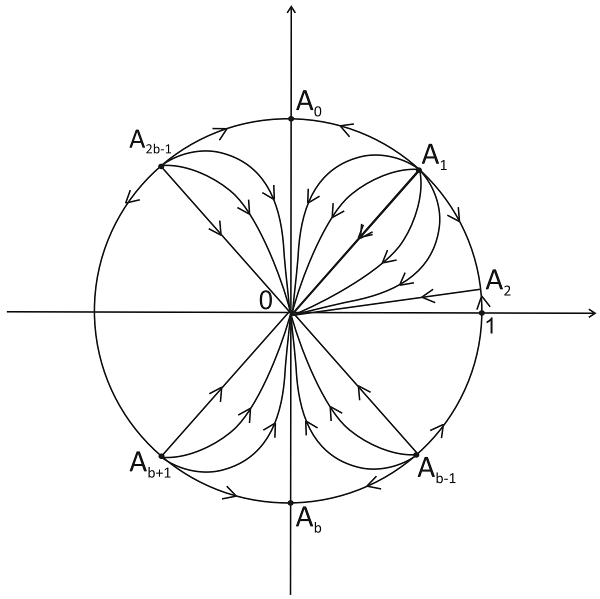

FIGURE 16. Trajectories of the flow χ_b^t

8.2. Realization of diffeomorphisms of the class G_2.

To realize a diffeomorphism with the periodic data $m_\omega = m,\ \ m_\sigma = km, m_{\alpha_1} = (k,j+1)\left(\frac{k}{(k,j+1)}, m\right), m_{\alpha_2} = (k,j)\left(\frac{k}{(k,j)}, m\right)$, $k \in \mathbb{N}, j \in \{0,\dots,k-1\}$, let $b = 2k$. Define the diffeomorphism $\bar{f} : D\times\mathbb{Z}_m \to D\times\mathbb{Z}_m$ by $\bar{f}(d,q) = (e^{i\frac{\pi}{k}}\cdot\chi_{2k}(d), (q+1) \sim mod\ m)$. In order to obtain the surface S_g of genus

$$g = 1 + \frac{1}{2}\left((k-1)m - (k,j+1)\left(\frac{k}{(k,j+1)}, m\right) - (k,j)\left(\frac{k}{(k,j)}, m\right)\right)$$

from the disks $D \times \mathbb{Z}_m$, we identify the arcs on $\partial D \times \mathbb{Z}_m$ by the diffeomorphisms

$$\Psi_\mu(1,\varphi,q) = (1, k_\mu(\varphi - \varphi_{4\mu-1}) + \varphi_{(3+4(\mu+j))\sim mod4k}, (q+1) \sim mod\ m),$$

$\varphi \in [\varphi_{4\mu+1}, \varphi_{4\mu-1}]$ for $\mu \in \{0,\dots,k-1\}$, where $\varphi_{-1} = \frac{\pi}{2} + \frac{\pi}{2k}$ and

$$k_\mu = \frac{\varphi_{(1+4(\mu+j))\sim mod4k} - \varphi_{(3+4(\mu+j))\sim mod4k}}{\varphi_{4\mu+1} - \varphi_{4\mu-1}}.$$

By construction, the diffeomorphism $\bar{f}$ commutes with the identification; therefore, it induces the homeomorphism $f : S_g \to S_g$ which is smooth except at the source point. Using the technique of [5], it is possible to introduce such a local chart in a neighborhood of the source that the surface becomes smooth and f induces the desired Morse–Smale diffeomorphism on it.

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