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by

Daria Michalik

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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SUSPENSIONS OF LOCALLY CONNECTED CURVES: HOMOGENEITY DEGREE AND UNIQUENESS

DARIA MICHALIK

Abstract. The homogeneity degree of a space X is the number of orbits for the action of the group of homeomorphisms of X onto itself. We determine the homogeneity degree of the suspension over a locally connected curve X not being a local dendrite in terms of that of X. Using the main result of Alicia Santiago-Santos's *Degree of homogeneity on suspensions* (Topology Appl. **158** (2011), no. 16, 2125–2139) gives us a formula for the homogeneity degree of the suspension over any locally connected curve X.

We also prove that the suspensions over locally connected curves not being local dendrites X and Y are homeomorphic if and only if X and Y are homeomorphic.

1. Introduction

A *continuum* is a nondegenerate compact connected metric space. A *curve* is a one-dimensional continuum. An *arc* is a continuum homeomorphic to the interval $\mathbb{I} = [0, 1]$. A *simple closed curve* is a continuum homeomorphic to the unit circle S^1.

Let X be a topological space. The *cone* of X is the quotient space defined by

$$\mathrm{Cone}(X) = X \times \mathbb{I} \diagup \{X \times \{1\}\},$$

and the *suspension* of X is the quotient space defined by

$$\mathrm{Sus}(X) = X \times \mathbb{I} \diagup \{X \times \{0\},\ X \times \{1\}\}.$$

Let $\mathcal{H}(X)$ denote the group of homeomorphisms of X onto itself. An *orbit of* X is an orbit under the action of $\mathcal{H}(X)$. Given a point $x \in X$,

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we consider the following set

$$\mathcal{O}_X(x) = \{h(x) : \ h \in \mathcal{H}(X)\}.$$

We call it the *orbit of the point* x. Let $\mathcal{O}_X$ be the set of orbits in X. We say that the *homogeneity degree of* X is n provided that X has exactly n orbits (in symbols $d_H(X) = n$).

The homogeneity degree has been studied recently in many papers (see, e.g., [4]–[6] and [8]–[10]).

For every $n \in \mathbb{N}$, let Θ_n denote the suspension over an n-point set. In [11], Alicia Santiago-Santos proved the following result.

Theorem 1.1. *Let X be a local dendrite.*

(1) *If $d_H(X)$ is infinite or $X \in \{\mathbb{I}, S^1\} \cup \{\Theta_n : n \in \mathbb{N}\}$, then $d_H(\mathrm{Sus}(X)) = d_H(X)$.*
(2) *If $d_H(X)$ is finite and $X \notin \{\mathbb{I}, S^1\} \cup \{\Theta_n : n \in \mathbb{N}\}$, then $d_H(\mathrm{Sus}(X)) = d_H(X) + 1$.*

In this paper we give a formula for the homogeneity degree of the suspension over X in terms of that of X for a locally connected curve X not being a local dendrite.

Theorem 1.2. *Let us assume that X is a locally connected curve not being a local dendrite. If $d_H(X)$ is finite, then $d_H(\mathrm{Sus}(X)) = d_H(X) + 1$. If $d_H(X)$ is infinite, then $d_H(X) = d_H(\mathrm{Sus}(X))$.*

By theorems 1.1 and 1.2, we obtain a formula for the homogeneity degree of the suspension over X in terms of that of X for any locally connected curve X.

In [6] and [10], the reader can find formulas for the homogeneity degree of the cones over locally connected curves.

An important step toward proving Theorem 1.2 is Lemma 2.5, which can be found in §2. The proof of Theorem 1.2 also involves techniques and ideas developed in [2] and employs the notation of isotopy components. This part of our work is contained in §3.

It is well known that cones and suspensions of non-homeomorphic spaces can be homeomorphic. In [7], we proved that the cones over locally connected curves not being local dendrites X and Y are homeomorphic if and only if X and Y are homeomorphic. In this paper, we present the analogous result for suspensions, which was already announced in [7]. Namely, we prove the following theorem.

Theorem 1.3. *Let us assume that X and Y are locally connected curves not being local dendrites. Then $\mathrm{Sus}(X)$ and $\mathrm{Sus}(Y)$ are homeomorphic if and only if X is homeomorphic to Y.*

Its proof can be found in §5.

The crucial step in the proof of Theorem 1.3 is the following result.

Theorem 1.4. *Let C and C' be locally connected curves. Then $C \times (0,1)$ is homeomorphic to $C' \times (0,1)$ if and only if C is homeomorphic to C'.*

Since, in general, the cancellation law in Cartesian products does not hold, especially for non-compact factors, Theorem 1.4 seems to be interesting. Its proof can be found in §4.

The proofs of all results presented here are similar in spirit to the proofs in [2], [6], and [7].

2. Notation and Tools

Our terminology follows [3]. All spaces are assumed to be metric. A *dendrite* is a locally connected continuum without simple closed curves.

By $\alpha(X)$, we denote the set of Euclidean points of X, i.e., the points having a neighborhood homeomorphic to an Euclidean space E^n for some $n \in \mathbb{N}$. By $\beta(X)$, we denote the set of semi-Euclidean points of $X \setminus \alpha(X)$, namely the points $(x_1, \ldots, x_n) \in E^n$ with $x_n \geq 0$, and $\gamma(X) = X \setminus (\alpha(X) \cup \beta(X))$. A component of $\alpha(X)$ is called a *Euclidean component* of X.

A space M is a *manifold* if M is a compact and connected space such that $\gamma(M) = \emptyset$.

Remark 2.1. If C is a locally connected curve, then

$$\alpha(C \times (-1,1)) = \alpha(C) \times (-1,1).$$

Proof. Obviously, $\alpha(C \times (-1,1)) \supseteq \alpha(C) \times (-1,1)$. Hence, it is enough to prove the converse inclusion. Let $(x,y) \in \alpha(C \times (-1,1))$. By the definition of α, the point (x,y) has a neighborhood in $C \times (-1,1)$ homeomorphic to the Euclidean 2-dimensional space. By [1, p. 275], $x \in \alpha(C)$. □

A point $p \in X$ is *approximately Euclidean* if, for every $\epsilon > 0$, there exists a map $f : X \times \mathbb{I} \to X$ such that

(1) $f(x,0) = x$,
(2) $\mathrm{dist}(f(x,t),x) < \epsilon$ for every $(x,t) \in X \times \mathbb{I}$,
(3) $p \in \alpha(f(X \times \{1\}))$,
(4) the dimension of $f(X \times \{1\})$ in the point p is equal to the dimension of X in p.

Let κ be a cardinal number. A point $x \in X$ is *of order less than or equal to κ* provided that x has a basis of open neighborhoods whose boundaries have at most κ elements. The smallest cardinal number κ with the above property is called the *order of a point x* in X.

Since the property of a point being approximately Euclidean is a local one (see [2, p. 145]), by [2, theorems 7 and 9], we obtain the following proposition.

Proposition 2.2. *Let C be a locally connected curve. A point $(x,t) \in C \times (0,1)$ is approximately Euclidean in $C \times (0,1)$ if and only if x is of order 2 and C is locally contractible at x.*

For every pair of points x and y in a locally connected curve C, we denote by $\nu_C(x,y)$ the number (finite or not) of Euclidean components A in C such that the boundary of A contains only the points x and y.

In this paper we will use the following result.

Lemma 2.3 ([2, p. 155])**.** *Let C and C' be two locally connected curves and $h : \beta(C) \cup \gamma(C) \to \beta(C') \cup \gamma(C')$ be a homeomorphism. Then h can be extended to a homeomorphism between C and C' if and only if $\nu_C(x,y) = \nu_{C'}(h(x),h(y))$ for every pair of points $x, y \in \beta(C) \cup \gamma(C)$.*

Recall that an orbit in X is a set $\mathcal{O}_X(x) = \{h(x) : \ h \in \mathcal{H}(X)\}$ for some $x \in X$. Obviously, if $A \subseteq X$ and $B \subseteq Y$ are orbits in X and Y, respectively, then $A \times B$ is contained in some orbit of $X \times Y$.

Remark 2.4. Let $(x,y) \in X \times Y$. Then $\mathcal{O}_X(x) \times \mathcal{O}_Y(y) \subseteq \mathcal{O}_{X\times Y}((x,y))$.

The crucial step in the proof of Theorem 1.2 is the following lemma. The analogous result for cones can be found in [7].

Lemma 2.5. *If C and C' are locally connected curves not being local dendrites and $h\colon \mathrm{Sus}(C) \to \mathrm{Sus}(C)$ is a homeomorphism, then h maps the vertices of $\mathrm{Sus}(C)$ onto the vertices of $\mathrm{Sus}(C')$.*

Proof. Let C be a locally connected curve not being a local dendrite and x_{-1} and x_1 be the vertices of $\mathrm{Sus}(C)$. Observe that $\mathrm{Sus}(C)$ is locally contractible in x_{-1} and in x_1 but, since C is not a local dendrite, in every neighborhood of x_{-1} and in every neighborhood of x_1 there are the points y_{-1} and y_1, respectively, such that $\mathrm{Sus}(C)$ is not locally contractible in y_{-1} and in y_1. The vertices of $\mathrm{Sus}(C)$ are the only points of $\mathrm{Sus}(C)$ with these properties. Since the vertices of $\mathrm{Sus}(C')$ are also the only points in $\mathrm{Sus}(C')$ with these properties, every homeomorphism maps the vertices of $\mathrm{Sus}(C)$ onto the vertices of $\mathrm{Sus}(C')$. $\square$

3. Isotopic Components

A continuous map $h\colon X \times \mathbb{I} \to X$ is a *homotopic deformation in X* if $h(x,0) = x$ for every $x \in X$. A map $h\colon X \times \mathbb{I} \to X$ is an *isotopic deformation in X* if h is a homotopic deformation and the map $h_t(x) = h(x,t)$ is a homeomorphic embedding in X for every $t \in \mathbb{I}$. If, for every $t \in$

$\mathbb{I}$, the map $h_t(x) = h(x,t)$ is a homeomorphism on X, then $h\colon X \times \mathbb{I} \to X$ is an *isotopic deformation on* X. Two points x_1 and x_2 are *isotopic in* X (in symbols $x_1 \sim x_2$) if there exists an isotopic deformation $h\colon X \times \mathbb{I} \to X$ on X such that $h(x_1, 1) = x_2$.

The relation of being isotopic is an equivalence relation on X (for details see [7]). An equivalence class of the relation $\sim$ is called an *isotopy component of* X and the equivalence class of a point x is called the *isotopy component of* x. We denote it by $K(x)$.

The idea of isotopy components comes from K. Borsuk's paper [2], where isotopy components are used in the proof of the decomposition uniqueness for the products of a locally connected curve and a manifold. Our proofs of lemmas 3.6 and 3.8 are based on Borsuk's proofs in [2].

Below the reader can find some elementary properties of isotopy components.

Lemma 3.1. *Let X be a topological space. Then*

(a) *Every isotopy component of X is arcwise connected.* [2, p. 151]
(b) *If $x \in X$ and $h\colon X \to Y$ is a homeomorphism, then the image of the isotopy component of the point x is the isotopy component of the point $h(x)$.* [7, Remark 3.4.b]
(c) *If x and y belong to the same isotopy component of X, then X is locally homeomorphic in x and in y.* [2, p. 151]
(d) *Every Euclidean component of X is an isotopy component of X.* [2, p. 152]
(e) *Let C be a locally connected curve. The isotopy components of C containing at least two points are identical with the Euclidean components of C.* [2, p. 152]
(f) [7, Corollary 3.8] *A locally connected curve C has isotopy components*
  - *identical to the Euclidean components,*
  - *the individual points of $\beta(C)$,*
  - *the individual points of $\gamma(C)$.*

A point $p \in X$ is *isotopically labile* if, for every $\epsilon > 0$, there exists an isotopic deformation $h(x,t)$ in X satisfying the following conditions:

(1) $\operatorname{dist}(h(x,t), x) < \epsilon$, for every $(x,t) \in X \times \mathbb{I}$, and
(2) $h(x,1) \neq p$, for every $x \in X$.

The points which are not isotopically labile are said to be *isotopically stable.*

Remark 3.2 (see [2, p. 149])**.** If C is a locally connected curve, then the isotopically labile points are the same as the semi-Euclidean points.

Lemma 3.3. *Let C be a locally connected curve. The set of isotopically labile points in $C \times (0,1)$ is the same as the set $\beta(C) \times (0,1)$.*

Proof. Note that every point in $\beta(C) \times (0,1)$ is isotopically labile.

To prove the converse implication, let $p \in C \times (0,1)$ be an isotopically labile point in $C \times (0,1)$. First, we will prove that $p \in C \times (0,1) \subset C \times [0,1]$ is also an isotopically labile point in $C \times [0,1]$. Let us fix $\epsilon > 0$. There exists an isotopic deformation $h : C \times (0,1) \times \mathbb{I} \to C \times (0,1)$ such that $\operatorname{dist}(h(x,t),x) < \epsilon/2$ for every $(x,t) \in C \times (0,1) \times \mathbb{I}$, and $h(x,1) \neq p$ for every $x \in C \times (0,1)$. Let $g : C \times [0,1] \times \mathbb{I} \to C \times [0,1]$ be an isotopic deformation in $C \times [0,1]$ such that $g(C \times [0,1] \times \{1\}) \subseteq C \times (0,1)$ and $\operatorname{dist}(g(x,t),x) < \epsilon/2$ for $x \in C \times [0,1]$ and $t \in \mathbb{I}$. One can see that $h_g(x,t) : C \times [0,1] \times \mathbb{I} \to C \times [0,1]$, defined by the formula

$$h_g(x,t) = \begin{cases} g(x,2t) & \text{for } t \in [0,1/2] \\ h(g(x,1),2t-1) & \text{for } t \in (1/2,1], \end{cases}$$

is an isotopic deformation in $C \times [0,1]$, $\operatorname{dist}(h_g(x,t),x) < \epsilon$ for $(x,t) \in C \times [0,1] \times \mathbb{I}$, and $h_g(x,1) \neq p$, for $x \in C \times [0,1]$. Hence, p is an isotopically labile point in $C \times [0,1]$. Recall that $p \in C \times (0,1)$. By [2, Lemma 11], $p \in \beta(C) \times (0,1)$. □

The following result will be used in the proof of Lemma 3.6. Its statement and proof are analogous to [2, Lemma 13].

Lemma 3.4. *Let C be a locally connected curve. Two points $(x_0,y_0) \in \gamma(C) \times (0,1)$ and $(x_1,y_1) \in C \times (0,1)$ are isotopic in $C \times (0,1)$ if and only if $x_0 = x_1$.*

Proof. Note that if $x_0 = x_1$, then (x_0,y_0) and (x_1,y_1) are isotopic.

Assume now that $(x_0,y_0) \in \gamma(C) \times (0,1)$ and $(x_1,y_1) \in C \times (0,1)$ are isotopic in $C \times (0,1)$. Hence, there exists an isotopic deformation $\phi : C \times (0,1) \times \mathbb{I} \to C \times (0,1)$ such that

$$\phi(x,y,t) = (\phi_C(x,y,t), \phi_{(0,1)}(x,y,t)) \text{ and}$$

$$\phi(x_0,y_0,1) = (x_1,y_1).$$

Assume for a contradiction that $x_0 \neq x_1$. Hence, $x_0 \neq \phi_C(x_0,y_0,1) = x_1$.

Observe that $\phi_C(x,y_0,t)$ is a homotopic deformation in C. Since in a locally connected curve the homotopically fixed points are the same as the points in which the curve is not a local dendrite (see [2, p. 143]), x_0 has a neighborhood U in C being a dendrite. Since the point (x_0,y_0) is isotopic to the point $(\phi_C(x_0,y_0,t), \phi_{(0,1)}(x_0,y_0,t))$ for every $t \in [0,1]$, the point $\phi_C(x_0,y_0,t)$ belongs to $\gamma(C)$ for every $t \in [0,1]$. The set of points of order greater than or equal to 3 of a dendrite is always finite or countable.

Hence, there exist t_1 and t_2 in $[0,1]$ such that $\phi_C(x_0,y_0,t_1) \in U$ is of order 2 and $\phi_C(x_0,y_0,t_2) \in U$ is of order greater than or equal to 3. By Proposition 2.2, the point $\phi_C(x_0,y_0,t_1)$ is approximately Euclidean and the point $\phi_C(x_0,y_0,t_2)$ is not approximately Euclidean in $C \times (0,1)$. Since $\phi(x_0,y_0,t_1)$ and $\phi(x_0,y_0,t_2)$ are isotopic, we obtain a contradiction. Hence, $x_0 = x_1$. □

Let C be a locally connected curve where $S^1 \neq C \neq \mathbb{I}$, and let $\mathcal{A}$ be the set of all Euclidean components of C. Let us observe that every Euclidean component A of C is homeomorphic to the open interval $(0,1)$ and $\bar{A} \cap \gamma(C)$ contains one or two points. Hence, there are three types of Euclidean components of C, namely, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, where the $\mathcal{A}_i$ are pairwise disjoint and

(1) $A \in \mathcal{A}_1$ if $\bar{A} \cap \beta(C) = \emptyset$ and $|\bar{A} \cap \gamma(C)| = 1$,
(2) $A \in \mathcal{A}_2$ if $\bar{A} \cap \beta(C) = \emptyset$ and $|\bar{A} \cap \gamma(C)| = 2$,
(3) $A \in \mathcal{A}_3$ if $\bar{A} \cap \beta(C) \neq \emptyset$.

Remark 3.5. Let C be a locally connected curve.

(1) If A and B are Euclidean components of C of the same type and $\bar{A} \cap \gamma(C) = \bar{B} \cap \gamma(C)$, then A and B are contained in the same orbit of C.
(2) Every Euclidean component of C is contained in some orbit of C.

Now, we will use the sets $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ from above to classify the isotopy components of the product $C \times (-1,1)$. An analogous result for the product of a manifold and a locally connected curve can be found in [2].

Lemma 3.6. *Let C be a locally connected curve where $S^1 \neq C \neq \mathbb{I}$, and let $\mathcal{B}$ be the set of all isotopy components of $C \times (-1,1)$. Then $\mathcal{B} = \bigcup_{i=1}^{5} \mathcal{B}_i$, where $\mathcal{B}_i$ are pairwise disjoint and*

1. *$B \in \mathcal{B}_1$, if $B = A \times (-1,1)$, where $A \in \mathcal{A}_1$;*
2. *$B \in \mathcal{B}_2$, if $B = A \times (-1,1)$, where $A \in \mathcal{A}_2$;*
3. *$B \in \mathcal{B}_3$, if $B = A \times (-1,1)$, where $A \in \mathcal{A}_3$;*
4. *$B \in \mathcal{B}_4$, if $B = \{x\} \times (-1,1)$, where $x \in \beta(C)$;*
5. *$B \in \mathcal{B}_5$, if $B = \{x\} \times (-1,1)$, where $x \in \gamma(C)$.*

Before the proof of Lemma 3.6, let us introduce some notation and one remark.

If $B \in \mathcal{B}_i$, we say that B is an *isotopy component of type i* for $i \in \{1,2,\ldots,5\}$. Let $i \in \{1,\ldots,5\}$ and $h\colon C \times (-1,1) \to C' \times (-1,1)$ be a homeomorphism such that $h(B) \in \mathcal{B}_i$ if and only if $B \in \mathcal{B}_i$ for every isotopy component B. Then we say that h *preserves type i* of the isotopy components.

Remark 3.7. If $B \in \bigcup_{i=1}^{3} \mathcal{B}_i$, then B is 2-dimensional and if $B \in \mathcal{B}_4 \cup \mathcal{B}_5$, then B is 1-dimensional.

Proof of Lemma 3.6. It is clear that, for every set $B \in \bigcup_{i=1}^{5} \mathcal{B}_i$, there exists an isotopy component of $C \times (0,1)$ containing B.

Take $B \in \bigcup_{i=1}^{3} \mathcal{B}_i$. By Remark 2.1, B is a Euclidean component of $C \times (0,1)$. Hence, by Lemma 3.1(d), B is an isotopy component of $C \times (0,1)$.

Take $B \in \mathcal{B}_5$. By Lemma 3.4, B is an isotopy component of $C \times (0,1)$.

Now, observe that $\beta(C \times (0,1)) = \bigcup_{B \in \mathcal{B}_4} B$. By lemmas 3.3 and 3.1(a), every $B \in \mathcal{B}_4$ is an isotopy component of $C \times (0,1)$. $\square$

Lemma 3.8. *Let C and C' be locally connected curves, where $S^1 \neq C \neq \mathbb{I}$ and $S^1 \neq C' \neq \mathbb{I}$, $\mathcal{B} = \bigcup_{i=1}^{5} \mathcal{B}_i$, and $\mathcal{B}' = \bigcup_{i=1}^{5} \mathcal{B}'_i$ defined in Lemma 3.6, are the sets of isotopy components of $C \times (0,1)$ and $C' \times (0,1)$, respectively. If $h : C \times (-1,1) \to C' \times (-1,1)$ is a homeomorphism, then h maps $B \in \mathcal{B}_i$ onto $h(B) \in \mathcal{B}'_i$ for every $B \in \mathcal{B}_i$ and $i \in \{1,2,\dots,5\}$.*

Proof. By Lemma 3.1(b), if $B \in \mathcal{B}$, then $h(B) \in \mathcal{B}'$.

Take $B \in \mathcal{B}_1$. By the definition of $\mathcal{B}_1$, $B = A \times (-1,1)$, where $A \in \mathcal{A}_1$. Since A is a Euclidean component of C such that $\overline{A} \cap \beta(C) = \emptyset$ and $|A \cap \gamma(C)| = 1$, $\overline{A}$ is homeomorphic to S^1 and $\overline{A} \cap \overline{C \setminus \overline{A}} = \{a\} \subset \gamma(C)$. Hence, $\overline{B} = \overline{A} \times (-1,1)$ is homeomorphic to $S^1 \times (0,1)$ and $\overline{B} \cap \overline{(C \times (0,1)) \setminus \overline{B}}$ is homeomorphic to $(0,1)$.

Take $B \in \mathcal{B}_2$. By the definition of $\mathcal{B}_2$, $B = A \times (-1,1)$, where $A \in \mathcal{A}_2$. Since A is a Euclidean component of C such that $\bar{A} \cap \beta(C) = \emptyset$ and $|\bar{A} \cap \gamma(C)| = 2$, $\overline{A}$ is homeomorphic to $[0,1]$ and $\overline{A} \cap \overline{C \setminus \overline{A}} = \{a,b\} \subset \gamma(C)$. Hence, $\overline{B} = \overline{A} \times (-1,1)$ is homeomorphic to $[0,1] \times (0,1)$ and $\overline{B} \cap \overline{(C \times (0,1)) \setminus \overline{B}}$ is homeomorphic to $\{0,1\} \times (0,1)$.

Take $B \in \mathcal{B}_3$. By the definition of $\mathcal{B}_3$, $B = A \times (-1,1)$, where $A \in \mathcal{A}_3$. Since A is a Euclidean component of C such that $\bar{A} \cap \beta(C) \neq \emptyset$, $\overline{A}$ is homeomorphic to $[0,1]$, and $\overline{A} \cap \overline{C \setminus \overline{A}} = \{a\} \subset \gamma(C)$. Hence, $\overline{B} = \overline{A} \times (-1,1)$ is homeomorphic to $[0,1] \times (0,1)$ and $\overline{B} \cap \overline{(C \times [0,1)) \setminus \overline{B}}$ is homeomorphic to $(0,1)$.

Since only the isotopy components from $\bigcup_{i=1}^{3} \mathcal{B}_i$ are 2-dimensional and the above topological properties distinguish each of them, we can conclude that h preserves type i of isotopy components for $i \in \{1,2,3\}$.

If $B \in \mathcal{B}_4$, then B lies on the boundary of exactly one isotopy component C_B of $C \times (0,1)$ where $C_B \in \mathcal{B}_3$. Since h preserves type 3 of isotopy components, it also preserves type 4.

Since type i of isotopy components is preserved by the homeomorphism h for $i \in \{1,2,3,4\}$, type 5 is preserved by h as well. □

4. Uniqueness of the Decomposition for the Products of a Locally Connected Curve and the Open Interval $(-1,1)$

Recall that $K(x)$ denotes the isotopy component of the point x.

Lemma 4.1. *Let C and C' be locally connected curves and let $h : C \times (-1,1) \to C' \times (-1,1)$ be a homeomorphism. Then there exists a homeomorphism $h_0 : C \times \{0\} \to C' \times \{0\}$ such that $K(h_0(x,0)) = K(h(x,0))$ for every $x \in \beta(C) \cup \gamma(C)$.*

Proof. If $x \in \gamma(C)$, then $(x,0) \in B = \{x\} \times (-1,1) \in \mathcal{B}_5$ and $h(x,0) = (y,t) \in B' = \{y\} \times (-1,1) \in \mathcal{B}'_5$ for $y \in \gamma(C')$ and $t \in (-1,1)$.

Analogously, if $x \in \beta(C)$, then $(x,0) \in B = \{x\} \times (-1,1) \in \mathcal{B}_4$ and $h(x,0) = (y,t) \in B' = \{y\} \times (-1,1) \in \mathcal{B}'_4$ for $y \in \beta(C')$ and $t \in (-1,1)$.

Define $h_0 \colon (\beta(C) \cup \gamma(C)) \times \{0\} \to (\beta(C') \cup \gamma(C')) \times \{0\}$ by the formula $h_0(x,0) = (y,0)$, where y satisfies the above equations. It is clear that h_0 is a homeomorphism of $(\beta(C) \cup \gamma(C)) \times \{0\}$ onto $(\beta(C') \cup \gamma(C')) \times \{0\}$ and $K(h_0(x,0)) = K(h(x,0))$.

Recall that $\nu_C(p,q)$ denotes the number of Euclidean components A in C such that the boundary of A contains only the points p and q. Now observe that for $x, y \in (\gamma(C) \cup \beta(C)) \times \{0\}$, the number $\nu_{C\times\{0\}}(x,y)$ is equal to the number of Euclidean components of $C \times (-1,1)$ for which the boundary contains both sets $K(x)$ and $K(y)$ and does not contain any other sets $K(z)$ for $z \in (\gamma(C) \cup \beta(C)) \times \{0\}$. The same equality holds for $\nu_{C'\times\{0\}}(x',y')$ where $x', y' \in (\gamma(C') \cup \beta(C')) \times \{0\}$. By Lemma 3.8, we obtain $\nu_{C\times\{0\}}(x,y) = \nu_{C'\times\{0\}}(h_0(x), h_0(y))$. Now, using Lemma 2.3, we can extend h_0 onto $C \times \{0\}$. Note that h_0 satisfies the desired conditions. □

Observe that using Lemma 4.1, we immediately obtain Theorem 1.4.

Lemma 4.2. *Let C and C' be locally connected curves with $(x,0) \in C \times (-1,1)$ and $(y,0) \in C' \times (-1,1)$. If there exists a homeomorphism $h \colon C \times (-1,1) \to C' \times (-1,1)$ such that $h(x,0) = (y,0)$, then there exists a homeomorphism $h_0 \colon C \times \{0\} \to C' \times \{0\}$ such that $h_0(x,0) = (y,0)$.*

Proof. Let h_0 be a homeomorphism as defined in Lemma 4.1. If $x \in \beta(C) \cup \gamma(C)$, then $h_0(x,0) = (y,0)$.

Assume that $x \in \alpha(C)$. Thus, $x \in A \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$, where $\mathcal{A}_i$ is the set of Euclidean components of C of type i for $i \in \{1,2,3\}$.

If $x \in A \in \mathcal{A}_i$, then $(x,0) \in B \in \mathcal{B}_i$, $h(x,0) = (y,0) \in B' \in \mathcal{B}_i'$, and $y \in A' \in \mathcal{A}_i'$ for $i \in \{1,2,3\}$. Moreover, $h(\overline{A \times (-1,1)}) = \overline{A' \times (-1,1)}$. Thus, $h_0((\bar{A} \cap \gamma(C)) \times \{0\}) = (\bar{A}' \cap \gamma(C)) \times \{0\}$ and, by Remark 3.5, we can assume that $h_0(x,0) = (y,0)$. □

5. Proof of Theorem 1.3

It is clear that if X and Y are homeomorphic, then $\mathrm{Sus}(X)$ and $\mathrm{Sus}(Y)$ are also homeomorphic.

To prove the converse implication, assume that $h : \mathrm{Sus}(X) \to \mathrm{Sus}(Y)$ is a homeomorphism. By Lemma 2.5, $h(\{x_{-1}, x_1\}) = \{y_{-1}, y_1\}$ where x_i and y_i, for $i \in \{-1,1\}$, are the vertices of $\mathrm{Sus}(X)$ and $\mathrm{Sus}(Y)$, respectively. Thus, $h(X \times (-1,1)) = Y \times (-1,1)$. Using Theorem 1.4, we conclude that X is homeomorphic to Y.

6. Proof of Theorem 1.2

The following proof is similar in spirit to the proofs of the main results in [6] and [10].

Proof of Theorem 1.2. Let $\mathcal{O}_X$ be the set of orbits in X. Let $\mathcal{G}$ be a minimal subset of X such that

$$\mathcal{O}_X = \{\mathcal{O}_X(x) : x \in \mathcal{G}\}.$$

By Remark 2.4, for every orbit $\mathcal{O}_{\mathrm{Sus}(X)}(x,t)$ of $\mathrm{Sus}(X)$, different from the orbit of the vertices,

$$\mathcal{O}_X(x) \times (0,1) \subseteq \mathcal{O}_{\mathrm{Sus}(X)}(x,t).$$

Thus, by Lemma 2.5,

$$\mathcal{O}_{\mathrm{Sus}(X)} = \mathcal{O}_v \cup \{\mathcal{O}_{\mathrm{Sus}(X)}(x,0) : x \in \mathcal{G}\},$$

where $\mathcal{O}_v$ is the orbit of the vertices of $\mathrm{Sus}(X)$.

Define a function

$$\nu \colon \mathcal{G} \to \{\mathcal{O}_{\mathrm{Sus}(X)}(x,0) : x \in \mathcal{G}\}$$

by the formula

$$\nu(x) = \mathcal{O}_{\mathrm{Sus}(X)}(x,0).$$

Note that the function ν is surjective.

Let us prove that ν is one-to-one. Assume for a contradiction that there exist $x \neq y \in \mathcal{G}$ such that $\nu(x) = \nu(y)$. By the definition of ν, there exists a homeomorphism $h \colon \mathrm{Sus}(X) \to \mathrm{Sus}(X)$ such that $h(x,0) = (y,0)$. By Lemma 2.5, h maps the vertices of $\mathrm{Sus}(X)$ onto the vertices. Hence, there exists a homeomorphism $h^* \colon X \times (-1,1) \to X \times (-1,1)$ satisfying

$h^*(x,0) = (y,0)$. By Lemma 4.2, $\mathcal{O}_X(x) = \mathcal{O}_X(y)$, and we obtain a contradiction with the minimality of $\mathcal{G}$.

Hence, if $\mathcal{G}$ is finite,

$$d_H(\mathrm{Sus}(X)) = 1 + |\mathcal{G}| = 1 + d_H(X).$$

If $\mathcal{G}$ is infinite, we easily obtain

$$d_H(\mathrm{Sus}(X)) = d_H(X). \qquad \square$$

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Faculty of Mathematics and Natural Sciences; College of Science; Cardinal Stefan Wyszyński University; Wóycickiego 1/3; 01-938 Warszawa, Poland

E-mail address: d.michalik@uksw.edu.pl