http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 54, 2019

Pages 81–102

http://topology.nipissingu.ca/tp/

On $M-$Metric Spaces and Fixed Point Theorems

by

Samer Assaf

Electronically published on September 27, 2018

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS

Volume 54 (2019)
Pages 81-102

E-Published on September 27, 2018

ON $M-$METRIC SPACES AND FIXED POINT THEOREMS

SAMER ASSAF

Abstract. In this paper we make some observations concerning $M-$metric spaces and point out some discrepancies in some proofs found in the literature. To remedy this, we propose a new topological construction and prove that it is, in fact, a generalization of a partial metric space. Then, using this construction, we present our main theorem, having as its corollaries the fixed point theorems found in previous publications.

1. Introduction

In 2014, Mehdi Asadi, Erdal Karapinar, and Peyman Salimi [1] proposed the $M-$metric, an intended generalization of a partial metric. In their paper, the proof of Lemma 2.5 does not hold, as we demonstrate in Example 2.4. Although it is a small lemma, its assertion was crucial to the proof of their main theorems: Theorem 3.1 and Theorem 3.2. Our main concern in their approach lies in the open balls they proposed. We go more in depth on the subject in §4.

In §2, we introduce the $M-$metric presented in [1] and generalize it to allow negative values. We also present examples that show why some assumptions proposed in [1], including Lemma 2.5, are not accurate.

In §3, we present the partial metric found in [2], [6], and [7]. We also show how to induce a partial metric from an $M-$metric. The purpose of this section is to put in perspective the generalization from a partial metric to an $M-$metric.

2010 *Mathematics Subject Classification.* Primary 47H10, Secondary 37C25, 54H25.

Key words and phrases. fixed point theorem, generalized metric, $M-$metric, partial metric.

©2018 Topology Proceedings.

In §4, we discuss why the proposed open balls in [1] are not optimal. We then present an alternative definition of open balls and discuss the resulting topology.

In §5, we use the topology presented in §4 to define limits and Cauchy-like sequences in $M-$metric spaces. We then present some of their topological properties.

In §6, we present contractive criteria on functions, allowing them to generate Cauchy-like sequences.

In §7, we discuss weak orbital continuity, non-expansiveness, and the lower bound of a space. These properties are needed for our main theorem.

Finally, in §8, we introduce our main theorem:

> **Theorem 8.1.** *Let (X, σ) be an $M-$metric space with $x_o \in X$. Let $f : X \to X$ be a function such that f is $r-$Cauchy at x_0 with special limit $a \in X$. Further assume at least one of the following conditions holds:*
> (1) *f is weakly orbitally continuous at x_0 and non-expansive.*
> (2) *f is weakly orbitally continuous at x_0 and (X, σ) is bounded below by $\sigma(f(a), f(a))$.*
> (3) *f is non-expansive and (X, σ) is bounded below by $\sigma(a, a)$.*
>
> *Then a is a fixed point of f.*

We then use Theorem 8.1 to present a valid proof of [1, Theorem 3.1 and Theorem 3.2].

2. $M-$Metric

Definition 2.1. Consider a set X and a function $\sigma : X \times X \to \mathbb{R}$. Let

$$m_{x,y} = \min\{\sigma(x, x), \sigma(y, y)\}$$

and

$$M_{x,y} = \min\{\sigma(x, x), \sigma(y, y)\}.$$

We say that σ is an *$M-$metric* on X if it satisfies the following axioms:

For all $x, y, z \in X$,

($\sigma-$lbnd): $m_{x,y} \leq \sigma(x, y)$;
($\sigma-$sym): $\sigma(x, y) = \sigma(y, x)$;
($\sigma-$sep): $\sigma(x, x) = \sigma(x, y) = \sigma(y, y) \iff x = y$;
($\sigma-$inq): $\sigma(x, y) - m_{x,y} \leq \sigma(x, z) - m_{x,z} + \sigma(z, y) - m_{z,y}$.

It is important to notice that for all $x, y \in X$,

$$m_{x,x} = M_{x,x} = \sigma(x, x).$$

Remark 2.2. In [1], the $M-$metric was restricted to having non-negative values. In Definition 2.1, we remove that restriction to expand on the generalization. The reader may notice that ($\sigma-$lbnd) is redundant since it can be obtained from ($\sigma-$inq). Nevertheless, we chose to state ($\sigma-$lbnd) mainly because we use it often enough to warrant giving it its own name and to be consistent with [1].

In [1, Example 1.2], the function

$$\sigma^\star(x,y) = \sigma(x,y) - m_{x,y} \text{ for } x \neq y \text{ and } \sigma^\star(x,x) = 0$$

was proposed to be a metric. We present a counterexample below.

Example 2.3. *Let σ be an $M-$metric on the set $X = \{a,b\}$ defined as*

$$\sigma(a,a) = \sigma(a,b) = \sigma(b,a) = 1 \text{ and } \sigma(b,b) = 2.$$

Hence, $m_{a,b} = \min\{\sigma(a,a), \sigma(b,b)\} = 1$. Therefore,

$$\sigma^\star(a,b) = \sigma(a,b) - m_{a,b} = 1 - 1 = 0, \text{ but } a \neq b.$$

Since $\sigma(a,b) - m_{a,b} = 0$ with $a \neq b$, $\sigma^\star$ fails to satisfy the metric separation axiom.

One of the basic ideas behind the $M-$metric is ($\sigma-$lbnd). This axiom ensures that $m_{x,y} = \min\{\sigma(x,x), \sigma(y,y)\}$ is bounded above by $\sigma(x,y)$. Alternatively, $M_{x,y} = \max\{\sigma(x,x), \sigma(y,y)\}$ remains free from any restrictions. This idea is reinforced by ($\sigma-$inq) which cannot be used to bound $M_{x,y}$. That is why the claim in [1, Lemma 2.5(B)],

$$\lim_{n\to+\infty} \sigma(x_n, x_{n-1}) = 0 \Rightarrow \lim_{n\to+\infty} \sigma(x_n, x_n) = 0,$$

is incorrect. We present the counterexample below.

Example 2.4. *Consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ on a set $X = \{a,b\}$ such that*

$$x_n = \begin{cases} a & \text{if n is odd} \\ b & \text{if n is even.} \end{cases}$$

Let σ be an $M-$metric on X defined by

$$\sigma(a,a) = \sigma(a,b) = \sigma(b,a) = 0 \text{ and } \sigma(b,b) = 1.$$

For example,

$$\sigma(x_1,x_3) = \sigma(x_1,x_1) = \sigma(a,a) = 0 = \sigma(a,b) = \sigma(x_1,x_2) = \sigma(x_2,x_3).$$

On the other hand,

$$\sigma(x_2,x_4) = \sigma(x_2,x_2) = \sigma(b,b) = 1.$$

Therefore, for all i,

$$\sigma(x_{i+2}, x_{i+1}) \leq c\sigma(x_{i+1}, x_i),$$

satisfying the requirement of [1, Lemma 2.5]. It is clear that

$$\lim_{n\to+\infty} \sigma(x_n, x_{n-1}) = 0,$$

while $\lim_{n\to+\infty}\sigma(x_n,x_n)$ *does not exist, as it alternates between* 0 *and* 1.

For [1, Lemma 2.5(D)] and the fixed point theorems presented in [1] to hold, [1, Lemma 2.5(B)] is crucial. Therefore, the techniques used to prove the theorems found in [1] are no longer valid.

3. Partial Metric

As mentioned in §1, the $M-$metric was proposed to generalize the partial metric. In Definition 2.1, we expanded on the definition of an $M-$metric space found in [1] to allow negative values. Hence, our $M-$metric is a generalization of the partial metric as defined by S. J. O'Neill in [7].

Definition 3.1. A partial metric p on a set X is a function $p: X\times X\to\mathbb{R}$ satisfying the following axioms:

For all $x,y,z\in X$,

($p-$lbnd): $p(x,x)\le p(x,y)$;
($p-$sym): $p(x,y)=p(y,x)$;
($p-$sep): $p(x,x)=p(x,y)=p(y,y)\iff x=y$;
($p-$inq): $p(x,y)\le p(x,z)+p(z,y)-p(z,z)$.

Remark 3.2. Notice that ($p-$inq) self-regulates when $x=z$; i.e., for any arbitrary function $s: X\times X\to\mathbb{R}$,

$$s(x,y)=s(x,x)+s(x,y)-s(x,x).$$

For examples on partial metrics, we refer the reader to [2], [3], [6], and [7].

In [1], the authors show that any partial metric is an $M-$metric. Another approach to their proof is by using a well-known property which we present in Lemma 3.1.

Lemma 3.1. *Let* $(\Gamma,+,\le)$ *be an ordered commutative group. Then, for every* $\{a,b,c\}\in\Gamma$,

$$\min\{c,a\}+\min\{c,b\}\le c+\min\{a,b\} \text{ and}$$

$$c+\max\{a,b\}\le\max\{c,a\}+\max\{c,b\}.$$

Hence, for an $M-$metric σ on a set X, for every $x,y,z\in X$,

$$-\sigma(z,z)\le -m_{x,z}-m_{z,y}+m_{x,y} \text{ and}$$

$$-M_{x,z}-M_{z,y}+M_{x,y}\le-\sigma(z,z).$$

In Example 3.3, we slightly adapt [1, Example 1.1] to give an $M-$metric that is not a partial metric.

Example 3.3. *Consider the set $X = \mathbb{R}$. Let $\sigma : X \times X \to \mathbb{R}$ be defined by setting for all $x, y \in X$,*

$$\sigma(x, y) = x + y.$$

Then σ is an $M-$metric on X.

Proof. Except for ($\sigma-$inq), the proof of the other axioms is quite straightforward. Without loss of generality, assume $x \leq y \in X$. Then,

$$m_{x,y} = 2x \text{ and}$$

$$\sigma(x, y) - m_{x,y} = x + y - 2x = y - x.$$

Let $z \in X$ such that

Case 1. $x \leq y \leq z$. Then

$$[\sigma(x, z) - m_{x,z}] + [\sigma(z, y) - m_{z,y}]$$

$$= [z - x] + [z - y] \geq z - x \geq y - x = \sigma(x, y) - m_{x,y}.$$

Case 2. $x \leq z \leq y$. Then

$$[\sigma(x, z) - m_{x,z}] + [\sigma(z, y) - m_{z,y}]$$

$$= [z - x] + [y - z] = y - x = \sigma(x, y) - m_{x,y}.$$

Case 3. $z \leq x \leq y$. Then

$$[\sigma(x, z) - m_{x,z}] + [\sigma(z, y) - m_{z,y}]$$

$$= [x - z] + [y - z] \geq y - z \geq y - x = \sigma(x, y) - m_{x,y}.$$

Clearly, if $x < y \in X$, then

$$2x < x + y < 2y;$$

$$\text{i.e., } m_{x,y} < \sigma(x, y) < \sigma(y, y).$$

Hence, σ is not a partial metric. □

We now show that an $M-$metric on a set X induces a partial metric on X.

Theorem 3.2. *Let σ be an $M-$metric on a set X. As in Definition 2.1, we denote $M_{x,y} = \max\{\sigma(x, x), \sigma(y, y)\}$ and $m_{x,y} = \min\{\sigma(x, x), \sigma(y, y)\}$. For $x, y \in X$, let*

$$p^\sigma(x, y) = \sigma(x, y) + M_{x,y} - m_{x,y}.$$

Then p^σ is a partial metric on X.

Proof. For all $x \in X$, $p^{\sigma}(x,x) = \sigma(x,x) + M_{x,x} - m_{x,x} = \sigma(x,x)$.

($p-$sym): The proof is trivial.

($p-$lbnd): For all $x, y \in X$, from ($\sigma-$lbnd), we have $\sigma(x,y) - m_{x,y} \geq 0$. Hence,

$$p^{\sigma}(x,x) = \sigma(x,x) \leq M_{x,y} \leq M_{x,y} + \sigma(x,y) - m_{x,y} = p^{\sigma}(x,y).$$

($p-$sep): Assume that $p^{\sigma}(x,x) = p^{\sigma}(x,y) = p^{\sigma}(y,y)$. Then

$$\sigma(x,x) = p^{\sigma}(x,x) = p^{\sigma}(y,y) = \sigma(y,y);$$
$$\text{i.e., } \sigma(x,x) = \sigma(y,y) = m_{x,y} = M_{x,y}.$$

Therefore,

$$p^{\sigma}(x,y) = \sigma(x,y) + M_{x,y} - m_{x,y} = \sigma(x,y).$$

Hence, $\sigma(x,x) = \sigma(x,y) = \sigma(y,y)$ and, therefore, by ($\sigma-$sep), $x = y$.

($p-$inq): For all $x, y, z \in X$,

$$p^{\sigma}(x,y) = [\sigma(x,y) - m_{x,y}] + M_{x,y};$$

by ($\sigma-$inq),

$$\begin{aligned} &\leq [\sigma(x,z) - m_{x,z} + \sigma(z,y) - m_{z,y}] + M_{x,y} \\ &= p^{\sigma}(x,z) + p^{\sigma}(z,y) - M_{x,z} - M_{z,y} + M_{x,y}; \end{aligned}$$

by Lemma 3.1,

$$\begin{aligned} &\leq p^{\sigma}(x,z) + p^{\sigma}(z,y) - \sigma(z,z) \\ &= p^{\sigma}(x,z) + p^{\sigma}(z,y) - p^{\sigma}(z,z). \end{aligned}$$ □

As shown in Example 2.3, given σ, an $M-$metric on a set X,

$$\sigma^{\star}(x,y) = \sigma(x,y) - m_{x,y}$$

need not be a metric. However, we can guarantee that $\sigma^{\star}$ is a metric in the special case where σ is a partial metric.

Lemma 3.3. *Let σ be a partial metric on a set X. For all $x, y \in X$, let*

$$\sigma^{\star}(x,y) = \sigma(x,y) - m_{x,y}.$$

Then $\sigma^{\star}$ is a metric on X.

Proof. The major issue in Example 2.3 is the metric separation axiom. Since σ is a partial metric, and by ($p-$lbnd), for all $x, y \in X$

$$\sigma(x,y) - \sigma(x,x) \geq 0.$$

Therefore,

$$0 \leq \sigma(x,y) - \sigma(x,x) \leq \sigma(x,y) - m_{x,y} = \sigma^{\star}(x,y).$$

Hence, if $\sigma^{\star}(x,y) = 0$, then $\sigma(x,x) = \sigma(x,y) = \sigma(y,y)$, and by ($p-$sep), $x = y$. The rest of the axioms are straightforward and easy to check. □

4. Topology

Let σ be an $M-$metric on a set X. For every $x \in X$ and $\epsilon > 0$, an $A-$open ball is defined in [1] as

$$B_\epsilon^A(x) = \{y \in X | \sigma^\star(x,y) = \sigma(x,y) - m_{x,y} < \epsilon\}.$$

Therefore, for the special case of σ being a partial metric and from Lemma 3.3, the $A-$open balls span a metric space. Moreover, in [1, Theorem 2.1], the authors state that the topology generated by the $A-$open balls is not Hausdorff. This is a faux pas since a metric is an $M-$metric.

On the other hand, given a partial metric p on a set X, S. G. Matthews [6] defines the $p-$open ball as

$$B_\epsilon^p(x) = \{y \in X | p(x,y) - p(x,x) < \epsilon\}.$$

Matthews [6] also shows that the $p-$open balls span a T_0 topology that need not be T_1. We will call the $p-$open balls the standard partial metric balls. In [2], we show that the standard partial metric balls still work when allowing the partial metric to have negative values, i.e., taken in the sense of O'Neill [7].

Lemma 4.1. *Let σ be a partial metric on a set X. Then $\mathcal{T}_A$, the topology generated by B_ϵ^A balls, is finer than $\mathcal{T}_{\sigma^s}$, the standard partial metric topology generated by the B_ϵ^p balls.*

Proof. For all $x \in X$ and for each $y \in B_\epsilon^p(x)$, i.e., $\sigma(x,y) - \sigma(x,x) < \epsilon$, let

$$\delta = \epsilon - \sigma(x,y) + \sigma(x,x) > 0.$$

We show that $B_\delta^A(y) \subseteq B_\epsilon^p(x)$. If $z \in B_\delta^A(y)$, i.e., $\sigma(y,z) - m_{y,z} < \delta$, and using ($p-$inq) and -$\sigma(y,y) \leq -m_{y,z}$, we get

$$\begin{aligned}
\sigma(x,z) - \sigma(x,x) &\leq \sigma(x,y) + \sigma(y,z) - \sigma(y,y) - \sigma(x,x) \\
&\leq \sigma(x,y) - \sigma(x,x) + \sigma(y,z) - m_{y,z} \\
&< \sigma(x,y) - \sigma(x,x) + \delta = \epsilon.
\end{aligned}$$

And, hence, $B_\delta^A(y) \subseteq B_\epsilon^p(x)$; i.e., $\mathcal{T}_A$ is finer than $\mathcal{T}_{\sigma^s}$. □

Lemma 4.1 shows why $B_\epsilon^A(x)$ is not an optimal generalization of $B_\epsilon^p(x)$. In the special case where σ is a partial metric, $B_\epsilon^A(x)$ becomes a metric ball. Hence, in that case, the topology generated by $B_\epsilon^A(x)$ is much finer than the one generated by $B_\epsilon^p(x)$.

If an $M-$metric theory is to be developed as a generalization of the partial metric one, the topology proposed should not be finer than the standard partial metric topology. The $M-$open balls presented below accomplish just that. We define them and show that the collection of $M-$open balls form a basis.

Definition 4.1. Let σ be an $M-$metric on a set X. For every $x \in X$ and $\epsilon > 0$, the *$M-$open ball* around x of radius ϵ is

$$B_\epsilon^\sigma(x) = \{y \in X | \sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) < \epsilon\}.$$

Remark 4.2. We notice that $\sigma(x,x) - \sigma(x,x) + \sigma(x,x) - m_{x,x} = 0$; i.e., for every $\epsilon > 0$, $x \in B_\epsilon^\sigma(x)$. Additionally, if for some $x, y \in X$

$$m_{x,y} = \sigma(y,y) \leq \sigma(x,y) \leq \sigma(x,x),$$

then

$$\begin{aligned} &\sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) \\ &= [\sigma(x,y) - \sigma(x,x)] + [\sigma(y,y) - m_{x,y}] \\ &= \sigma(x,y) - \sigma(x,x) \leq 0. \end{aligned}$$

Hence, every $M-$open ball centered at x contains y.

Lemma 4.2. *Let σ be an $M-$metric on a set X. The collection of all $M-$open balls on X, $\mathcal{B}^\sigma = \{B_\epsilon^\sigma(x)\}_{x\in X}^{\epsilon>0}$ forms a basis on X.*

Proof. For every $x \in X$ and $\epsilon > 0$, let $y \in B_\epsilon^\sigma(x)$. Then,

$$\sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) < \epsilon.$$

Take

$$(\star) \qquad \delta = \epsilon - \sigma(x,y) - \sigma(y,y) + m_{x,y} + \sigma(x,x) > 0.$$

We claim that $B_\delta^\sigma(y) \subseteq B_\epsilon^\sigma(x)$. If $z \in B_\delta^\sigma(y)$, then

$$(\oplus) \qquad \sigma(y,z) + \sigma(z,z) - m_{y,z} - \sigma(y,y) < \delta.$$

Hence, by ($M-$inq) (see Definition 2.1),

$$\begin{aligned} &\sigma(x,z) + \sigma(z,z) - m_{x,z} - \sigma(x,x) \\ &= [\sigma(x,z) - m_{x,z}] + \sigma(z,z) - \sigma(x,x) \\ &\leq [\sigma(x,y) - m_{x,y} + \sigma(y,z) - m_{y,z}] + \sigma(z,z) - \sigma(x,x); \end{aligned}$$

by adding and subtracting $\sigma(y,y)$, we get

$$= [\sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x)] + [\sigma(y,z) + \sigma(z,z) - m_{z,z} - \sigma(y,y)].$$

By ($\oplus$) and ($\star$), we get

$$\sigma(x,z) + \sigma(z,z) - m_{x,z} - \sigma(x,x) < \sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) + \delta = \epsilon.$$

Therefore, $B_\delta^\sigma(y) \subseteq B_\epsilon^\sigma(x)$ and $\mathcal{B}^\sigma$ is a basis on X. $\square$

Notation 4.3. Given an $M-$metric σ on a set X, we denote by

$\underline{\mathcal{T}_\sigma}$ the topology generated by the $M-$open balls

$$B_\epsilon^\sigma(x) = \{y \in X | \sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) < \epsilon\};$$

$\underline{\mathcal{T}_{p^\sigma}}$ the standard partial metric topology spanned by the $p-$open balls

$$B_\epsilon^{p^\sigma}(x) = \{y \in X | p^\sigma(x,y) - p^\sigma(x,x) < \epsilon\},$$

where p^σ is the induced partial metric defined in Theorem 3.2; i.e., $B_\epsilon^{p^\sigma}(x) = \{y \in X | \sigma(x,y) + M_{x,y} - m_{x,y} - \sigma(x,x) < \epsilon\}$.

In the special case where σ is a partial metric, we denote by

$\underline{\mathcal{T}_{\sigma_s}}$ the standard partial metric topology generated by the $p-$open balls

$$B_\epsilon^{\sigma_s}(x) = \{y \in X | \sigma(x,y) - \sigma(x,x) < \epsilon\}.$$

We now move to comparing the topologies defined in Notation 4.3. Given an $M-$metric σ on a set X, we show in Lemma 4.3 that $\mathcal{T}_\sigma$ is coarser than $\mathcal{T}_{p^\sigma}$. In the special case where σ is a partial metric, we show in Lemma 4.4 that $\mathcal{T}_{\sigma_s} = \mathcal{T}_\sigma$. Lemma 4.3 and Lemma 4.4 shed light as to why we consider $\mathcal{T}_\sigma$ to be a proper generalization of $\mathcal{T}_{\sigma_s}$.

Lemma 4.3. *Let σ be an $M-$metric on a set X. Then $\mathcal{T}_\sigma$ is coarser than $\mathcal{T}_{p^\sigma}$.*

Proof. Consider the $M-$open ball $B_\epsilon^\sigma(x)$. Let $y \in B_\epsilon^{p^\sigma}(x)$. Then, from Notation 4.3,

$$\sigma(x,y) + M_{x,y} - m_{x,y} - \sigma(x,x) < \epsilon.$$

Hence,

$$\sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) \leq \sigma(x,y) + M_{x,y} - m_{x,y} - \sigma(x,x) < \epsilon.$$

Therefore, $B_\epsilon^{p^\sigma}(x) \subseteq B_\epsilon^\sigma(x)$ and, hence, $\mathcal{T}_\sigma \subseteq \mathcal{T}_{p^\sigma}$. □

Lemma 4.4. *Let σ be a partial metric on a set X. Then $\mathcal{T}_{\sigma_s} = \mathcal{T}_\sigma$.*

Proof. Consider the standard $p-$open ball $B_\epsilon^{\sigma_s}(x)$. Let $y \in B_\epsilon^\sigma(x)$. Then $\sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) < \epsilon$. Thus,

$$\sigma(x,y) - \sigma(x,x) \leq \sigma(x,y) - \sigma(x,x) + \sigma(y,y) - m_{x,y} < \epsilon.$$

Hence, $B_\epsilon^\sigma(x) \subseteq B_\epsilon^{\sigma_s}(x)$. Therefore, $\mathcal{T}_{\sigma_s} \subseteq \mathcal{T}_\sigma$.

Conversely, let $y \in B_{\frac{\epsilon}{2}}^{\sigma_s}(x)$. Then $\sigma(x,y) - \sigma(x,x) < \frac{\epsilon}{2}$.

Case 1. If $m_{x,y} = \sigma(x,x) \leq \sigma(y,y)$, then, from ($p-$lbnd) in Definition 3.1,

$$\sigma(y,y) \leq \sigma(x,y) \text{ and } m_{x,y} = \sigma(x,x).$$

Hence,

$$\begin{aligned}&\sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x)\\ &\leq \sigma(x,y) + \sigma(x,y) - \sigma(x,x) - \sigma(x,x) < 2(\frac{\epsilon}{2}) = \epsilon.\end{aligned}$$

Case 2. If $m_{x,y} = \sigma(y,y) \leq \sigma(x,x)$, then $\sigma(y,y) - m_{x,y} = 0$ and, hence,

$$\sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) = \sigma(x,y) - \sigma(x,x) < \frac{\epsilon}{2} < \epsilon.$$

Therefore, $B^{\sigma_s}_{\frac{\epsilon}{2}}(x) \subseteq B^{\sigma}_{\epsilon}(x)$ and, hence, $\mathcal{T}_\sigma = \mathcal{T}_{\sigma_s}$. $\square$

Notation 4.4. Let σ be an $M-$metric on a set X. We denote by
(X, σ) the $M-$metric space $(X, \mathcal{T}_\sigma)$;
(X, p^σ) the partial metric space $(X, \mathcal{T}_{p^\sigma})$.
We remind the reader that if σ is a partial metric, then the standard partial metric space $(X, \mathcal{T}_{\sigma_s}) = (X, \mathcal{T}_\sigma) = (X, \sigma)$.

All our work would be useless if, for every $M-$metric σ, $\mathcal{T}_\sigma = \mathcal{T}_{p^\sigma}$. We use the $M-$metric defined in Example 3.3 to give an example where $\mathcal{T}_\sigma \subsetneq \mathcal{T}_{p^\sigma}$.

Example 4.5. *Let σ be an $M-$metric on $X = \mathbb{R}$ as defined in Example 3.3 by*

$$\sigma(x,y) = x + y.$$

Then, $\mathcal{T}_\sigma \subsetneq \mathcal{T}_{p^\sigma}$.

Proof. We remind our reader that if $x \leq y \in X$, then

$$\sigma(y,y) = 2y \text{ and } \sigma(x,x) = m_{x,y} = 2x.$$

From Notation 4.3, $\mathcal{T}_\sigma$ is generated by the $M-$open balls

$$B^\sigma_\epsilon(x) = \{y \in X | \sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) < \epsilon\}.$$

If $y \leq x$, then $\sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) = y - x \leq 0 < \epsilon$.
If $x < y$, then $\sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) = 3(y - x)$. Therefore,

$$B^\sigma_\epsilon(x) = (-\infty, x + \frac{\epsilon}{3}).$$

Again from Notation 4.3, $\mathcal{T}_{p^\sigma}$ is generated by the $p-$open balls

$$B^{p^\sigma}_\epsilon(x) = \{y \in X | \sigma(x,y) + M_{x,y} - m_{x,y} - \sigma(x,x) < \epsilon\}.$$

If $y \leq x$, then $\sigma(x,y) + M_{x,y} - m_{x,y} - \sigma(x,x) = x - y$.
If $x < y$, then $\sigma(x,y) + M_{x,y} - m_{x,y} - \sigma(x,x) = 3(y - x)$. Therefore,

$$B^{p^\sigma}_\epsilon(x) = (x - \epsilon, x + \frac{\epsilon}{3}).$$

Clearly, $\mathcal{T}_\sigma \subsetneq \mathcal{T}_{p^\sigma}$. $\square$

Lemma 4.5. *An $M-$metric space is T_o.*

Proof. Let (X,σ) be an $M-$metric space with two distinct elements $x, y \in X$. Without loss of generality, we can consider two cases.

Case 1. If $\sigma(x,x) = \sigma(y,y)$, then by ($\sigma-$lbnd) and ($\sigma-$sep) (see Definition 3.1), and since $x \neq y$, we have

$$m_{x,y} = \sigma(x,x) = \sigma(y,y) < \sigma(x,y).$$

Hence,

$$\sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) = \sigma(x,y) - \sigma(x,x) > 0.$$

Therefore, if $\epsilon = \sigma(x,y) - \sigma(x,x)$, then $y \notin B^{\sigma}_{\epsilon}(x)$.

Case 2. If $\sigma(x,x) < \sigma(y,y)$, then by ($\sigma-$lbnd)

$$\sigma(x,y) - m_{x,y} \geq 0.$$

Hence,

$$\sigma(x,y) + \sigma(y,y) - m_{x,y} - \sigma(x,x) \geq \sigma(y,y) - \sigma(x,x) > 0.$$

Therefore, if $\epsilon = \sigma(y,y) - \sigma(x,x)$, then $y \notin B^{\sigma}_{\epsilon}(x)$.

Since a partial metric is an $M-$metric, we refer the reader to [2] and [6] for examples of $M-$metric spaces that need not be T_1. □

5. $r-$Cauchy Sequences and Limits

We begin this section by defining a Cauchy-like sequence. We use the same approach found in [2] and [3] and apply it to the $M-$metric case.

Definition 5.1. Let (X,σ) be an $M-$metric space and r a real number. A sequence $\{x_i\}_{i\in\mathbb{N}}$ in X is said to be *$r-$Cauchy* if and only if

$$\lim_{i,j\to+\infty} \sigma(x_i, x_j) = r.$$

r is called the *central distance* of $\{x_i\}_{i\in\mathbb{N}}$.

Remark 5.2. Alternatively, we could have defined an $r-$Cauchy sequence as

$$\lim_{i\neq j\to+\infty} \sigma(x_i, x_j) = \lim_{i,j\to+\infty} m_{x_i,x_j} = r.$$

This definition is closer to the one presented in [1], but since it has a subsequence $\{y_i\}_{i\in\mathbb{N}}$ such that

$$\lim_{i,j\to\infty} \sigma(y_i, y_j) = r,$$

we find that there is very little point in using a more general definition.

Lemma 5.1. *Let $\{x_i\}_{i\in\mathbb{N}}$ be an $r-$Cauchy sequence in an $M-$metric space (X,σ). Then*

(a) $\lim\limits_{i\to+\infty} \sigma(x_i, x_i) = r$;

(b) $\lim_{i,j\to+\infty} m_{x_i,x_j} = r$;

(c) $\lim_{i,j\to+\infty} M_{x_i,x_j} = r$.

Proof. The proof of Lemma 5.1 is quite straightforward: (a) follows trivially from Definition 5.1; (b) and (c) follow trivially from (a). □

The limit is a topological definition which we translate into the language of $M-$metric spaces.

Definition 5.3. Let $\{x_i\}_{i\in\mathbb{N}}$ be an $r-$Cauchy sequence in an $M-$metric space (X,σ). We say that $a \in X$ is a *limit* of $\{x_i\}_{i\in\mathbb{N}}$ if and only if

$$\lim_{i\to+\infty} \sigma(a,x_i) + \sigma(x_i,x_i) - m_{a,x_i} = \sigma(a,a).$$

The natural question to ask here is: How does r relate to $\sigma(a,a)$? This question has been answered in the partial metric case in [2]. The result remains the same in an $M-$metric case.

Lemma 5.2. *Let $\{x_i\}_{i\in\mathbb{N}}$ be an $r-$Cauchy sequence in an $M-$metric space (X,σ). If a is a limit of $\{x_i\}_{i\in\mathbb{N}}$, then*

$$r \leq \sigma(a,a).$$

Proof. By ($\sigma-$lbnd) (see Definition 2.1), we know that for each i,

$$0 \leq \sigma(a,x_i) - m_{a,x_i}.$$

Adding $\sigma(x_i,x_i)$ on both sides, we get

$$\sigma(x_i,x_i) \leq \sigma(a,x_i) + \sigma(x_i,x_i) - m_{a,x_i}.$$

Now taking the limit of both sides, by Lemma 5.1(a) and Definition 5.3, we get

$$r \leq \sigma(a,a).$$ □

The limit of an $r-$Cauchy sequence need not be unique. An example is given for the partial metric case in [2].

Reading through the partial metric literature (see, for example, [2], [3], [4], [5], [7]), it becomes obvious that a stronger version of a limit is needed. The $M-$metric space, being a generalization of a partial metric space, is no exception.

Definition 5.4. Let $\{x_i\}_{i\in\mathbb{N}}$ be an $r-$Cauchy sequence in an $M-$metric space (X,σ). An element $a \in X$ is a *special limit* of $\{x_i\}_{i\in\mathbb{N}}$ if and only if

$$a \text{ is a limit of } \{x_i\}_{i\in\mathbb{N}} \text{ and } \sigma(a,a) = r.$$

Unlike a regular limit, a special limit is unique.

Lemma 5.3. *Let $\{x_i\}_{i\in\mathbb{N}}$ be an $r-$Cauchy sequence in an $M-$metric space (X,σ). If a is a special limit of $\{x_i\}_{i\in\mathbb{N}}$, then a is unique.*

Proof. Let a and b be two special limits of $\{x_i\}_{i\in\mathbb{N}}$. From Definition 5.4,

$$r = \sigma(a,a) = \sigma(b,b) = m_{a,b}.$$

From ($\sigma-$lbnd) (see Definition 2.1), we know that

$$r = \sigma(a,a) = m_{a,b} \leq \sigma(a,b).$$

Hence,

$$\sigma(a,b) = [\sigma(a,b) - m_{a,b}] + r;$$

using ($\sigma-$inq), for all i,

$$\sigma(a,b) \leq [\sigma(a,x_i) - m_{a,x_i} + \sigma(b,x_i) - m_{b,x_i}] + r.$$

Therefore, by adding and subtracting $\sigma(x_i,x_i)$, we get

$$\sigma(a,b) \leq [\sigma(a,x_i) + \sigma(x_i,x_i) - m_{a,x_i}]$$
$$+[\sigma(b,x_i) + \sigma(x_i,x_i) - m_{b,x_i}] - 2\sigma(x_i,x_i) + r.$$

Since a special limit is also a limit (see Definition 5.4) and by Lemma 5.1,

$$\sigma(a,x_i) + \sigma(x_i,x_i) - m_{a,x_i} \to \sigma(a,a) = r,$$
$$\sigma(b,x_i) + \sigma(x_i,x_i) - m_{b,x_i} \to \sigma(b,b) = r, \text{ and}$$
$$\sigma(x_i,x_i) \to \sigma(a,a) = r.$$

Hence,

$$\sigma(a,b) \leq r + r - 2r + r = r = \sigma(a,a);$$

therefore,

$$\sigma(a,a) = \sigma(a,b) = \sigma(b,b).$$

By ($\sigma-$inq) (see Definition 2.1), $a = b$. □

Lemma 5.4. *Let $\{x_i\}_{i\in\mathbb{N}}$ be an $r-$Cauchy sequence in an $M-$metric space (X,σ). Let a be the special limit of $\{x_i\}_{i\in\mathbb{N}}$. Then*

(a) $\lim\limits_{i\to+\infty} M_{a,x_i} = \sigma(a,a)$;

(b) $\lim\limits_{i\to+\infty} m_{a,x_i} = \sigma(a,a)$; *and*

(c) $\lim\limits_{i\to+\infty} \sigma(a,x_i) = \sigma(a,a)$.

Proof. Parts (a) and (b) are straightforward.

For (c), using Definition 5.3, we get

$$\lim_{i\to+\infty} \sigma(a,x_i) + \sigma(x_i,x_i) - m_{a,x_i} = \sigma(a,a).$$

Hence, using Definition 5.4 and Lemma 5.1(b), we get

$$\lim_{i\to+\infty} \sigma(a,x_i) = \lim_{i\to+\infty} ([\sigma(a,x_i) + \sigma(x_i,x_i) - m_{a,x_i}] - \sigma(x_i,x_i) + m_{a,x_i})$$
$$= \sigma(a,a) - \sigma(a,a) + \sigma(a,a) = \sigma(a,a).$$ □

Lemma 5.5. *Let $\{x_i\}_{i\in\mathbb{N}}$ be an $r-$Cauchy sequence in an $M-$metric space (X,σ). If a is a special limit of $\{x_i\}_{i\in\mathbb{N}}$, then for every $y \in X$,*

$$\lim_{i\to+\infty} \sigma(y,x_i) - m_{y,x_i} = \sigma(y,a) - m_{y,a}.$$

Proof. The proof follows directly from ($\sigma-$lbnd) and Lemma 5.4. □

What is left is to guarantee the existence of a special limit. Therefore, we present the notion of completeness in an $M-$metric space.

Definition 5.5. An $M-$metric space (X,σ) is said to be *complete* if and only if for every real number r, every $r-$Cauchy sequence in X has a special limit in X.

6. $r-$Cauchy Functions

One of the cornerstones of Banach-like fixed point theorems is that the function f in question has a Cauchy-like orbit $\{f^i(x_o)\}_{i\in\mathbb{N}}$ for some $x_o \in X$.

Throughout the literature, different criteria on a function f were investigated for f to be an $r-$Cauchy function. Many of those cases boil down to two main ones which we present in Definition 6.1 and Definition 6.2.

Definition 6.1. Let (X,σ) be an $M-$metric space with $x_o \in X$. Suppose $f : X \to X$ is a function on X. We say that f is an *$r-$Cauchy function at x_o* if and only if $\{f^i(x_o)\}_{i\in\mathbb{N}}$ is an $r-$Cauchy sequence in X.

Definition 6.2. Let (X,σ) be an $M-$metric space with $x_o \in X$. Let $f : X \to X$ be a function on X. Let r and $0 < c < 1$ be two real numbers. We say that f is an *orbital c_r-contraction at x_o* (or f is *orbitally c_r-contractive at x_o*) if and only if for all natural numbers i,

$$r \le \sigma(f^{i+1}(x_o), f^{i+1}(x_0)) \le r + c^i|\sigma(f(x_o),x_o)|$$

and

$$\sigma(f^{i+2}(x_o), f^{i+1}(x_0)) \le r + c^{i+1}|\sigma(f(x_o),x_o)|.$$

Lemma 6.1. *Let (X,σ) be an $M-$metric space with $x_o \in X$. Let $f : X \to X$ be a function on X. Let r and $0 < c < 1$ be two real numbers. If f is an orbital c_r-contraction at x_o, then f is an $r-$Cauchy function at x_o.*

Proof. The proof is quite straightforward by first showing that

$$\lim_{i\to+\infty} \sigma(x_i,x_i) = r \text{ and, hence, } \lim_{i,j\to+\infty} m_{x_i,x_j} = r.$$

For a detailed similar proof, we refer the reader to [2, Lemma 6.2]. □

Definition 6.3. Let (X, σ) be an $M-$metric space with $x_o \in X$. Let $f : X \to X$ be a function on X. Let r be a real number and $\varphi : [r, +\infty) \subset \mathbb{R} \to [0, +\infty)$ be a non-decreasing function such that

$$\varphi(t) = 0 \text{ iff } t = r.$$

We say that f is an *orbital* φ_r-*contraction at* x_o if and only if for all i and j,

$$r \leq \sigma(f^{i+1}(x_o), f^{j+1}(x_o)) \leq \sigma(f^i(x_o), f^j(x_o)) - \varphi(\sigma(f^i(x_o), f^j(x_o)).$$

Lemma 6.2. *Let (X, σ) be an $M-$metric space with x_o in X. Let $f : X \to X$ be a function on X. If f is an orbital φ_r-contraction at x_o, then f is an $r-$Cauchy function at x_o.*

Proof. The proof of Lemma 6.2 is quite delicate. We will give it in its most explicit form while repeatedly clarifying any ambiguous notation.

Let $x_o \in X$ and suppose $f : X \to X$ is an orbital φ_r-contraction at x_o. Denote $x_i = f^i(x_o)$. To remedy any possible ambiguity, we will be adding parentheses to differentiate between $x_{(n_k+1)}$ and $x_{n_{(k+1)}}$ when the need arises.

Step 1: Let $t_i = \sigma(x_{i+1}, x_i)$. In this step, we will show that in the topological space $\mathbb{R}$ (endowed with the standard topology) $\{t_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence that converges to r.

From Definition 6.3, for all i,

$$\sigma(x_{i+1}, x_{i+1}) \leq \sigma(x_i, x_i) - \varphi(\sigma(x_i, x_i));$$

hence, $\{\sigma(x_i, x_i)\}_{i\in\mathbb{N}}$ forms a decreasing chain since, for all i, $\varphi(t_i) \geq 0$; i.e.,

$$m_{x_i, x_{i+1}} = \sigma(x_{i+1}, x_{i+1}).$$

Moreover, from $(\sigma-$lbnd$)$,

$$r \leq \sigma(x_{i+2}, x_{i+2}) = m_{x_{i+2}, x_{i+1}} \leq \sigma(x_{i+2}, x_{i+1}) = t_{i+1} \quad \text{and}$$

$$t_{i+1} = \sigma(x_{i+2}, x_{i+1}) \leq \sigma(x_{i+1}, x_i) - \varphi(\sigma(x_{i+1}, x_i)) = t_i - \varphi(t_i) \leq t_i.$$

Hence, for all i,

$$r \leq t_{i+1} \leq t_i;$$

i.e., $\{t_i\}_{i\in\mathbb{N}}$ is a non-increasing sequence in $\mathbb{R}$ bounded below by r and, therefore, $\{t_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ with the standard topology is a complete metric space, $\{t_i\}_{i\in\mathbb{N}}$ has a limit L such that for all i,

$$t_i \geq L \geq r,$$

and, since φ is a non-decreasing function,

$$\varphi(t_i) \geq \varphi(L) \geq \varphi(r) = 0;$$

$$\text{i.e., } -\varphi(t_i) \leq -\varphi(L) \leq 0.$$

Hence, by Definition 6.3,

$$\begin{aligned} r \leq t_{i+1} &\leq t_i - \varphi(t_i) \leq t_i - \varphi(L) \\ &\leq t_{i-1} - \varphi(t_{i-1}) - \varphi(L) \leq t_{i-1} - 2\varphi(L); \end{aligned}$$

by induction,

$$t_{i+1} \leq t_1 - i\varphi(L).$$

Assume that $L > r$. Then, by Definition 6.3, $\varphi(L) > 0$. By taking $i > \frac{t_1 - r}{\varphi(L)}$, we get

$$t_{i+1} \leq t_1 - i\varphi(L) < t_1 - \frac{t_1 - r}{\varphi(L)}\varphi(L) = r,$$

a contradiction, since $t_i \geq r$. Therefore,

$$(\bar{\bigcirc}) \qquad \lim_{i \to +\infty} t_i = \lim_{i \to +\infty} \sigma(x_i, x_{i+1}) = \lim_{i,j \to +\infty} m_{x_i, x_j} = r.$$

Step 2: We now show that $\{x_i\}_{i \in \mathbb{N}}$ is an $r-$Cauchy sequence by supposing that it is not (a contrapositive approach).

Suppose that $\{x_i\}_{i \in \mathbb{N}}$ is not an $r-$Cauchy sequence. Since $r \leq \sigma(x_i, x_j)$, there exists a positive real number δ such that, for every natural number N, there exists $i, j > N$ where

$$\sigma(x_i, x_j) \geq r + \delta > r.$$

From step 1, by choosing N big enough, for all $i > N$,

$$r \leq \sigma(x_i, x_i) = m_{x_{i-1}, x_i} \leq \sigma(x_i, x_{i-1}) < r + \delta.$$

Then there exist $j_1 > h_1 > N$ such that

$$\sigma(x_{h_1}, x_{j_1}) \geq r + \delta > r.$$

Let n_1 be the smallest number with $n_1 > h_1$ and

$$\sigma(x_{h_1}, x_{n_1}) \geq r + \delta.$$

Note that

$$\sigma(x_{h_1}, x_{(n_1 - 1)}) < r + \delta.$$

There exist $j_2 > h_2 > n_1$ such that

$$\sigma(x_{h_2}, x_{j_2}) \geq r + \delta > r.$$

Let n_2 be the smallest number with $n_2 > m_2$ and

$$\sigma(x_{h_2}, x_{n_2}) \geq r + \delta.$$

Then

$$\sigma(x_{h_2}, x_{(n_2 - 1)}) < r + \delta.$$

Continuing this process, we build two increasing sequences in $\mathbb{N}$, $\{h_k\}_{k \in \mathbb{N}}$ and $\{n_k\}_{k \in \mathbb{N}}$, such that for all k,

$$(\nabla) \qquad \sigma(x_{h_k}, x_{(n_k - 1)}) < r + \delta \leq \sigma(x_{h_k}, x_{n_k})$$

and

$$m_{x_{h_k},x_{n_k}} = \sigma(x_{n_k}, x_{n_k}).$$

For all k, denote $s_k = \sigma(x_{h_k}, x_{n_k})$. We should note that

$$\sigma(x_{(h_k+1)}, x_{(n_k+1)}) \neq s_{k+1} = \sigma(x_{h_{(k+1)}}, x_{n_{(k+1)}}),$$

but rather, from Definition 6.2,

$$(\otimes) \quad \sigma(x_{(h_k+1)}, x_{(n_k+1)}) \leq \sigma(x_{h_k}, x_{n_k}) - \varphi(\sigma(x_{h_k}, x_{n_k})) = s_k - \varphi(s_k).$$

Therefore, for all $k > N$ (N defined in the beginning of step 2),

$$r + \delta \leq s_k = \sigma(x_{h_k}, x_{n_k}) = [\sigma(x_{h_k}, x_{n_k}) - m_{x_{h_k},x_{n_k}}] + m_{x_{h_k},x_{n_k}};$$

by ($\sigma-$inq),

$$\leq [\sigma(x_{h_k}, x_{(n_k-1)}) - m_{x_{h_k},x_{n_k-1}} + \sigma(x_{(n_k-1)}, x_{n_k}) - m_{x_{(n_k-1)},x_{n_k}}] + m_{x_{h_k},x_{n_k}};$$

by ($\triangledown$) and step 1,

$$\leq r + \delta - r + t_{(n_k-1)} - r + m_{x_{h_k},x_{n_k}}.$$

Hence, taking $k \to +\infty$ by ($\bar{\bigcirc}$) and step 1,

$$r + \delta \leq \lim_{k\to+\infty} s_k \leq r + \delta - r + r - r + r = r + \delta.$$

We have just shown that there exists $\delta > 0$ such that

$$\lim_{k\to+\infty} s_k = r + \delta.$$

Next we prove that $\delta = 0$, giving us our contradiction. By applying ($\sigma-$inq), we get

$$\begin{aligned} s_k &= \sigma(x_{h_k}, x_{n_k}) = [\sigma(x_{h_k}, x_{n_k}) - m_{x_{h_k},x_{n_k}}] + m_{x_{h_k},x_{n_k}} \\ &\leq [\sigma(x_{h_k}, x_{(n_k+1)}) - m_{x_{h_k},x_{(n_k+1)}} + \sigma(x_{(n_k+1)}, x_{n_k}) - m_{x_{(n_k+1)},x_{n_k}}] \\ &+ m_{x_{h_k},x_{n_k}} \leq \sigma(x_{h_k}, x_{(n_k+1)}) - m_{x_{h_k},x_{(n_k+1)}} + t_{n_k} - r + m_{x_{h_k},x_{n_k}}. \end{aligned}$$

Using ($\sigma-$inq) again on $\sigma(x_{h_k}, x_{(n_k+1)}) - m_{x_{h_k},x_{(n_k+1)}}$, we get

$$\begin{aligned} s_k &\leq \sigma(x_{h_k}, x_{(h_k+1)}) - m_{x_{h_k},x_{(h_k+1)}} + \sigma(x_{(h_k+1)}, x_{(n_k+1)}) \\ &- m_{x_{(h_k+1)},x_{(n_k+1)}} + t_{n_k} - r + m_{x_{h_k},x_{n_k}}. \end{aligned}$$

Therefore, by ($\otimes$) and step 1,

$$s_k \leq t_{h_k} - r + s_k - \varphi(s_k) - r + t_{n_k} - r + m_{x_{h_k},x_{n_k}};$$

$$\text{i.e., } 0 \leq \varphi(s_k) \leq t_{h_k} - r + -r + t_{n_k} - r + m_{x_{h_k},x_{n_k}}.$$

Taking the limit as $k \to +\infty$, we get

$$0 \leq \lim_{k\to\infty} \varphi(s_k) \leq r - r - r + r - r + r = 0.$$

Hence, and since φ is non-decreasing with $r + \delta \leq s_k$,

$$0 \leq \varphi(r + \delta) \leq \lim_{k \to +\infty} \varphi(s_k) = 0;$$

i.e., $r + \delta = r$ and, therefore, $\delta = 0$, a clear contradiction. Therefore, the assumption considered at the beginning of step 2 is incorrect, proving that $\{x_i\}_{i \in \mathbb{N}}$ is an $r-$Cauchy sequence. □

7. Continuity and Non-Expansiveness

Definition 7.1. Let (X, σ) be an $M-$metric space with $x_o \in X$. A function $f : X \to X$ is *weakly orbitally continuous at x_o* if and only if

if a is the special limit of $\{f^i(x_o)\}_{i \in \mathbb{N}}$, then $f(a)$ is a limit of $\{f^i(x_o)\}_{i \in \mathbb{N}}$.

Remark 7.2. Notice that $f(a)$ is not required to be a special limit of $\{f^i(x_o)\}_{i \in \mathbb{N}}$, but rather only required to be its limit.

Lemma 7.1. *Let (X, σ) be an $M-$metric space with $x_o \in X$. Let $f : X \to X$ be a weakly orbitally continuous function at x_o. If a is a special limit of $\{f^i(x_o)\}_{i \in \mathbb{N}}$, then*

$$m_{a,f(a)} = \sigma(a, a) \leq \sigma(a, f(a)) \leq \sigma(f(a), f(a)).$$

Proof. Denote for all natural numbers i, $x_i = f^i(x_o)$. Since a is a special limit of $\{f^i(x_o)\}_{i \in \mathbb{N}}$ and f is weakly orbitally continuous at x_o, then $f(a)$ is a limit of $\{f^i(x_o)\}_{i \in \mathbb{N}}$. Therefore, by Lemma 5.2,

$$\sigma(a, a) \leq \sigma(f(a), f(a)),$$

and, hence, by ($\sigma-$lbnd),

$$m_{a,f(a)} = \sigma(a, a) \leq \sigma(a, f(a)).$$

Furthermore, for all i,

$$\sigma(a, f(a)) = [\sigma(a, f(a)) - m_{a,f(a)}] + \sigma(a, a);$$

by ($\sigma-$inq),

$$\leq [\sigma(a, x_i) - m_{a,x_i} + \sigma(x_i, f(a)) - m_{x_i,f(a)}] + \sigma(a, a);$$

by adding and subtracting $\sigma(x_i, x_i)$,

$$= \sigma(a, x_i) - m_{a,x_i} + \underline{\underline{\sigma(x_i, f(a)) + \sigma(x_i, x_i) - m_{x_i,f(a)}}} - \sigma(x_i, x_i) + \sigma(a, a);$$

taking the limit as $i \to +\infty$,

$$= \sigma(a, a) - \sigma(a, a) + \underline{\underline{\sigma(f(a), f(a))}} - \sigma(a, a) + \sigma(a, a) = \sigma(f(a), f(a)).$$ □

In Lemma 7.1, and by ($\sigma-$sep), for the special limit a to be a fixed point of f, we need $\sigma(f(a), f(a)) \leq \sigma(a, a)$. This can be obtained in various ways. In this paper we discuss two: the first is non-expansiveness and the second is the space having $\sigma(f(a), f(a))$ as a lower bound.

Definition 7.3. Let (X,σ) be an $M-$metric space. Let $f : X \to X$ be a function on X. We say that f is *non-expansive* if and only if for all $x, y \in X$,

$$\sigma(f(x), f(y)) \leq \sigma(x, y).$$

Definition 7.4. Let (X,σ) be an $M-$metric space and r_o a real number. We say (X,σ) is *bounded below by* r_o if and only if for all $x, y \in X$,

$$r_o \leq \sigma(x, y).$$

8. Main Theorem and Corollaries

We now present our main theorem. The reader will notice that we tried, as much as possible, to state it in its most general form.

Theorem 8.1. *Let (X,σ) be an $M-$metric space with $x_o \in X$. Let $f : X \to X$ be a function such that f is $r-$Cauchy at x_0 with special limit $a \in X$. Further assume that at least one of the following conditions holds:*

(1) *f is weakly orbitally continuous at x_0 and non-expansive.*
(2) *f is weakly orbitally continuous at x_0 and (X,σ) is bounded below by $\sigma(f(a), f(a))$.*
(3) *f is non-expansive and (X,σ) is bounded below by $\sigma(a, a)$.*

Then a is a fixed point of f.

Proof. In (1) and (2), since f is weakly orbitally continuous at x_o, then by Lemma 7.1,

$$\sigma(a, a) \leq \sigma(a, f(a)) \leq \sigma(f(a), f(a)).$$

Both (1) f is non-expansive and (2) (X,σ) is bounded below by $\sigma(f(a), f(a))$ assert $\sigma(f(a), f(a)) \leq \sigma(a, a)$. Therefore,

$$\sigma(a, a) = \sigma(a, f(a)) = \sigma(f(a), f(a)),$$

and, hence, by $(\sigma-\text{sep})$, $f(a) = a$.

As for (3), f is non-expansive and (X,σ) is bounded below by $\sigma(a, a)$ assert that $\sigma(f(a), f(a)) = \sigma(a, a)$ and, hence, for all i,

$$(\ominus) \qquad \begin{aligned} \lim_{i\to+\infty} m_{f(a),f(x_i)} &= \lim_{i\to+\infty} m_{a,x_i} \\ &= \sigma(a, a) = \sigma(f(a), f(a)) = m_{f(a),a}. \end{aligned}$$

Therefore, by $(\sigma-\text{lbnd})$,

$$\sigma(a, a) = m_{a,f(a)} \leq \sigma(a, f(a)).$$

By $(\sigma-\text{inq})$, for all i,

$$\sigma(f(a), a) - m_{f(a),a} \leq \sigma(f(a), f(x_i)) - m_{f(a),f(x_i)} + \sigma(a, f(x_i)) - m_{a,f(x_i)}.$$

Hence, by non-expansiveness,

$$\sigma(f(a),a) \leq m_{f(a),a} + \sigma(a,x_i) - m_{f(a),f(x_i)} + \sigma(a,f(x_i)) - m_{a,f(x_i)}.$$

Using ($\ominus$) and by taking $i \to +\infty$, we get

$$\sigma(f(a),a) \leq \sigma(a,a) + \sigma(a,a) - \sigma(a,a) + \sigma(a,a) - \sigma(a,a) = \sigma(a,a).$$

Therefore, by ($\sigma-$sep), $a = f(a)$ □

Both Lemma 6.1 and Lemma 6.2 assert that under their respective conditions, f is an $r-$Cauchy sequence. Unfortunately, Theorem 8.1 presents us with the conditions to obtain a fixed point without guaranteeing its uniqueness. We now present a valid proof of [1, Theorem 3.1 and Theorem 3.2] in Corollary 8.2 and Corollary 8.3, respectively.

We remind our reader that the definition presented for the $M-$metric in [1] restricts σ to non-negative values. This section is presented with that premise in mind.

Corollary 8.2. *Let (X,σ) be a complete $M-$metric space. Let $f : X \to X$ be a continuous function satisfying the following condition: There exists $0 \leq k < 1$ such that for all $x, y \in X$,*

$$(\circ) \qquad 0 \leq \sigma(f(x), f(y)) \leq k\sigma(x,y).$$

Then f has a unique fixed point.

Proof. Consider any arbitrary $x_o \in X$. The function f is φ_0-contractive at x_o (φ_r with $r = 0$) where $\varphi(t) = (1-k)t$. Hence, using Lemma 6.2, f is a $0-$Cauchy function at x_o. Since (X,σ) is complete, let a be the special limit of $\{f^i(x_o)\}_{i\in\mathbb{N}}$. Since f is continuous, f is weakly orbitally continuous at x_o. Additionally, ($\circ$) also asserts that f is non-expansive. Therefore, by Theorem 8.1(1), the special limit a is a fixed point. Now to prove uniqueness. Assume that a and b are both fixed points of f. Hence, by ($\circ$),

$$\sigma(a,a) = \sigma(f(a),f(a)) \leq k\sigma(a,a) < \sigma(a,a),$$
$$\sigma(b,b) = \sigma(f(b),f(b)) \leq k\sigma(b,b) < \sigma(b,b), \text{ and}$$
$$\sigma(a,b) = \sigma(f(a),f(b)) \leq k\sigma(a,b) < \sigma(a,b).$$

Therefore,

$$\sigma(a,a) = \sigma(b,b) = \sigma(a,b) = 0,$$

and, hence, by ($\sigma-$sep), $a = b$. □

Corollary 8.3. *Let (X,σ) be a complete $M-$metric space. Let $f : X \to X$ be a continuous function satisfying the following condition: There exists $0 \leq k < \frac{1}{2}$ such that for all $x, y \in X$,*

$$(\triangle) \qquad 0 \leq \sigma(f(x), f(y)) \leq k[\sigma(x, f(x)) + \sigma(y, f(y))].$$

Then f has a unique fixed point.

Proof. Consider any arbitrary $x_o \in X$ and denote $x_i = f^i(x_o)$. We first show that f is a c_0-contraction at x_o (c_r with $r = 0$) and $c = 2k > \frac{k}{1-k}$.

By ($\triangle$), we get for every i,

$$\sigma(x_{i+1}, x_{i+1}) \leq 2k\sigma(x_{i+1}, x_i).$$

Moreover, for every i,

$$\sigma(x_{i+2}, x_{i+1}) \leq k(\sigma(x_{i+1}, x_{i+2}) + \sigma(x_i, x_{i+1}));$$

$$\text{i.e., } \sigma(x_{i+2}, x_{i+1}) \leq \frac{k}{1-k}\sigma(x_{i+1}, x_i) < c\sigma(x_{i+1}, x_i).$$

Hence,

$$0 \leq \sigma(x_{i+2}, x_{i+2}) < \sigma(x_{i+2}, x_{i+1}) \leq c^{i+1}\sigma(x_1, x_o),$$

and by Lemma 6.1, f is a $0-$Cauchy function at x_o. Since (X, σ) is complete, $\{x_i\}_{i\in\mathbb{N}}$ has a special limit a. Hence, by Definition 5.4, $\sigma(a, a) = 0$.

The function f is continuous and, hence, weekly orbitally continuous at x_o. Therefore, by Lemma 7.1, we have

$$m_{a,f(a)} = \sigma(a, a) \leq \sigma(a, f(a)) \leq \sigma(f(a), f(a)).$$

Additionally, by ($\triangle$),

$$\sigma(f(a), f(a)) \leq 2k\sigma(a, f(a)) \leq 2k\sigma(f(a), f(a)).$$

Hence, $\sigma(f(a), f(a)) = 0$, completing the requirement for Theorem 8.1(2). As for uniqueness, assume both a and b are fixed points of f. Hence, by ($\triangle$),

$$\sigma(a, b) = \sigma(f(a), f(b)) \leq 2k\sigma(a, b) \text{ and}$$

$$\sigma(a, a) = \sigma(f(a), f(a)) \leq 2k\sigma(a, a);$$

similarly,

$$\sigma(b, b) = \sigma(f(b), f(b)) \leq 2k\sigma(b, b).$$

Therefore,

$$\sigma(a, a) = \sigma(b, b) = \sigma(a, b) = 0,$$

and, by ($\sigma-$sep), $a = b$. $\square$

References

[1] Mehdi Asadi, Erdal Karapinar, and Peyman Salimi, *New extension of p-metric spaces with some fixed-point results on M-metric spaces*, J. Inequal. Appl. **2014** (2014), no. 1, Article 18.

[2] Samer Assaf and Koushik Pal, *Partial metric spaces with negative distances and fixed point theorems*, Topology Proc. **49** (2017), 19–40.

[3] Michael Bukatin, Ralph Kopperman, Steve Matthews, and Homeira Pajoohesh, *Partial metric spaces*, Amer. Math. Monthly **116** (2009), no. 8, 708–718.

[4] Erdal Karapinar and İnci M. Erhan, *Fixed point theorems for operators on partial metric spaces*, Appl. Math. Lett. **24** (2011), no. 11, 1894–1899.

[5] Erdal Karapinar and Salvador Romaguera, *Nonunique fixed point theorems in partial metric spaces*, Filomat **27** (2013), no. 7, 1305–1314.

[6] S. G. Matthews, *Partial metric topology*, Ann. New York Acad. Sci. **728**, (1994), no. 1, 183–197.

[7] S. J. O'Neill, *Partial metrics, valuations, and domain theory*, Ann. New York Acad. Sci. **806** (1996), no. 1, 304–315.

Department of Mathematics and Natural Sciences; American University of Kuwait; Salem Mubarak Street; Salmiya, Kuwait

E-mail address: sassaf@auk.edu.kw