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TOPOLOGY PROCEEDINGS

Volume 54, 2019

Pages 103–108

http://topology.nipissingu.ca/tp/

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by

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Electronically published on November 16, 2018

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/
TOPOLOGY
PROCEEDINGS
Volume 54 (2019)
Pages 103-108
E-Published on November 16, 2018

LUSTERNIK–SCHNIRELMANN CATEGORY OF THE CONFIGURATION SPACE OF COMPLEX PROJECTIVE SPACE

CESAR A. IPANAQUE ZAPATA

Abstract. The Lusternik–Schnirelmann category $cat(X)$ is a homotopy invariant which is a numerical bound on the number of critical points of a smooth function on a manifold. Another similar invariant is the topological complexity $TC(X)$ (a la Farber) which has interesting applications in robotics, specifically, in the robot motion planning problem. In this paper we calculate the Lusternik–Schnirelmann category and, as a consequence, we calculate the topological complexity of the two-point ordered configuration space of $\mathbb{CP}^n$ for every $n \geq 1$.

1. Introduction

The *ordered configuration space* of k distinct points of a topological space X (see [4]) is the subset

$$F(X,k) = \{(x_1, \ldots, x_k) \in X^k \mid \ x_i \neq x_j \text{ for all } i \neq j\}$$

topologized as a subspace of the Cartesian power X^k. This space has been used in robotics to try to avoid collisions when one controls multiple objects simultaneously [6].

The first definition of category, given by L. Lusternik and L. Schnirelmann [9], was a consequence of an investigation to obtain numerical

2010 *Mathematics Subject Classification.* Primary 55R80, 55M30; Secondary 55T10.

Key words and phrases. complex projective space, configuration space, Lusternik–Schnirelmann category, topological complexity.

The author wishes to acknowledge support for this research from FAPESP 2016/18714-8.

bounds for the number of critical points of a smooth function on a manifold.

Here we follow a definition of category, one greater than that given in [3]. We say that the *Lusternik–Schnirelmann category* or *category* of a topological space X, denoted $cat(X)$, is the least integer m such that X can be covered with m open sets, which are all contractible within X. One of the basic properties of $cat(X)$ is its homotopy invariance [3, Theorem 1.30].

Proposition 1.1 below gives the general lower and upper bound of the category of a space X.

Proposition 1.1. (1) [11, Proposition 2.1(5)] *If X is an $(n-1)$-connected CW-complex, then*

$$cat(X) \leq \frac{dim(X)}{n} + 1.$$

(2) [3, Theorem 1.5] *Let R be a commutative ring with unit and X be a space. We have*

$$1 + cup_R(X) \leq cat(X)$$

where $cup_R(X)$ is the least integer n such that all $(n+1)$-fold cup products vanish in the reduced cohomology $\widetilde{H^\star}(X;R)$.

On the other hand, we recall the definition of topological complexity (see [5] for more details). The *topological complexity* (TC) of a path-connected space X is the least integer m such that the Cartesian product $X \times X$ can be covered with m open subsets U_i,

$$X \times X = U_1 \cup U_2 \cup \cdots \cup U_m,$$

such that for any $i = 1, 2, \ldots, m$ there exists a continuous function $s_i : U_i \longrightarrow PX$, $\pi \circ s_i = id$ over U_i. If no such m exists we will set $TC(X) = \infty$. Where PX denotes the space of all continuous paths $\gamma : [0,1] \longrightarrow X$ in X and $\pi : PX \longrightarrow X \times X$ denotes the map associating to any path $\gamma \in PX$, the pair of its initial and end points $\pi(\gamma) = (\gamma(0), \gamma(1))$. Equip the path space PX with the compact-open topology.

The central motivating result of this paper is the Lusternik–Schnirelmann category of the configuration space of two distinct points in complex projective n-space for all $n \geq 1$.

Theorem 1.2. *For $n \geq 1$,*

$$cat(F(\mathbb{CP}^n, 2)) = 2n.$$

As an application, we have the following statement.

Corollary 1.3. For $n \geq 1$,

$$TC(F(\mathbb{CP}^n, 2)) = 4n - 1.$$

2. Proof

In this section we prove Theorem 1.2 and Corollary 1.3. We begin with two lemmas needed for our proofs.

Lemma 2.1. *For $n \geq 1$,*

$$H_q(F(\mathbb{CP}^n, 2); \mathbb{Z}) = \begin{cases} \mathbb{Z}^{\oplus(\frac{q}{2}+1)}, & q = 0, 2, 4, \cdots, 2(n-1); \\ \mathbb{Z}^{\oplus(2n-\frac{q}{2})}, & q = 2n, 2n+2, \cdots, 2n+2(n-1); \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. By the Leray–Serre spectral sequence [10, Theorem 5.4] of the fibration $F(\mathbb{CP}^n, 2) \longrightarrow \mathbb{CP}^n,\ (x, y) \mapsto x$ with fibre $\mathbb{CP}^{n-1}$ ([4, Theorem 1], we have the E^2-term

$$E^2_{p,q} = H_p(\mathbb{CP}^n; \mathbb{Z}) \otimes H_q(\mathbb{CP}^{n-1}; \mathbb{Z})$$

and all those differentials are zero (see Figure 1). So Lemma 2.1 follows. □

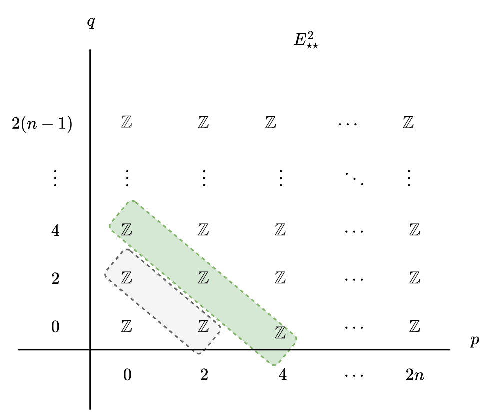

Figure 1. E^2-term.

We recall that $F(\mathbb{CP}^n, 2)$ is simply connected since $\mathbb{CP}^n$ and $\mathbb{CP}^{n-1}$ are. By [8, Proposition 4C.1], we have the following corollary.

Corollary 2.2. The configuration space of complex projective space $F(\mathbb{CP}^n, 2)$ has the homotopy type of a CW complex which has $j+1$ $2j$-cells ($j = 0, 1, \ldots, n-1$), and $2n - j$ $2j$-cells ($j = n, n+1, n+2, \ldots, n+(n-1)$). In particular, $F(\mathbb{CP}^n, 2)$ has the homotopy type of a $2(2n-1)$-dimensional finite CW complex.

The multiplicative structure of the cohomological algebra of the configuration space $F(\mathbb{C}P^n, 2)$ is given in [12, Theorem 2]:

$$H^\star(F(\mathbb{CP}^n,2);\mathbb{C}) = \frac{\mathbb{C}[a_1,a_2]}{\langle r_n(a_1,a_2); a_1^{n+1}; a_2^{n+1}\rangle},$$

where $deg(a_1) = deg(a_2) = 2$ and $r_n(x,y) = x^n + x^{n-1}y + \cdots + y^n$. Thus, we can conclude $a_1^n a_2^n = 0$ and $a_1^{n-1}a_2^n \neq 0$, since $a_1^n a_2^n = r_n(a_1,a_2)a_2^n = 0$ and $a_1^{n-1}a_2^n$ is a unique (up to sign) generator of $H^{4n-2} = \mathbb{C}$.

Lemma 2.3. $cup_{\mathbb{C}}(F(\mathbb{CP}^n,2)) = 2n-1$.

Proof. We just have to note that $a_1^n a_2^n = 0$ and $a_1^{n-1}a_2^n \neq 0$. □

Proof of Theorem 1.2. Using Corollary 2.2, Lemma 2.3, and Proposition 1.1, the proof follows. □

Proof of Corollary 1.3. Since $F(\mathbb{CP}^n,2)$ is path-connected and paracompact, the inequality

$$TC(F(\mathbb{CP}^n,2)) \leq 4n-1$$

follows from Theorem 1.2 and [5, Theorem 5]. On the other hand, $1\otimes a_1 - a_1\otimes 1$ and $1\otimes a_2 - a_2 \otimes 1 \in H^\star(F(\mathbb{CP}^n,2);\mathbb{C})\otimes H^\star(F(\mathbb{CP}^n,2);\mathbb{C})$ are zero-divisors whose $(2n-1)^{\text{th}}$ power

$$\begin{aligned}(1\otimes a_1 - a_1\otimes 1)^{2n-1} &= pa_1^{n-1}\otimes a_1^n + qa_1^n \otimes a_1^{n-1};\\ (1\otimes a_2 - a_2\otimes 1)^{2n-1} &= pa_2^{n-1}\otimes a_2^n + qa_2^n \otimes a_2^{n-1},\end{aligned}$$

where $p = (-1)^{n-1}\binom{2n-1}{n-1}$ and $q = (-1)^n\binom{2n-1}{n}$.

Thus, we have that

$$(1\otimes a_1 - a_1\otimes 1)^{2n-1}(1\otimes a_2 - a_2\otimes 1)^{2n-1} = 2p^2 a_1^{n-1}a_2^n \otimes a_1^{n-1}a_2^n$$

does not vanish. The opposite inequality

$$TC(F(\mathbb{CP}^n,2)) \geq 4n-1$$

now follows from [5, Theorem 7]. □

Remark 2.4. Corollary 1.3 in the case $n = 1$ was also calculated by Daniel C. Cohen and Michael Farber in [2, Theorem A].

Remark 2.5. Theorem 1.2 shows that the configuration space $F(\mathbb{CP}^n,2)$ satisfies Ganea's conjecture because $cat(F(\mathbb{CP}^n,2)) = cup_{\mathbb{C}}(F(\mathbb{CP}^n,2)) + 1$.

Remark 2.6. By [7, Corollary 3.2], we have

$$TC(M) = dim(M) + 1 \tag{2.1}$$

when M is a closed simply connected symplectic manifold. Corollary 1.3 shows that (2.1) for noncompact cases does not hold.

Remark 2.7. We will compare the result stated in Corollary 1.3 with the topological complexity of the Cartesian product $\mathbb{CP}^n \times \mathbb{CP}^n$. By [7, Corollary 3.2], we have

$$TC(\mathbb{CP}^n \times \mathbb{CP}^n) = 4n + 1.$$

Thus, on the complex projective space $\mathbb{CP}^n$, the complexity of the collision-free motion planning problem for two robots is *less* complicated than the complexity of the similar problem when the robots are allowed to collide. This example also provides an illustration of the fact that the concept $TC(X)$ reflects only the *topological* complexity, which is just a part of the *total* complexity of the problem.[1]

Remark 2.8. We note that the configuration space $F(\mathbb{CP}^n, 2)$ is the space of all lines in the complex projective space $\mathbb{CP}^n$ since two points in $\mathbb{CP}^n$ generate a subspace of dimension 1. More generally, in [1] the ordered configuration space $F(\mathbb{CP}^n, k)$ has a stratification with complex submanifolds as follows:

$$F(\mathbb{CP}^n, k) = \coprod_{i=1}^{n} F^i(\mathbb{CP}^n, k),$$

where $F^i(\mathbb{CP}^n, k)$ is the ordered configuration space of all k points in $\mathbb{CP}^n$ generating a subspace of dimension i.

Remark 2.9. There is no discussion of what might happen for more than two points. Thus, it is interesting to calculate the TC for the ordered configuration space $F(\mathbb{CP}^n, k)$ when $k \geq 3$. In general, calculate the TC for the ordered configuration space $F(V, k)$ where V is a smooth complex projective variety.

Acknowledgments. I am very grateful to Jesús González and my advisor Denise de Mattos for their comments and encouraging remarks which were of invaluable mental support. I also would like to thank the referee for very helpful suggestions.

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