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by

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NOTES ON LINEARLY H-CLOSED SPACES AND OD-SELECTION PRINCIPLES

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ABSTRACT. A space is called *linearly H-closed* if and only if any chain cover possesses a dense member. This property lies strictly between feeble compactness and H-closedness. While regular H-closed spaces are compact, there are non-compact linearly H-closed spaces which are even collectionwise normal and Fréchet–Urysohn. We give examples in other classes and ask whether there is a first countable normal linearly H-closed non-compact space in ZFC. We show that PFA implies a negative answer if the space is moreover either locally separable or both locally compact and locally ccc. An Ostaszewski space (built with $\diamondsuit$) is an example which is even perfectly normal. We also investigate Menger-like properties for the class of od-covers, that is, covers whose members are open and dense.

1. INTRODUCTION

This note is mainly about a property (to our knowledge not investigated before) we decided to call linear H-closedness, which lies strictly between H-closedness and feeble compactness. Since it came up while investigating simple instances of od-selection properties (see below), and all have a common "density of open sets" flavor, we include a section about this latter topic although they are not related more than on a superficial level.

By "space" we mean "topological space." We take the convention that "regular" and "normal" imply "Hausdorff." A *cover* of a space always means a cover by open sets, and a cover is a *chain cover* if it is linearly ordered by the inclusion relation. In any Hausdorff space (of cardinality

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at least 2), each point has a non-dense neighborhood, and, thus, the space has the property of possessing a cover by open non-dense sets, but the chain-generalization of this property may fail.

Definition 1.1. A space X is *linearly H-closed* if and only if any chain cover has a member which is dense in X (or, equivalently, if and only if any chain cover has a finite subfamily with a dense union).

Recall that a Hausdorff space any of whose covers has a finite subfamily with a dense union is called *H-closed*, whence the name "linearly H-closed." While H-closed regular spaces are compact (see [22, Corollary 4.8(c)] for a simple proof), there are plenty of Tychonoff linearly H-closed non-compact spaces, perhaps the most simple being the Tychonoff plank (see Example 2.8). We will give examples in various classes, such as first countable, normal, collectionwise normal, etc., but while there are consistent examples of non-compact, perfectly normal, first countable, linearly H-closed spaces, we were unable to determine whether a non-compact, first countable, normal, linearly H-closed space exists in ZFC alone. A partial result is that PFA prevents such a space from existing if it is moreover either locally separable or both locally compact and locally ccc (see Theorem 2.13). These results are contained in §2.

In §3, we investigate Menger-like properties for *od-covers* of topological spaces, that is, covers whose members are open and dense. In our short study, we show, in particular, that the class of non-compact spaces satisfying $\mathsf{U}_{fin}(\mathcal{O},\Delta)$ does contain some Hausdorff spaces but no regular spaces, and that a separable space satisfies $\mathsf{U}_{fin}(\Delta,\mathcal{O})$ if and only if it satisfies $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$, where Δ is the class of od-covers. We defer the definition of $\mathsf{U}_{fin}(\mathcal{A},\mathcal{B})$ until §3. Research on selection principles (such as Menger-like properties) currently flourishes and sees an impressive flow of new results (see, for instance, [23] and [26] for surveys about recent activity in the field). Since the author is not an expert on the subject and admits to feeling a bit lost in its numerous subtleties, we shall content ourselves with a humble introduction to the class of od-covers and derive only basic properties.

For convenience, we now give a grouped definition: the *(od-)[linear-] Lindelöf number* $(odL(X))$ $[\ell L(X)]$ $L(X)$ of a space X is the smallest cardinal κ such that any (od-)[chain] cover of X has a subcover of cardinality $\leq \kappa$. A space is *od-compact* if and only if any od-cover has a finite subcover, and we define similarly *od-Lindelöf*, *linearly-Lindelöf*, etc. We do not assume separation axioms in any of these properties. It happens that the od-Lindelöf number and the Lindelöf number almost always coincide; the only exception is when the space contains a "big" clopen discrete subspace. See §3 (especially Theorem 3.1) for details and remarks about the ignorance of past results. For the information of the reader, we note that our definitions above of $odL(X)$, $\ell L(X)$, and $L(X)$, are different from the ones we found convenient to give in [1], where, for example, $L(\mathbb{R}) = \omega_1$, not ω.

2. Linearly H-Closed Spaces

In this section, each space is assumed to be Hausdorff, even though that property is not needed for every assertion, and we will repeat the assumption often (for clarity). Any chain cover possesses a subcover indexed by a regular cardinal and, for simplicity, we will always use such indexing. It is immediate that the continuous image of a linearly H-closed space is linearly H-closed. Our first lemma is almost trivial.

Lemma 2.1. *A space is linearly H-closed if and only if any infinite cover of it has a subfamily of strictly smaller cardinality with a dense union.*

Proof. Given a chain cover indexed by a regular cardinal, a subcover of strictly smaller cardinality is contained in some member, so the latter implies the former. If X is linearly H-closed, given a cover $\{U_\alpha : \alpha \in \kappa\}$, then the sets $V_\alpha = \cup_{\beta<\alpha} U_\beta$ form a chain cover and some V_α is dense. □

It is well known that a space is H-closed if and only if any open filter base on X (that is, a filter base containing only open subsets of X) has an adherent point. See, for instance, (the proof of) [22, Proposition 4.8(b),(2) $\Leftrightarrow$ (3)]. The referee pointed out to us that a similar result holds for linear H-closedness. By a *chain filter base* we mean an open filter base which is linearly ordered by the inclusion relation. The proof we just mentioned can be easily adapted to show the following.

Lemma 2.2. *A space X is linearly H-closed if and only if any chain filter base on X has an adherent point.*

Likewise, the following result (also suggested by the referee) can be proved as in [22, Proposition 4.8(e)].

Lemma 2.3. *If X is linearly H-closed and U is open, then $\overline{U}$ is linearly H-closed.*

However, not every closed subset of a linearly H-closed space is linearly H-closed; see, for instance, Example 2.8. Linear H-closedness is linked to other generalized compactness properties, as seen in Figure 1, where plain straight arrows denote implications that hold for Hausdorff spaces (and most of them for any space), while additional properties (for instance, those written on their side) are needed for those denoted by dotted curved arrows.

Recall that a space is *feebly compact* if and only if every locally finite family of open sets is finite. This turns out to be equivalent to "every locally finite cover is finite" and to "every countable cover of X has a finite subfamily with a dense union" (see [22, Theorem 1.11(b)]). The term "feebly compact" is due to Mardešić and Papić (see [25, p. 902]). A space is *pseudocompact* if and only if any continuous real valued function on it is bounded. All implications in Figure 1 are classical except linearly H-closed $\longrightarrow$ feebly compact and its converse whose proofs are given in Lemma 2.4. An example of condition (*) is given in the statement of the lemma.

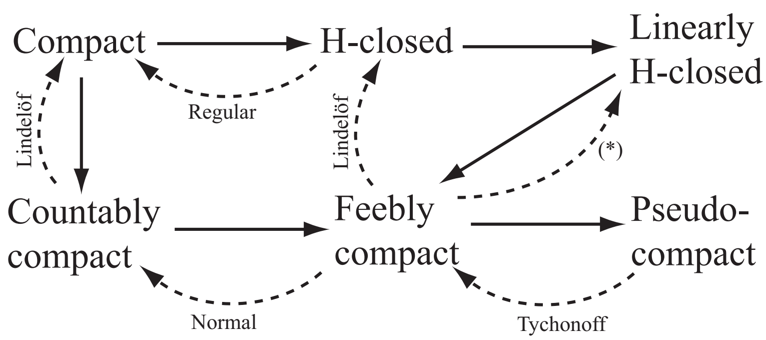

FIGURE 1. Some implications for Hausdorff spaces.

We decided to state this lemma in an almost absurd amount of generality, so we need some definitions. The good news is that more readable corollaries do follow quite easily. Given an infinite cardinal κ, a space is *initially κ-[linearly] Lindelöf* if and only if any open [chain] cover of cardinality $\leq \kappa$ has a countable subcover. Notice that any space is initially ω-[linearly] Lindelöf. The *weak Lindelöf number* $wL(X)$ of a space X is the least cardinal κ such that any open cover of X has a subfamily of cardinality $\leq \kappa$ whose union is dense. Notice that if $Y \subset X$ is dense, then $wL(X) \leq wL(Y)$ and if Y is feebly compact, then so is X.

Lemma 2.4. (1) *A linearly H-closed space is feebly compact.*

(2) *Let X be a Hausdorff space, $Y \subset X$ be dense in X, and κ be an infinite cardinal. Assume that $wL(X) \leq \kappa$ and that Y is both initially κ-linearly Lindelöf and feebly compact. Then X is linearly H-closed.*

Proof. (1) Given a countable cover $\mathcal{U} = \{U_n : n \in \omega\}$ of a linearly H-closed X, set $V_n = \cup_{m \leq n} U_m$. Then V_n is dense for some n, and the result follows.

(2) Let $\mathcal{U} = \{U_\alpha : \alpha \in \lambda\}$ be an infinite chain cover of X, with λ a regular cardinal. Assume first that $\lambda \leq \kappa$. Thus, there is a countable subfamily that covers Y, and then some U_α is dense in it by feeble compactness. It follows that U_α is dense in X as well. Now, suppose that $\lambda > \kappa$. Since $wL(X) \leq \kappa$, there is some subfamily of cardinality $\leq \kappa < \lambda$ whose union is dense in X and, by regularity of λ, its union is contained in some U_α. □

A case not covered by this lemma is the following easy fact.

Lemma 2.5. *Let X be a Hausdorff space containing a dense feebly compact linearly Lindelöf subspace Y. Then X is linearly H-closed.*

Proof. Given a chain cover of Y, linear Lindelöfness gives a countable subcover and then feeble compactness gives a finite subfamily of the subcover which contains a dense member. □

For a cardinal κ, a space is *κ-cc* (or *ccc* if $\kappa = \omega$) if and only if any disjoint collection of open sets has cardinality at most κ. A space with a dense subset of cardinality κ is obviously κ-cc.

Corollary 2.6. *If X is Hausdorff and possesses a dense feebly compact ccc subspace Y, then X is linearly H-closed.*

Proof. It is well known that a κ-cc space has a weak Lindelöf number $\leq \kappa$; hence, $wL(Y) \leq \omega$. Invoking the vacuousness of the definition, Y is also initially ω-Lindelöf, and the conditions of Lemma 2.4(2) are thus fulfilled. □

Recall that a space is *perfect* if and only if any closed subset is a G_δ.

Corollary 2.7. *Let X be a feebly compact regular perfect space. Then X is first countable and linearly H-closed.*

Proof. In [10, p. 378, (b)], Irving Glicksberg (using different terminology) provides a proof that a G_δ point in a regular feebly compact space has a countable neighborhood base. For another proof, see [21, Lemma 2.2]. Moreover, [21, Lemma 2.3] shows that if each closed set in X is a G_δ, then X is ccc. □

We can use Lemma 2.4 to obtain simple examples.

Example 2.8. There are linearly H-closed Tychonoff spaces of arbitrarily high weak Lindelöf number and cellularity.

Details. A very classical example is the Tychonoff plank of a regular cardinality. Let us recall the construction and its properties for convenience. Fix a regular cardinal κ. Let X be the subspace of the product

$(\kappa^+ + 1) \times (\kappa + 1)$ obtained by removing the point $\{\langle \kappa^+, \kappa \rangle\}$. Each ordinal is given the order topology.

As a subspace of a compact space, X is Tychonoff. The cellularity of X is at least κ^+ since $\{\alpha\} \times (\kappa+1)$ for successor $\alpha \in \kappa^+$ is a disjoint collection of open subsets. The cover $\{\alpha \times (\kappa+1) : \alpha \in \kappa^+\} \cup \{(\kappa^+ + 1) \times \beta : \beta \in \kappa\}$ shows that $wL(X) \geq \kappa$. Since $(\kappa^+ + 1) \times \kappa$ is the union of κ compact sets and is dense in X, $wL(X) \leq \kappa$. Recall that κ^+ with the order topology is initially κ-compact, and so is its product with the compact space $(\kappa + 1)$ (see, e.g., [24, Theorem 2.2]). Thus, the dense subset $Y = \kappa^+ \times (\kappa+1)$ is, in particular, feebly compact and initially κ-Lindelöf. This implies that X is linearly H-closed by Lemma 2.4(2). $\square$

Of course, these spaces are not first countable. Let us give more elaborate examples. All are "classical" spaces which happen to be linearly H-closed. In the following, we refer to [27] for the definitions of the "small" uncountable cardinals $\mathfrak{p}$ and $\mathfrak{b}$, but recall that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{b} \leq 2^{\aleph_0}$ and that each inequality may be strict. The diamond axiom $\diamondsuit$ implies the continuum hypothesis CH and is defined in any book on set theory.

Example 2.9. There are linearly H-closed non-compact spaces with the following additional properties:
(a) First countable, Tychonoff, Lindelöf number ω_1. [3, Example 1]
(b) First countable, locally compact (and thus Tychonoff), perfect. [9, Exercise 5I] and [15]
(c) ($\mathfrak{p} = \omega_1$) First countable, locally compact, normal. [7]
(d) ($\diamondsuit$) First countable, locally compact, perfectly normal. [20]
(e) Frechet–Urysohn, collectionwise normal. [Folklore]

Details. Linear H-closedness follows from Corollary 2.6 in each case except (b) where Corollary 2.7 is used.

(a) Murray G. Bell [3, Example 1] constructed a first countable countably compact ccc (non-separable) Tychonoff space X. Since X is an increasing union of $\aleph_1$-many compact spaces, it has Lindelöf number ω_1.

(b) The space Ψ, due independently to J. Isbell [9, Exercise 5I] and S. Mrówka [15], is first countable, perfect, Tychonoff, and feebly compact. This space is not countably compact, and thus non normal.

(c) S. P. Franklin and M. Rajagopalan [7] introduced a class of spaces called $\gamma\mathbb{N}$ spaces, which consist of a dense discrete countable set to which is "attached" a copy of ω_1 in such a way that the space is locally compact and normal, with various additional properties depending on how the attachment is done. In particular, their Example 1.4 is countably compact under CH. The constructions were later simplified and generalized by various authors including van Douwen, P. Nyikos, and Vaughan, and a version of $\gamma\mathbb{N}$ which is countably compact and first countable can be built if and only if $\mathfrak{p} = \omega_1$. For more details, see, for instance, Nyikos's account in [16, Theorem 2.1 and Example 3.4] and [18]).

(d) The celebrated space attributed to A. J. Ostaszewski [20] is a first countable, perfectly normal, hereditarily separable, countably compact, locally compact, non-compact space built with $\diamondsuit$.

(e) The sigma-product of 2^{ω_1}, i.e., the subspace of the compact space 2^{ω_1} where at most countably many coordinates have value 1, is collectionwise normal, Frechet–Urysohn, countably compact, and ccc (see, for instance, Henno Brandsma's answer on the MathOverflow question [5]). □

More than ZFC is necessary for the construction in (c); see Theorem 2.14 below. Bell's space in (a) cannot be shown to be locally compact in ZFC by Theorem 2.15. It is also not separable, and no separable regular example with Lindelöf number ω_1 can be found in ZFC, as the next lemma shows.

Lemma 2.10. *A first countable separable linearly H-closed Hausdorff space of Lindelöf number $< \mathfrak{p}$ is H-closed (and thus compact if regular).*

Notice the similarity with the fact (proved in [12]) that a regular separable countably compact space of Lindelöf number $< \mathfrak{p}$ is compact.

Proof. A first countable separable space has countable π-weight, as easily seen. Since X is linearly H-closed, it is feebly compact. A feebly compact space with countable π-weight and Lindelöf number $< \mathfrak{p}$ is H-closed ([21, Lemma 3.1]). □

Likewise, Example 2.9(d) cannot be constructed in $\mathsf{ZFC} + \mathsf{CH}$ alone.

Lemma 2.11. *It is consistent with* ZFC *(and even with* $\mathsf{ZFC} + \mathsf{CH}$*) that a perfectly normal linearly H-closed space is compact. In particular, it follows from* $\mathsf{MA} + \neg\mathsf{CH}$.

Proof. A linearly H-closed normal space is countably compact; William Weiss [28] shows that $\mathsf{MA} + \neg\mathsf{CH}$ implies that a countably compact regular perfect space is compact, and Todd Eisworth [6] shows that this latter result is compatible with CH. □

Question 2.12. Is there a normal first countable linearly H-closed non-compact space in ZFC?

The following theorem is a partial answer.

Theorem 2.13. (PFA) *Let X be a normal linearly H-closed space. If either* (a) *X is countably tight and locally separable, or* (b) *X is first countable, locally compact, and locally ccc, then X is compact.*

Our use of PFA is indirect. Indeed, we need only two of its classical consequences. Recall that PFA implies $\mathsf{MA} + \neg\mathsf{CH}$.

Theorem 2.14 ([2, Corollary 2])**.** (PFA) *Every separable, normal, countably tight, countably compact space is compact.*

Theorem 2.15 ([11])**.** ($\mathsf{MA} + \neg\mathsf{CH}$) *Every first countable, locally compact, ccc space is separable.*

Proof of Theorem 2.13. (a) If X is not compact, we will show that it is possible to define open subsets $U_\alpha \subset X$ for each $\alpha < \omega_1$, such that $\overline{U_\beta} \subsetneqq U_\alpha$ whenever $\beta < \alpha$. Then $Y = \cup_{\alpha<\omega_1} U_\alpha$ is a clopen subset of X. Openness is immediate. To see that it is closed, notice that given a point $x \in \overline{Y}$ by countable tightness, there is a countable subset of Y having x in its closure. But a countable subset of Y is contained in some U_α, so $x \in \overline{U_\alpha} \subset U_{\alpha+1} \subset Y$. It follows that X is not linearly H-closed, since no member of the chain cover $\{(X - Y) \cup U_\alpha : \alpha \in \omega_1\}$ is dense in X.

To find U_α, we proceed by induction. Each will be a separable open subset of X. Let U_0 be any such open separable subset. Assume that U_β is defined for each $\beta < \alpha$. Recall that by normality and linear H-closedness X is countably compact. Thus, $Z = \overline{\cup_{\beta<\alpha} U_\beta}$, being separable, is compact by Theorem 2.14. If $Z = X$, then X is compact. Otherwise, choose a point $x \notin Z$, cover $\{x\} \cup Z$ by open separable sets, and take the union of a finite subcover to obtain a separable U_α properly containing Z. In particular, $\overline{U_\beta} \subset U_\alpha$ for all $\beta < \alpha$. This defines U_α for each $\alpha < \omega_1$ with the required properties.

(b) We proceed as in (a), defining U_α to be ccc with compact closure. The successor stages are the same. If α is limit, then $\overline{\cup_{\beta<\alpha} U_\beta}$, having a dense ccc subspace, is ccc. By Theorem 2.15, it is separable under $\mathsf{MA} + \neg\mathsf{CH}$ and thus compact under PFA. $\square$

Remark 2.16. In [16, Theorem 5.4], Nyikos seems to indicate that there are models of $\mathsf{MA} + \neg\mathsf{CH}$ or even PFA^- with separable, locally compact, locally countable, countably compact, countably tight normal spaces, but we do not know to which spaces this assertion refers. The referee kindly informed us that the preprint [19], where these spaces were probably described, was never published.

We now briefly investigate how far a first countable linearly H-closed space is from being sequentially compact and show in Lemma 2.17 below that there are restrictions on the Lindelöf number. (The result seems well known; see the remarks before Problem 359 in [27], but we include the proof for completeness.) We first need some vocabulary. A collection of subsets of X is a *discrete collection* if each point of X possesses a neighborhood intersecting at most one member of the collection. This implies that given any subcollection, the union of the closures of its members is closed. A space satisfies the condition wD if, given any infinite closed discrete subspace D of X, there is an infinite $D' \subset D$ which expands to a discrete collection of open sets; that is, for each $x \in D'$, there is an open $U_x \ni x$ such that $\{U_x : x \in D'\}$ is a discrete collection.

Lemma 2.17. *A regular, first countable, feebly compact space either is countably compact or has Lindelöf number $\geq \mathfrak{b}$.*

Proof. Let X be a regular first countable space whose Lindelöf number is $< \mathfrak{b}$ and suppose that it is not countably compact. Thus, let $\{x_n \in X : n \in \omega\}$ be an infinite closed discrete subset. A regular first countable space with Lindelöf number $< \mathfrak{b}$ satisfies wD (see [17, Proposition 3.6 and Theorem 3.7]). Thus, let $E \subset \omega$ be infinite and $U_n \ni x_n$ $(n \in E)$ be open such that $\{\overline{U_n} : n \in E\}$ is discrete. In particular, $\{U_n : n \in E\}$ is an infinite locally finite family of open sets, which is impossible in a feebly compact space. □

We close this section with two results due to the referee who kindly gave us permission to include them in this note. First, notice that by continuity of the projections, if a product of spaces is linearly H-closed, then each factor space is linearly H-closed. But the converse may fail.

Proposition 2.18. *There is a linearly H-closed space G such that $G \times G$ is not linearly H-closed.*

Proof. It is well known (for example, see [9, Example 9.15]) that there exists a subspace G of the Stone–Čech compactification of the integers $\beta\omega$ such that $\omega \subset G$ (hence, G is separable), and G, but not $G \times G$, is feebly compact. Thus, G is linearly H-closed by Corollary 2.6, while $G \times G$ is not. □

However, the following holds.

Proposition 2.19. *If X is H-closed and Y is linearly H-closed, then $X \times Y$ is linearly H-closed.*

Proof. We use the characterization of linear H-closedness given by Lemma 2.2. Let $\mathcal{U}$ be a chain filter base on $X \times Y$. Since the projection on the Y factor π_Y is open, $\{\pi_Y(U) : U \in \mathcal{U}\}$ is a chain filter base on Y and hence has an adherent point $y \in Y$. Let

$$\mathcal{P} = \{(X \times W) \cap U : U \in \mathcal{U} \text{ and } W \text{ is an open neighborhood of } y\}.$$

Then $\mathcal{P}$ is an open filter base, and hence $\{\pi_X(V) : V \in \mathcal{P}\}$ must have an adherent point $x \in X$. For every neighborhood $V \subset X$ and $W \subset Y$ of x and y, respectively, and every $U \in \mathcal{U}$, we have by construction $U \cap (V \times W) \neq \varnothing$. It follows that $\langle x, y\rangle$ is an adherent point of $\mathcal{U}$. □

3. Od-Selection Properties

No separation axiom is assumed in this section. Allow us first a remark about the od-Lindelöf number. The author proves in [1] that a T_1 space is od-compact if and only if the subspace of non-isolated points is compact, and that a T_1 space with od-Lindelöf number $\leq \kappa$ either has a closed discrete subset of cardinality $> \kappa$, or $\ell L(X) \leq \kappa$ whenever κ is regular. We made the remark that since the methods were elementary, it would not be a surprise if similar results we were unaware of had appeared elsewhere. It was indeed the case: C. F. Mills and E. Wattel [14] have shown that a T_1 space without isolated points with $odL(X) \leq \kappa$ satisfies $L(X) \leq \kappa$ as well, which is much stronger (the compact case is actually due to Miroslav Katětov in 1947 [13]). Robert L. Blair [4] later improved their proof. (Both papers actually deal with $[\kappa, \lambda]$-compactness.) We show below that a very small modification of Blair's proof yields the following.

Theorem 3.1 ([14] and [4]). *Let κ be an infinite cardinal. Let X be a T_1 space with $odL(X) \leq \kappa$. Then either X contains a clopen discrete subset of cardinality $> \kappa$, or $L(X) \leq \kappa$. Moreover, the subspace of non-isolated points of X has Lindelöf number $\leq \kappa$.*

Proof. We follow Blair's proof. First, it is easy to see that a space has od-Lindelöf number $\leq \kappa$ if and only if any closed nowhere dense subset has Lindelöf number $\leq \kappa$. Let $\mathcal{U}$ be an open cover of X. Let $\mathcal{W}$ be a maximal family of disjoint open sets such that each member of $\mathcal{W}$ is contained in a member of $\mathcal{U}$. Then $\cup\mathcal{W}$ is dense. We may thus cover $X - \cup\mathcal{W}$ by a subfamily $\mathcal{V} \subset \mathcal{U}$ of cardinality $\leq \kappa$. Take one point in each member of $\mathcal{W}$ which is not entirely covered by $\cup\mathcal{V}$. This defines a closed discrete subset of $D \subset X$. Let $D_0 = \{d \in D : d \text{ is isolated in } X\}$. Then $D - D_0$ is nowhere dense and hence of cardinality at most κ. It follows that at most κ members of $\mathcal{U}$ cover $\cup\{W_d : d \in D - D_0\}$, where W_d is the unique member of $\mathcal{W}$ containing $d \in D$. The uncovered part of X is now contained in $\cup\{W_d : d \in D_0\}$. Then either $|D_0| \leq \kappa$, in which case we add $\leq \kappa$ members of $\mathcal{U}$ to complete the subcover, or $|D_0| > \kappa$ and X contains a clopen discrete subset of cardinality $> \kappa$. The "moreover" part follows easily from, e.g., [1, Lemma 4.8]. $\square$

For other results in the same spirit, see [8]. Let us now turn to selections properties. In what follows, $\mathcal{O}$ and Δ mean the collection of covers and od-covers, respectively, of some topological space, which will be clear from the context. Recall that a cover is an *od-cover* if and only if every member is dense. Given collections $\mathcal{A}$ and $\mathcal{B}$ of covers of a space X, we define the following property:

$\mathsf{U}_{fin}(\mathcal{A},\mathcal{B})$: For each sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of members of $\mathcal{A}$ which do not have a finite subcover, there are finite $\mathcal{F}_n \subset \mathcal{U}_n$ such that $\{\cup\mathcal{F}_n : n \in \omega\} \in \mathcal{B}$.

Recall that the classical Menger property is (equivalent to) $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$, and that

$$\sigma\text{-compact} \longrightarrow \mathsf{U}_{fin}(\mathcal{O},\mathcal{O}) \longrightarrow \text{ Lindelöf.}$$

3.1. The property $\mathsf{U}_{fin}(\mathcal{O},\Delta)$.

Let us first show the following simple lemma.

Lemma 3.2. *The following equivalences hold for any space X.*

(a) *Lindelöf and linearly H-closed $\longleftrightarrow$ Lindelöf and H-closed,*
(b) $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$*and linearly H-closed* $\longleftrightarrow \mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$ *and H-closed* $\longleftrightarrow \mathsf{U}_{fin}(\mathcal{O},\Delta)$.

Moreover, the properties in (b) *imply those in* (a).

Proof. (a) Immediate by Lemma 2.1.

(b) The leftmost equivalence follows from (a) by Lindelöfness. Let us prove the rightmost equivalence. For the direct implication, let $\langle\mathcal{U}_n\rangle$ be a sequence of covers, and let $\mathcal{F}_n \subset \mathcal{U}_n$ be finite such that $\{\cup\mathcal{F}_n : n \in \omega\}$ is a cover of X. By H-closedness, we can choose finite $\mathcal{G}_n \subset \mathcal{U}_n$ such that $\cup\mathcal{G}_n$ is dense. Taking $\mathcal{F}_n \cup \mathcal{G}_n$ yields the result. For the converse implication, $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$ trivially holds. We prove that X is linearly H-closed and use the leftmost equivalence to obtain H-closedness. Suppose that there is a chain cover $\mathcal{U} = \{U_n : n \in \omega\}$ without any dense member. A finite union of members of $\mathcal{U}$, being contained in a member of $\mathcal{U}$, is therefore not dense; taking $\mathcal{U}_n = \mathcal{U}$ for all $n \in \omega$ gives a sequence of open covers violating $\mathsf{U}_{fin}(\mathcal{O},\Delta)$.

The "moreover" part is immediate since $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O}) \longrightarrow$ Lindelöf. □

The situation is then very simple for regular spaces.

Proposition 3.3. *The following properties are equivalent for regular spaces.*
(a) *Lindelöf and linearly H-closed,*
(b) $\mathsf{U}_{fin}(\mathcal{O},\Delta)$,
(c) *compact.*

Proof. (b) $\to$ (a) by Lemma 3.2 and (c) $\to$ (b) is trivial. Since a regular H-closed space is compact, (a) $\to$ (c) follows again by Lemma 3.2. □

We will show that both (a) $\to$ (b) and (b) $\to$ (c) may fail for Hausdorff spaces; that is, we shall exhibit Hausdorff examples of Lindelöf (linearly) H-closed spaces which do not satisfy $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$ and non-compact spaces which satisfy $\mathsf{U}_{fin}(\mathcal{O},\Delta)$. Recall that a space Y is an *extension* of the space X if and only if Y contains a copy of X which is dense in Y. H-closed extensions of Hausdorff spaces are well studied; see, for instance, [22]. The examples we describe below are very similar to the ones given in Chapter 7 of this book. They can be seen as modifications of the half disk topology.

Let X be a space equipped with two topologies τ and ρ. Denote by $\widehat{X}(\tau,\rho)$ the space whose underlying set is $X\times[0,1]$ topologized as follows. The topology on $X\times(0,1]$ is the product topology of τ and the usual metric topology on $(0,1]$. Neighborhoods of $\langle x,0\rangle$ are then defined to be $U\times\{0\}\sqcup V\times(0,a)$ for $U\in\rho$ and $V\in\tau$ with $x\in U\cap V$, and $0<a\leq 1$.

Lemma 3.4. *Assume $\tau\subset\rho$; that is, ρ is finer than τ.*
(1) *If X is Hausdorff for τ (and thus for ρ), then so is $\widehat{X}(\tau,\rho)$.*
(2) *If X is H-closed for τ, then $\widehat{X}(\tau,\rho)$ is H-closed.*
(3) *X is Lindelöf for ρ if and only if $\widehat{X}(\tau,\rho)$ is Lindelöf.*
(4) *If X is first countable for both τ and ρ, then so is $\widehat{X}(\tau,\rho)$.*

Proof. Denote by $\tau\times\mu$ the product topology of τ on X and the usual metric topology μ on $[0,1]$. Notice that the topology on $\widehat{X}(\tau,\rho)$ is finer than $\tau\times\mu$ since $\tau\subset\rho$.

(1) Immediate since $\tau\times\mu$ is Hausdorff.

(2) A direct proof is not difficult, but let us give a more general argument suggested by the referee. Since the property H-closed is known to be productive (see, e.g., [22, Proposition 4.8(l)]), it follows that $Z=X\times[0,1]$, with topology $\tau\times\mu$, is an H-closed extension space of $Y=X\times(0,1]$. $\widehat{X}(\tau,\rho)$ is also an extension of Y with the same underlying set as Z and a finer topology. Moreover, $\widehat{X}(\tau,\rho)$ and Z have the same neighborhood filter trace on Y in the sense of [22, Definition 7.1(a)]. Then [22, Proposition 7.1(h) and (i)] imply that $\widehat{X}(\tau,\rho)$ is H-closed.

(3) The necessity is obvious since X with the topology ρ is a closed subspace of $\widehat{X}(\tau,\rho)$. For the sufficiency, assume that X is Lindelöf for ρ. Then X is Lindelöf for τ as well. Since $X\times(0,1]$ with topology $\tau\times\mu$ is the product of a Lindelöf space and a σ-compact space, it is Lindelöf. It follows that $\widehat{X}(\tau,\rho)=X\times\{0\}\cup X\times(0,1]$ is Lindelöf.

(4) Straightforward: a neighborhood basis for $\langle x,0\rangle$ is given by $\{U_n\times\{0\}\sqcup V_m\times(0,1/\ell)\,:\,\ell,m,n\in\omega,\ \ell>0\}$, where U_n and V_n are local bases for x in the ρ and τ topologies. $\square$

Notice that in most cases $\widehat{X}(\tau,\rho)$ is not regular.

Proposition 3.5. *The following hold.*

(1) *There are Hausdorff H-closed spaces of arbitrarily high Lindelöf number.*
(2) *There is a Hausdorff non-compact first countable space satisfying* $\mathsf{U}_{fin}(\mathcal{O}, \Delta)$.
(3) *There is a first countable Lindelöf H-closed Hausdorff space which does not satisfy* $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$.

Proof. The three examples are of the form $\widehat{X}(\tau, \rho)$; Hausdorffness, H-closedness, Lindelöfness, and first countability in (2) and (3) all follow from Lemma 3.4.

(1) This is well known, but let us give an example anyway. Take X to be the ordinal $\kappa+1$, τ the order topology (which makes it compact), and ρ the discrete topology. Then $L(\widehat{X}(\tau, \rho)) = \kappa$.

(2) Take $\kappa = \omega$ in (1). Then for each $\alpha \in \omega + 1$, $\{\alpha\} \times [0,1]$ is homeomorphic to $[0,1]$, so $\widehat{X}(\tau, \rho)$ is a σ-compact space and thus satisfies $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$. We apply Lemma 3.2 to obtain $\mathsf{U}_{fin}(\mathcal{O}, \Delta)$. Of course, $\omega+1$ is first countable in the order topology.

(3) Take X to be $[0,1]$ and τ its usual topology, while ρ is the coarsest refining of τ that makes $\mathbb{Q} \cap [0,1]$ clopen and discrete. Thus, a ρ-open set is the union of (i) some subset of $\mathbb{Q}$ and (ii) $U - \mathbb{Q}$ with U open for the usual topology. Denote as usual the irrational numbers by $\mathbb{P}$. It is well known that $\mathbb{P} \cap [0,1]$ is homeomorphic to the product space ω^ω and does not satisfy $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$. Indeed, fix a homeomorphism $h : \omega^\omega \to \mathbb{P} \cap [0,1]$. One way to easily obtain a sequence $\langle \mathcal{U}_n \rangle$ of covers of $\mathbb{P} \cap [0,1]$ violating $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$ is to set $\mathcal{U}_n = \{h(\pi_n^{-1}(\{m\})) : m \in \omega\}$ where π_n is the projection on the n^{th} coordinate. Then the sequence of covers $\mathcal{W}_n = \{(U \cup (\mathbb{Q} \cap [0,1])) \times \{0\} \sqcup [0,1] \times (0,1] : U \in \mathcal{U}_n\}$ shows that $\widehat{X}(\tau, \rho)$ does not satisfy $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$. $\square$

3.2. The property $\mathsf{U}_{fin}(\Delta, \mathcal{O})$.

We denote by Δ_1 the collection of open covers with at least one dense member. First, some easy facts.

Lemma 3.6. *Let X be a space. The items below are equivalent:*
(a) *X satisfies $\mathsf{U}_{fin}(\Delta, \mathcal{O})$;*
(b) *X satisfies $\mathsf{U}_{fin}(\Delta, \Delta)$;*
(c) *X satisfies $\mathsf{U}_{fin}(\Delta_1, \mathcal{O})$;*
(d) *any closed subset of X satisfies $\mathsf{U}_{fin}(\Delta, \mathcal{O})$;*
(e) *any closed nowhere dense subset of X satisfies $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$.*

Proof. (c) $\to$ (a) $\leftrightarrow$ (b) are immediate, and (d) $\to$ (a) as well.

(a) $\to$ (d) Let $Y \subset X$ be closed. Any od-cover of Y yields an od-cover of X by taking the union of the members with $X - Y$, and the result follows.

(a) $\to$ (e) Let $Y \subset X$ be closed and nowhere dense. If Y does not satisfy $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$, take a sequence of covers $\langle \mathcal{U}_n \rangle$ witnessing this fact. Set $\mathcal{V}_n = \{U \cup (X - Y) : U \in \mathcal{U}_n\}$. Then $\langle \mathcal{V}_n \rangle$ witnesses that X does not satisfy $\mathsf{U}_{fin}(\Delta,\mathcal{O})$.

(e) $\to$ (c) Let $\langle \mathcal{U}_n \rangle$ be a sequence of covers of X ($n \in \omega$) such that some $U \in \mathcal{U}_0$ is dense in X. Set $\mathcal{F}_0 = \{U\}$. Since $X - U$ is closed and nowhere dense, there are finite $\mathcal{F}_n \subset \mathcal{U}_n$, $n \geq 1$, such that $\bigcup_{n \geq 1} \cup \mathcal{F}_n \supset X - U$. Then $\bigcup_{n \geq 0} \cup \mathcal{F}_n = X$. $\square$

The following proposition settles most of the classical cases (such as sets of reals).

Proposition 3.7. *Let X be a separable space. Then X satisfies $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$ if and only if X satisfies $\mathsf{U}_{fin}(\Delta,\mathcal{O})$.*

Proof. One direction is trivial, so let us assume that X satisfies $\mathsf{U}_{fin}(\Delta,\mathcal{O})$. Let $D = \{d_i : i \in \omega\}$ be dense in X. Given a sequence of open covers $\langle \mathcal{U}_i : i \in \omega \rangle$, take $V_{2i} \in \mathcal{U}_{2i}$ containing d_i and set $\mathcal{F}_{2i} = \{V_{2i}\}$. Since $V = \cup_{i \in \omega} V_{2i}$ contains D, $X - V$ is closed and nowhere dense and satisfies $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$ by Lemma 3.6. Hence, there are finite $\mathcal{F}_{2i+1} \subset \mathcal{U}_{2i+1}$ such that $\bigcup_{i \in \omega} \cup \mathcal{F}_{2i+1} \supset X - V$. Then $\bigcup_{i \in \omega} \cup \mathcal{F}_i = X$. $\square$

Of course, od-compact spaces trivially satisfy $\mathsf{U}_{fin}(\Delta,\mathcal{O})$. Any non-Lindelöf such space (for instance, an uncountable discrete space) is a trivial example of a space satisfying $\mathsf{U}_{fin}(\Delta,\mathcal{O})$ but not $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$. But we do not know the answer to the following question.

Question 3.8. Is there a Lindelöf non-od-compact space satisfying $\mathsf{U}_{fin}(\Delta,\mathcal{O})$ but not $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$?

The following question was inspired by Theorem 3.1.

Question 3.9. Let X be a space and $D \subset X$ be the subspace of its isolated points. Does the following equivalence hold: X satisfies $\mathsf{U}_{fin}(\Delta,\mathcal{O})$ $\longleftrightarrow$ $X - D$ satisfies $\mathsf{U}_{fin}(\mathcal{O},\mathcal{O})$?

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