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RIGHT-ANGLED MOCK REFLECTION SURFACES

RICHARD SCOTT AND TIMOTHY SCHROEDER

Abstract. In articles by M. Davis, T. Januskiewicz, and R. Scott (see *Nonpositive curvature of blow-ups*, Selecta Math. (N.S.) **4** (1998), no. 4, 491–547 and *Fundamental groups of blow-ups*, Adv. Math. **177** (2003), no. 1, 115–179) and another article by Richard Scott (see *Right-angled mock reflection and mock Artin groups*, Trans. Amer. Math. Soc. **360** (2008), no. 8, 4189–4210), a generalization of right-angled Coxeter groups, so-called right-angled mock reflection groups (RAMRGs), are introduced and explored. As is the case with Coxeter groups, these groups can be defined by a finite simple graph (one that is now "decorated" in the language of Colin Hagemeyer and Richard Scott in *On groups with Cayley graph isomorphic to a cube* [Comm. Algebra **42** (2014), no. 4, 1484–1495]), and they act on CAT(0) cubical complexes such that the stabilizer of every edge is $\mathbb{Z}_2$ and the 1-skeleton of the link of every vertex is isomorphic to the defining graph. This note examines the case where the defining graph Γ is homeomorphic to $\mathbb{S}^1$, and thus the corresponding cubical complex Σ is a 2-manifold. We show, explicitly, that these RAMRGs are virtually torsion-free and we describe the resulting quotient surfaces.

1. Introduction

Let Γ be a finite, simple graph with vertex set S. Γ encodes the data for a presentation of a *right-angled Coxeter group* (*RACG*) W_Γ:

$$W_\Gamma = \left\langle S \mid s^2 = 1 \text{ for each } s \in S \text{ and } (st)^2 = 1 \text{ for each edge } \{s,t\} \text{ of } \Gamma \right\rangle.$$

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If two vertices s and t are not connected by an edge in Γ, then their product has infinite order and they generate the infinite dihedral group. Any clique of size m in Γ defines a subgroup isomorphic to $\mathbb{Z}_2^m$. The pair (W_Γ, S) ((W, S) when the graph Γ is clear) is called a *Coxeter system.* Note that the correspondence between a Coxeter system and a simple graph is often taken in reverse, beginning with some Coxeter system (W, S), whose presentation defines a simplicial complex L called the *nerve* of (W, S), where vertex sets of simplices correspond to subsets of generators that generate finite subgroups. The 1-skeleton of L then encodes the presentation for (W, S) as described above.

In several papers, Michael W. Davis describes a construction which associates with any Coxeter system (W, S), a simplicial complex $\Sigma(W, S)$, or simply Σ when the Coxeter system is clear, on which W acts properly and cocompactly. The complex Σ is referred to as the *Davis complex.* (For a full description of Σ, see, for example, [2] by Davis and [6] by Davis and G. Moussong.) It is proven by Moussong [12] that, with a suitable piecewise-Euclidean metric, Σ is CAT(0). The other salient features of Σ are that (1) it is contractible, and (2) that it admits a cellulation, called the *Coxeter cellulation* for which the link of each vertex is L. It follows from (2), that if L is a triangulation of $\mathbb{S}^{n-1}$, Σ is an n-dimensional manifold. In this case, if G is a torsion-free subgroup of finite-index in W, then G acts freely on Σ, and Σ/G is a finite complex. By (1), Σ/G is aspherical. Hence, if L is homeomorphic to an $(n-1)$-sphere, Davis's construction gives examples of closed aspherical n-manifolds. See [3] for a complete description of Coxeter systems and their nerves. Of particular note here is that when W is right-angled, the Coxeter cellulation is cubical, and with respect to this cellulation, W acts transitively on the vertex set of Σ and the stabilizer of each edge is isomorphic to $\mathbb{Z}_2$. We also remark that in the right-angled case, Moussong's CAT(0) result is Gromov's Lemma: A cube complex is CAT(0) if and only if the link of every vertex is a flag complex. (See [9, p. 120]. Recall that a simplicial complex is a *flag complex* if whenever the 1-skeleton of a simplex is in the complex, then so is the entire simplex.)

In [4] and [5], Davis, T. Januszkiewicz, and R. Scott observe that certain CAT(0) complexes arising from "blow-ups" of real hyperplane arrangements also have the features similar to those of RACGs described above. There is a "mock reflection group" acting simply-transitively on the vertex set and edge-stabilizers are isomorphic to $\mathbb{Z}_2$. In [15], Scott puts a combinatorial description to a special class of these groups, called right-angled mock reflection groups (RAMRGs). As in the case with right-angled Coxeter groups, the presentations of these RAMRGs can be defined from the combinatorics of a finite, simple graph Γ, but with "local involutions" encoded on the vertex set. Such a graph is called a right-angled mock reflection system (RAMRS). In fact, in [15], Scott shows that for any RAMRS Γ, with vertex set S, there exists a CAT(0) cube complex $\Sigma(\Gamma)$ and a group $W(\Gamma)$ such that (1) the 1-skeleton of $\Sigma(\Gamma)$ can be identified with the Cayley graph of $W(\Gamma)$ with respect to the generating set S, and (2) $W(\Gamma)$ acts simply-transitively on the vertex set of $\Sigma(\Gamma)$, and has edge-stabilizers isomorphic to $\mathbb{Z}_2$.

In [5], the authors describe a linear representation for mock reflection groups, but are not able to establish whether it is faithful or not; thus, the linearity of RAMRGs remains an open question. By Selberg's Lemma, a finitely generated linear group is virtually torsion-free (i.e., has a finite-index torsion-free subgroup); thus, a weaker form of the linearity question for RAMRGs is whether they are virtually torsion-free. This paper is part of ongoing work to answer that question in general, that is, proving that arbitrary RAMRGs are virtually torsion-free. It is shown by J. Tits [16] that every Coxeter group has a faithful linear representation, and thus is virtually torsion-free. A direct proof that RACGs are virtually torsion-free is straightforward, as the kernel of the map from W to its abelianization is a finite-index, torsion-free subgroup.

We employ a similar strategy in §3, though a general proof of the result for RAMRGs has remained elusive. This paper reflects some of the preliminary work in this direction with the main purpose being to identify finite-index torsion-free subgroups for those RAMRGs whose defining right-angled mock reflection systems are homeomorphic to $\mathbb{S}^1$. That is, we prove the following.

Main Theorem 1.1. *If Γ is a RAMRS homeomorphic to $\mathbb{S}^1$, then the group $W(\Gamma)$ is virtually torsion-free. In particular, if Γ is an n-gon, then $W(\Gamma)$ contains a torsion-free subgroup of index* 2, 4, *or* 8.

In the proof of Theorem 1.1, we specifically investigate the surfaces resulting from these torsion-free subgroups. Indeed, by recognizing a fundamental chamber of the action of $W(\Gamma)$ on $\Sigma(\Gamma)$, we use the orbihedral Euler characteristic to classify the surfaces resulting from the action of these finite-index, torsion-free subgroups.

2. Coxeter Groups and RAMRGs

We begin with a description of Σ in the RACG case. Let Γ denote a finite simple graph with vertex set S and edges $\{r, s\}$ for some pairs $r, s \in S$. We can define a Coxeter group W_Γ (or simply W, when the graph Γ is clear) by the presentation:

$$W_\Gamma = \left\langle S \mid s^2 = 1 \text{ for all } s \in S, (rs)^2 = 1 \text{ for all edges } \{r, s\} \right\rangle.$$

Let L denote the flag completion of Γ (i.e., the simplicial complex obtained from L by including any subset $T \subseteq S$ as a simplex whenever the elements of T are pairwise joined by edges). We let $\mathcal{S}$ denote the poset of simplices of L. For each full subcomplex A of L, let W_A denote the subgroup of W generated by the vertex set of A. Such a subgroup is called a *special subgroup*. Note that for each simplex $\sigma \in \mathcal{S}$, W_σ is finite. A *spherical coset* in W is a coset of the form wW_σ for some $\sigma \in \mathcal{S}$ and $w \in W$. The set of all spherical cosets we denote by $W\mathcal{S}$. It is partially ordered by inclusion. The space Σ is defined as the geometric realization of the poset $W\mathcal{S}$. This is the Davis complex. The natural W-action on $W\mathcal{S}$ induces a proper simplicial action on Σ. Let K denote the geometric realization of $\mathcal{S}$. K includes naturally into Σ, and K is the fundamental chamber for the action of W on Σ.

2.1. The cubical cellulation of Σ.

Let σ be a k-simplex of L, with vertex set T (the empty set taken to be a (-1)-simplex). The group $W_\sigma = (\mathbb{Z}_2)^{k+1}$, and $\Sigma_\sigma = [-1,1]^{k+1}$. W_σ acts simply-transitively on the vertices of Σ_σ, and for each $w \in W$, the subcomplex $w\Sigma_\sigma$ of Σ is homeomorphic to $[-1,1]^{k+1}$. This gives a decomposition of Σ into a family of cubical subcomplexes $\{w\Sigma_\sigma\}_{wW_\sigma \in W\mathcal{S}}$. Thus, Σ has the structure of a regular CW-complex in which each cell is a cube. There is a 0-dimensional cube (vertex) for each element of W (cosets in $W/\{1\}$), and a set of such 0-cubes is the vertex set of a $(k+1)$-cube $w\Sigma_\sigma$ if and only if it is the set of elements in the spherical coset wW_σ. We call the cells corresponding to the spherical cosets of W_σ cubes of *type* T. With respect to this cubical structure, the action of W on Σ is simply-transitive on the vertices (0-cubes), and the stabilizer of each edge is isomorphic to $\mathbb{Z}_2$. Moreover, with respect to this cubical structure, the link of each vertex of Σ is L.

Example 2.1. Let $\Gamma = L =$ a 6-gon. Figure 1 gives a representation of Σ in the unit disk model of $\mathbb{H}^2$. Σ is an infinite complex, and here we show six 2-cubes of the resulting cubical structure. K is the region enclosed by the six dotted lines, and the action can be understood geometrically as reflections over each of the dotted lines. One can see that the action is simply-transitive on the 0-cubes and the edge-stabilizers are isomorphic to $\mathbb{Z}_2$. Note that the full simplicial decomposition is not shown.

For more reference on the Davis construction for RACGs, see [3], [7], [13], and [14]. The moral here is that each RACG acts geometrically on a CAT(0) cubical complex simply-transitively on vertices and with edge-stabilizers isomorphic to $\mathbb{Z}_2$. These ideas are generalized in the notion of RAMRGs.

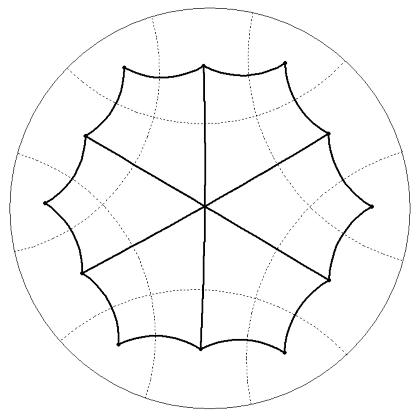


FIGURE 1. $\Sigma \cong \mathbb{H}^2$

Definition 2.2. A group G is a *right-angled mock reflection group* (*RAMRG*) if it acts isometrically and cellularly on a connected CAT(0)-cubical complex X such that the action is simply-transitive on the 0-skeleton, and the stabilizer of every edge is isomorphic to $\mathbb{Z}_2$.

The main theme of the first half of [15] is how to encode the presentations of RAMRGs in a finite simple graph. Citing relevant results from [15], we outline that argument here.

Let Γ be a simple graph with vertex set V. For each $v \in V$, let Γ_v denote the induced subgraph on the neighbors of v, and let $j_v : \Gamma_v \to \Gamma_v$ be an automorphism so that $(j_v)^2 =$ identity map on Γ_v. We call j_v a *local involution*. We illustrate this data by drawing the graph Γ with some edges at a vertex paired; that is, we connect two edges $\{u, v\}$ and $\{v, u'\}$ by an arc at the vertex v whenever $j_v(u) = u'$. It is clear that each involution j_v preserves adjacency to v. We call a graph Γ with arcs denoting local involutions a *graph with local involutions*.

Note that it is NOT the case that any graph, with arcs pairing any two edges at any vertex, defines a graph with local involutions. Indeed, consider the graph in Figure 2. Here, Γ_b is shown on the right, and a transposition of vertices c and a does not define a graph automorphism. So when we say a given Γ is a graph with local involutions, we will understand that at a given vertex v, the maps j_v defined by the indicated arcs do, in fact, define automorphisms of Γ_v.

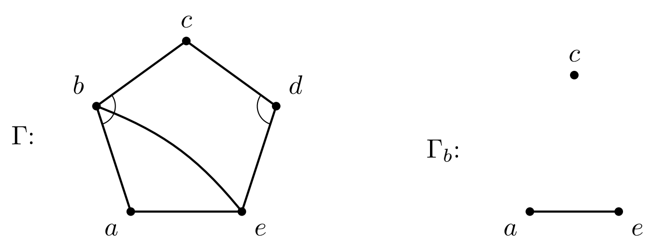


FIGURE 2. Involution $a \leftrightarrow c$ not defined.

2.2. Γ IS 4-PERIODIC.

Any pair u and v of adjacent vertices in a graph Γ with local involutions determines a sequence $\tau(u,v) = v_0, v_1, v_2, \ldots$ where $v_0 = u$, $v_1 = v$, and $v_{k+1} = j_{v_k}(v_{k-1})$ for $k \geq 1$. We call $\tau(u,v)$ a trajectory. We say Γ is *4-periodic* if for every trajectory, $v_n = v_{n+4}$ for all $n \geq 0$. This results in the following partition of the edges of Γ. Note that in the case Γ has no arcs, all involutions j are trivial and so each trajectory is 2-periodic, i.e., of the form $u, v, u, v, u, v, \ldots$, and therefore also 4-periodic.

RAMRG Result 1. *If Γ is a 4-periodic graph with local involutions, then the edges of Γ can be partitioned into sets of three types:*

- *a single edge (i.e., the local involution at each endpoint fixes the other endpoint),*
- *two edges paired by an arc at a shared vertex, or*
- *4-cycles of edges paired by arcs in cyclic order.*

We call an instance of one of the above sets in a graph Γ with local involutions an *edge-set* of Γ and call it a 1-, 2- or 4-edge-set, respectively. The different types of edge-sets are shown in Figure 3.

Example 2.3. Figure 4 shows an example of a graph Γ with local involutions and five vertices. There are three trajectories (up to cyclic permutation):

- $a, c, a, c, \ldots$;
- $a, e, c, e, \ldots$;
- $a, b, c, d, \ldots$.

These are all 4-periodic, and they correspond to a 1-, 2-, and 4-edge set, respectively, in the edge-set partition of Γ in Figure 3.

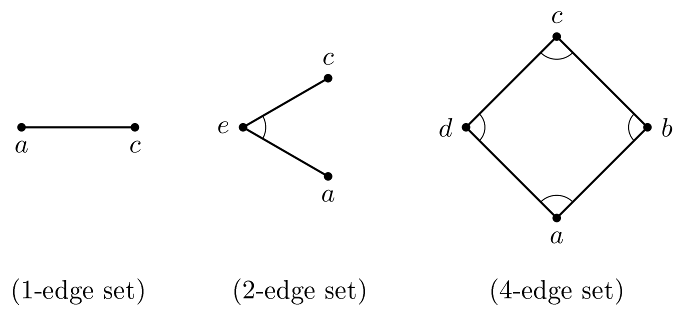


FIGURE 3. Edge-sets of Γ.

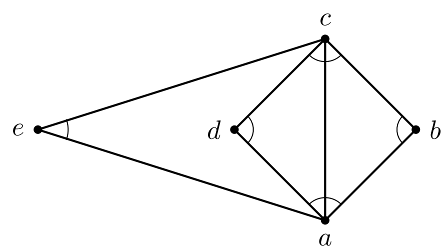


FIGURE 4. A 4-periodic graph Γ.

2.3. The presentation of $W(\Gamma)$.

Now suppose Γ is a 4-periodic graph with local involutions and with vertex set V. We use the data described above to define a group, $W(\Gamma)$. First, let $F(V)$ denote the free group on V. Next, for each vertex u, define $r(u,u) = u^2$. Finally, for each edge-set, select an edge $\{u,v\}$ in the edge-set and define the element $r(u,v) \in F(V)$ by

$$r(u,v) = uvu'v', \text{ where } u' = j_v(u) \text{ and } v' = j_{u'}(v).$$

Then define $W(\Gamma)$ as the group generated by V with relations $r(u,v)$. It is clear that every generator has order 2 and that for any two edges $\{u,v\}$ and $\{w,z\}$ in a given edge-set, the relation $r(u,v)$ can be obtained from $r(w,z)$ by cyclically permuting letters and taking inverses. Hence, the group $W(\Gamma)$ (up to isomorphism) is independent of the choice of edge within an edge-set. Specifically, we have the different types of edge-sets determining different types of relators. Indeed, 1-edge-sets produce relators of the form $r(a,c) = acac$ where a and c are vertices of an edge, 2-edge-sets produce relators of the form $r(a,e) = aece$ when $j_e(a) = c$, and lastly, 4-edge-sets produce relators of the form $r(a,b) = abcd$ where a, b, c, and d are the vertices of the edge-set read cyclically.

Example 2.4. The presentation for $W(\Gamma)$ for Example 2.3 is
$\left\langle a,b,c,d,e \mid a^2 = b^2 = c^2 = d^2 = e^2 = 1,\ acac = 1,\ aece = 1,\ abcd = 1 \right\rangle$.

2.4. Parallel transport and holonomy.

The idea is for the presentation described above to define a RAMRG; that is, it must act vertex transitively on a CAT(0) cubical complex. In order to ensure the CAT(0) criteria, a notion of parallel transport is required. The combinatorial description of this property is as follows. Let Γ_{uv} denote the induced subgraph on the set of vertices adjacent to both u and v. Given a trajectory $\tau(u,v) = v_0, v_1, v_2, v_3, \ldots$ and a vertex w_0 adjacent to both $u = v_0$ and $v = v_1$, define a sequence $w_k := j_{v_k}(w_{k-1})$, $k \geq 1$. Then, if Γ is 4-periodic, the correspondence $w_0 \mapsto w_4$ defines a map $\phi_{uv} : \Gamma_{uv} \to \Gamma_{uv}$. We call this map the *holonomy*, corresponding to the pair (u,v) of adjacent vertices. We say the holonomy map is *trivial* if $\phi_{uv}(w) = w$ for all vertices w adjacent to both u and v.

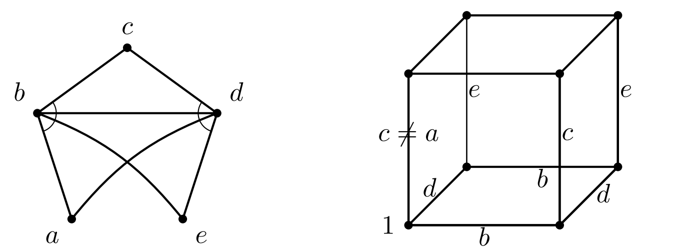
We are now ready for the following definition.

Definition 2.5. Let Γ be a graph with local involutions. Then Γ is a *right-angled mock reflection system* (RAMRS) if it is 4-periodic and all holonomy maps are trivial.

Example 2.6. Suppose Γ contains vertices a,b,c,d,e, where an arc at b connects edges $\{a,b\}$ and $\{b,c\}$, and an arc at d connects edges $\{d,c\}$ and $\{d,e\}$, and furthermore $\{b,e\}$ and $\{d,a\}$ are 1-edge-sets. See Figure 5A. Then b and d cannot be vertices of a 1-edge-set. The local involutions would be defined, but the trajectory determined by the 1-edge-set $\{b,d\}$ would not define a non-trivial holonomy. Indeed, $j_b j_d j_b j_d(c) \neq c$: thus, this graph is NOT a RAMRS. Note that the holonomy condition is satisfied by the graph in Example 2.3, so that graph is a RAMRS.

The following is the main result of [15].

RAMRG Result 2. *Let Γ be a RAMRS. Then $W(\Gamma)$ is a RAMRG. That is, there exists a CAT(0) cubical complex, denoted $\Sigma(\Gamma)$ or simply Σ when Γ is understood, on which $W(\Gamma)$ acts isometrically and simply-transitively on Σ^0 and such that the edge-stabilizers are all isomorphic to $\mathbb{Z}_2$. Moreover, the 1-skeleton of the link of each vertex of Σ is Γ.*



(A) The $\{b, d\}$ trajectory introduces non-trivial holonomy.

(B) The $\{b, d\}$ trajectory applied to edge a.

FIGURE 5. The holonomy condition.

Example 2.7. Suppose Γ is the graph shown in Figure 6. Then $\Sigma(\Gamma)$ is the square complex shown, drawn in the Poincaré disk model of hyperbolic two space, with vertices labeled according to the Cayley graph for $W(\Gamma)$. A fundamental chamber for the action of $W(\Gamma)$ is shaded.

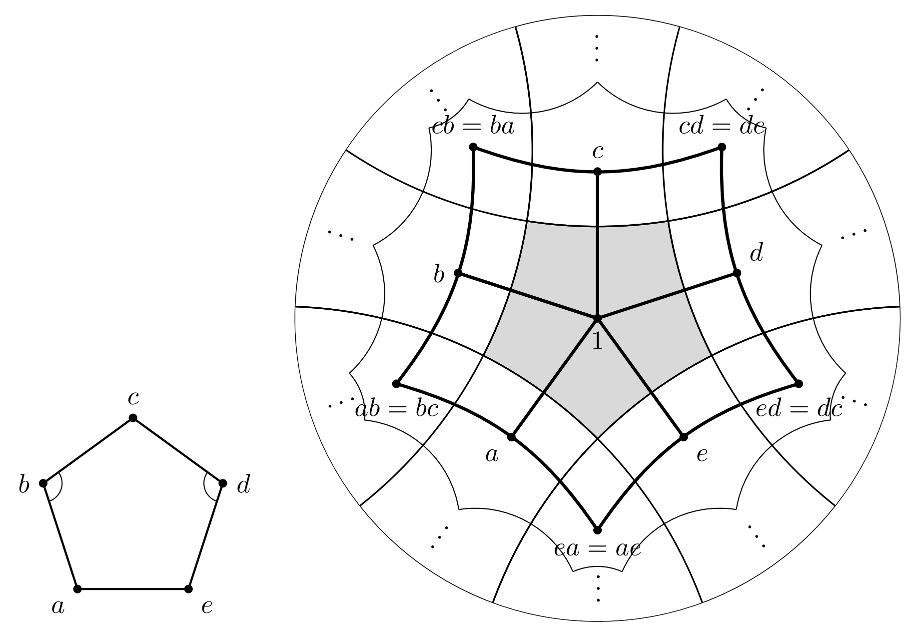


FIGURE 6. A RAMRS Γ and corresponding cubical complex $\Sigma(\Gamma)$.

To understand the "trivial holonomy" condition discussed above, consider the RAMRS Γ in Figure 5A. By the link condition of RAMRG Result 2, the clique $\{a, b, d\}$ corresponds to a 3-cube in $\Sigma(\Gamma)$ at the vertex 1 (see Figure 5B), and the trajectory $b, d, b, d, \ldots$ corresponds to the boundary of the square at the bottom. The "holonomy sequence" applied to the vertical edge a is a, $j_b(a) = c$, $j_d(c) = e$, $j_b(e) = e$, and $j_d(e) = c$. In order to consistently label the edges of Σ, these sequence elements $a, c, e, e, c, \ldots$ must be the labels on the vertical edges of the 3-cube emanating from the vertices of the bottom square (see Figure 5B). In particular, this holonomy sequence must be 4-periodic. In this case, it is NOT. The fourth term is c, while the 0^{th} term is a. Thus, there is no well-defined label on the vertical edge at the vertex 1.

Now before proceeding into a general discussion of the construction of Σ and, in particular, a discussion of the cell-stabilizers under the action of $W(\Gamma)$ on $\Sigma(\Gamma)$, we first point out a result from [15] regarding finite mock reflection groups.

RAMRG Result 3. *Let Γ be a RAMRS with associated RAMRG $W = W(\Gamma)$ and associated CAT(0) cubical complex $\Sigma = \Sigma(\Gamma)$. Then the following are equivalent:*

1. *W is finite.*
2. *Γ is a complete graph.*
3. *Σ is a cube.*

2.5. The space $\Sigma(\Gamma)$.

Given a RAMRS Γ and taking the existence of $\Sigma(\Gamma)$ from RAMRG Result 2, we now make some observations about its structure. Now, the cellulation of $\Sigma(\Gamma)$ is completely analogous to the Coxeter cellulation of the Davis complex in the case W is a Coxeter group. Note that with respect to this cellulation, the 1-skeleton of the Davis complex is isomorphic to the Cayley graph of W, and the cells of Σ are in one-to-one correspondence with the cosets of the subgroups generated by the cliques of Γ (see [3, Proposition 7.3.4].) Moreover, under the action of W, the cell-stabilizers correspond to conjugates of spherical subgroups generated by the vertices of the corresponding clique.

In our more general case, we again have that the 1-skeleton of $\Sigma(\Gamma)$ corresponds to the Cayley graph of $W(\Gamma)$. Indeed, identify a vertex v with the identity 1 in $W(\Gamma)$. Then for each $s \in S$, label the edge from 1 to $s \cdot v$ with s. Then use the transitivity of the action to extend this labeling to all of Σ. Thus, any two adjacent vertices are of the form w and ws where s is the label on the edge that connects the two. Thus, for simplicity of notation, we equate the vertices of Σ with group elements of W and the vertices of Γ with the generators of W.

The link condition and the transitivity of the action on vertices means that, as in the RACG case, we can again classify the cubes of $\Sigma(\Gamma)$ into types. Indeed, since the link of the vertex 1 in $\Sigma(\Gamma)$ is Γ, any n-clique T in Γ corresponds to the edges of an n-cube that are incident to the vertex 1 in $\Sigma(\Gamma)$. The CAT(0) condition means that this cube must, in fact, be an n-cell of $\Sigma(\Gamma)$ (filled in). We call this a cube of *type* T and realize that because of the transitive action on vertices, a cube of type T is actually present at every vertex.

Regarding cell-stabilizers, observe that by the action of $W(\Gamma)$ on its Cayley graph, we see immediately that the stabilizers of vertices are trivial and the edge $\{w, ws\}$ is stabilized by wW_sw^{-1}, where $W_s = \{1, s\}$ is the subgroup generated by some generator s. This is all that is required in the definition of a RAMRG: that it act on a cubical complex with edge-stabilizers isomorphic to $\mathbb{Z}_2$. But, whereas the stabilizers of cubes of type T of the Davis complex corresponding to a RACG are conjugates of (finite) subgroups generated by the vertices of T, the higher dimensional cells of $W(\Gamma)$ do not carry this exact relationship between cell type and cell-stabilizers.

Now, the significance of RAMRG Result 3, is that cells of type T for which T is itself a RAMRS have stabilizers conjugate to the subgroup $W(T)$ of $W(\Gamma)$ generated by the vertices of T. However, while every cube of $\Sigma(\Gamma)$ must correspond to a complete subgraph of Γ, the group structure of $W(\Gamma)$ resulting from the local involutions means that (appropriate conjugates of) the cell-stabilizers may not be generated by the vertices of the clique corresponding to the cube. Indeed, consider the RAMRS Γ and CAT(0) complex Σ depicted in Figure 6 and discussed in Example 2.7. The cell spanned by the edges $\{1, a\}$ and $\{1, b\}$ corresponds to the edge $\{a, b\}$ in Γ, but b does not stabilize this square. In this example, we say the complete subgraph of Γ spanned by the edges a and b is not "closed under edge-set decomposition." More generally, we say a subgraph $\Gamma' \subset \Gamma$ is *closed under edge-set decomposition* if whenever $\{x, y\}$ is an edge in Γ', then all of the edges in the edge-set containing $\{x, y\}$ are contained in Γ'.

So, regarding cell-stabilizers, if we are given a cell σ of $\Sigma(\Gamma)$ containing the vertex 1 and corresponding to a complete subgraph T of Γ, if T is not closed under edge-set decomposition, then there exists a generator $s \in T$ that doesn't stabilize σ. Indeed, the example above generalizes to the case that T contains the edge $\{a, b\}$ of a 2- or 4-edge-set, but does not include the edge $\{b, c\}$, where, in Γ, there is an arc at b from $\{a, b\}$ to $\{b, c\}$. This means that the square σ' spanned by the edges $\{1, a\}$ and $\{1, b\}$ in $\Sigma(\Gamma)$

is a face of σ, but $b \cdot \sigma'$, which is the square in $\Sigma(\Gamma)$ spanned by the edges $\{1, c\}$ and $\{1, b\}$, is not. So b is not in the stabilizer of σ.

The following lemma describes the cell-stabilizers of 2-cells of $\Sigma(\Gamma)$.

Lemma 2.8. *Let σ be a 2-cell of type $T = \{a, b\}$. Then the stabilizer of σ in $W(\Gamma)$ is gHg^{-1}, where either*

(i) *$H = 1$, if $\{a, b\}$ is part of a 4-edge-set $\{a, b\}, \{b, c\}, \{c, d\}$, and $\{d, a\}$,*
(ii) *$H = \langle a \rangle \cong \mathbb{Z}_2$ if $\{a, b\}$ is part of the 2-edge-set $\{a, b\}$, $\{b, c\}$, or*
(iii) *$H = \langle a, b \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ if there are no arcs at a or b connecting the edge $\{a, b\}$ to some other edge of Γ.*

Proof. By considering $g^{-1} \cdot \sigma$, where g is a group element corresponding to some vertex of σ, we can, without loss of generality, consider σ to be a cube containing the vertex 1. Because of this, any group element w which is not a vertex of σ cannot stabilize σ, since $w \cdot \sigma$ must contain the vertex w. So we need only to check group elements/vertices of σ.

In (i), σ has vertex set $\{1, a, b, ad = bc\}$. Then $a \cdot \sigma$ has vertex set $\{a, 1, ab = dc, d\}$, so $a \cdot \sigma \neq \sigma$. Similarly, $b \cdot \sigma \neq \sigma$. $ad \cdot \sigma$ has vertex set $\{ad, ada, adb, adad\}$, which shows $ad \cdot \sigma \neq \sigma$.

In (ii), σ has vertex set $\{1, a, b, ab = bc\}$, and $a \cdot \sigma$ has vertex set $\{a, 1, ab = bc, b\}$, so $a \cdot \sigma = \sigma$. But $b \cdot \sigma$ has vertex set $\{b, ba = cb, 1, c\} \neq \sigma$. Similarly, $ab \cdot \sigma$ has vertex set $\{ab, aba, a, ac\}$, so $ab \cdot \sigma \neq \sigma$.

In (iii), σ has vertex set $\{1, a, b, ab = ba\}$, and all of a, b, and $ab = ba$ fix the cell σ. This is the usual Coxeter group action $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on the 2-cell σ. $\square$

Similar observations can be made for higher dimensional cells, where to find the cell-stabilizer of σ corresponding to the complete subgraph T of Γ, one considers vertices of T that either do not have arcs or whose arcs connect two edges of T.

2.6. A simplicial and cellular decomposition of Σ.

We now demonstrate two different cellulations of Σ that enable us to realize a cellulation of a fundamental chamber of the action of W on Σ. Let Γ be a RAMRS and let Σ be the corresponding cubical complex. We can obtain a finer cellulation of Σ as follows. As above, let 1 denote the vertex of Σ corresponding to the identity of the Cayley graph for W. The cubes containing 1 are in one-to-one correspondence with complete subgraphs of Γ that are themselves RAMRS's, i.e., subsets of vertices of Γ. Refer to these subsets as *spherical* subsets and let $\mathcal{S}$ denote the poset of such subsets. Let $K = |\mathcal{S}|$ denote the geometric realization of $\mathcal{S}$. For $T \in \mathcal{S}$, by identifying T as the barycenter of the corresponding cube in Σ, we can view $K \subset \Sigma$, and we thus obtain a partial simplicial decomposition of the cubes containing 1. This simplicial decomposition extends to a decomposition of Σ via the action of W on Σ. That is, we identify $w \cdot T$ as the barycenter of the cube corresponding to the complete subgraph T containing the vertex w and extend linearly. We write $\Sigma_\triangle$ to denote Σ equipped with this simplicial decomposition. It is clear that K is a fundamental chamber of the action of W on $\Sigma_\triangle$.

Now if L, the flag completion of the RAMRS Γ, is a triangulation of an $(n-1)$-sphere, then, for each $T \in \mathcal{S}$, let K_T denote the geometric realization of the subposet $\mathcal{S}_{\geq T}$. K_T is then the triangulation of a k-cell, where $k = n - |T|$. We then define a new cell structure of K, and therefore on $\Sigma_\triangle$ (or Σ), by declaring the family $\{K_T\}_{T\in\mathcal{S}}$ to be the set of cells in K. For each generator v, we will let K_v denote $K_{\{v\}}$ and call it the *v-mirror* of $K := K_\emptyset$. For any $w \in W(\Gamma)$, we will denote by wK_T the w-translate of the cell K_T. To indicate Σ equipped with this cellulation, we write Σ_K. The simplicial triangulation of $\Sigma_\triangle$ is a subdivision of both Σ_K and Σ with the cubical cellulation.

In particular, if $\Gamma = L$ an n-gon, $n \geq 4$, $K = K_\emptyset$ is itself a 2-disk with the cell structure of an n-gon: one 2-cell, n 1-cells, or *mirrors*, each corresponding to a vertex of Γ, and n 0-cells, each corresponding to an edge of Γ. Note that Σ_K is "tiled" by K, and K is the fundamental chamber for the action of W on Σ_K. But, as we will see below, K is not Σ/W unless $W(\Gamma)$ is itself a RACG.

2.7. The orbihedral Euler characteristic.

If a group G acts on a complex X, then the *orbihedral Euler characteristic* of X/G is the rational number

$$\chi^{\mathrm{orb}}(X/G) = \sum_\sigma \frac{(-1)^{\dim \sigma}}{|G_\sigma|},$$

where the sum is over the cells of X/G and G_σ is the stabilizer in G of σ. (See [3] or [8].) Note that (1) the orbihedral Euler characteristic is the usual Euler characteristic in the case all cell-stabilizers are trivial and (2) the orbihedral Euler characteristic is multiplicative. That is, if $H \leq G$ of index m, then

$$\chi^{\mathrm{orb}}(X/H) = m\chi^{\mathrm{orb}}(X/G). \tag{2.1}$$

Also recall that surfaces are uniquely determined by their Euler characteristic. Indeed, orientable surfaces of genus g have $\chi = 2 - 2g$ and non-orientable surfaces have $\chi = 2 - g$. (The *non-orientable genus* is defined to be the number of crosscaps attached to a sphere in order to obtain the desired surface. See [11] and Figure 8 below.)

We apply this to the space Σ_K under the action of the RAMRG $W = W(\Gamma)$, in the case Γ is a n-gon. First, note that if Γ contains no arcs, then $\Sigma_K/W = K$ and

$$\chi^{\text{orb}}(\Sigma/W) = \frac{n}{4} - \frac{n}{2} + 1 = 1 - \frac{n}{4}.$$

Otherwise, $\Sigma_K/W \neq K$. Indeed, if a vertex u has an arc connecting edges $\{u, v\}$ and $\{u, w\}$ in Γ, then endpoints of the u-mirror are identified Σ_K/W (these endpoints corresponding to the 2-cells $\{u, v\}$ and $\{u, w\}$ in Σ), and we introduce a new vertex in the interior of the u-mirror, splitting the original edge of K into two "half-edges." In Σ_K/W these two half-edges are identified with trivial stabilizer. The stabilizer of the introduced vertex is $\mathbb{Z}_2$, corresponding to the involution u, and the stabilizer of the identified vertices is $\mathbb{Z}_2$ (from either W_v or W_w, which are conjugate in W).

Thus, working from no arcs, we can obtain Γ by cyclically introducing arcs at the appropriate vertices. Each introduction of an arc adjusts χ^{orb} in the following ways:

- $-\frac{1}{4} - \frac{1}{4} + \frac{1}{2}$: Two vertices with stabilizer $\mathbb{Z}_2 \times \mathbb{Z}_2$ are identified, with stabilizer $\mathbb{Z}_2$.
- $+\frac{1}{2} - 1 + \frac{1}{2}$: An edge with stabilizer $\mathbb{Z}_2$ has been replaced by a half-edge, with trivial stabilizer, but a vertex has been added, with stabilizer $\mathbb{Z}_2$.

Thus, the addition of an arc makes no change on the orbihedral Euler characteristic. As a result, if Γ is an n-gon, then

$$\chi^{\text{orb}}(\Sigma/W) = 1 - \frac{n}{4}. \tag{2.2}$$

3. Right-Angled Mock Reflection Surfaces

As stated in §1, the purpose of this paper is to identify finite-index torsion-free subgroups of RAMRGs acting on 2-manifolds, with predictable index. The general idea is to map the given group $W = W(\Gamma)$ to a finite group G, under which the cell-stabilizers inject. The kernel of this map is a finite-index torsion-free subgroup. Indeed, if $f : W \to G$ is such map and $w \in \text{Ker}(f)$ has finite order, then w is contained in a finite subgroup W' of W. The action of W' on Σ must have a fixed point, call it x (by [1, Part II, Corollary 2.8]). Let D denote the cube of lowest dimension in Σ that contains x. Since the action of W' on Σ is cellular, we must have that W' fixes the cube D, and therefore W' injects under f. Thus, since $w \in \text{Ker}(f)$, w must be the identity in W. So $\text{Ker}(f)$ is torsion-free. Moreover, since $\text{Ker}(f)$ is finite-index, $\text{Ker}(f)$ acts on Σ with resulting quotient space $\Sigma/\text{Ker}(f)$ a compact surface.

We seek out such maps when Γ is an n-gon. But we begin with some observations. First, there may be many maps to finite groups that fit the criteria described above, but when given the opportunity, we will define a map to as small a group as possible (thus resulting in a surface with as small as genus as possible). Second, note that by edge decomposition, successive vertices cannot have arcs unless Γ itself is a 4-gon with arcs at each vertex; that is, Γ is itself a 4-edge-set. Third, we can disregard Γ a 3-gon, since in this case W has order 8 and Σ is cube (a 3-manifold with boundary). Finally, as discussed above, in order for the kernel to be torsion-free, we need all cell-stabilizers to inject under our map. Referring to the cubical cellulation of $\Sigma(\Gamma) = \Sigma$, we note that the stabilizers of the 0-cells are trivial, so there is nothing to check there. The edge-stabilizers are conjugate to subgroups generated by vertices of Γ, so as long as individual generators have non-trivial image, edge-stabilizers inject. This will be clear from the definitions of our maps. Thus, our focus will be on verifying that the stabilizers of the 2-cells of Σ inject, which, by Lemma 2.8, amounts to verifying that the dihedral group of order 4 generated by the vertices of 1-edge-sets inject.

We are now ready to prove our Main Theorem. In doing so, we will not only identify a finite-index torsion-free subgroup, but also identify the genus of the resulting surface $\Sigma/\mathrm{Ker}(f)$.

Theorem 3.1. *Let Γ be an n-gon. Then $W(\Gamma)$ is virtually torsion-free.*

Proof. By edge-set decomposition, we have five cases to consider:

Case 1. Γ is a 4-edge-set

Case 2. n is even and Γ has arcs on every other vertex

Case 3. n-even, Γ is not as in 2

Case 4. n is odd and Γ is the graph in Figure 11

Case 5. n is odd, and Γ is not as in Figure 11.

Throughout this section, we will denote $\mathbb{Z}_2$ as the binary group $\{0, 1\}$, and we note that in order to show the resulting surface is non-orientable, it suffices to identify a Möbius strip embedded in the surface. Finally, in our images, for a given generator v, we indicate the v-mirror K_v of K simply with the letter v.

Case 1: Γ is a 4-edge-set. Let Γ have vertices r, s, t, u. By Lemma 2.8, the 2-cells of Σ have trivial stabilizers, so we need only to verify the edge-stabilizers inject.

Indeed, define $f : W(\Gamma) \to \mathbb{Z}_2$ by $f(r) = f(s) = f(t) = f(u) = 1$. It is clear that each generator injects, so $\mathrm{Ker}(f)$ is a torsion-free subgroup of index 2. Thus, $2\chi^{\mathrm{orb}}(\Sigma/W) = \chi^{\mathrm{orb}}(\Sigma/\operatorname{Ker}(f)) = \chi(\Sigma/\operatorname{Ker}(f))$, the usual Euler characteristic. But by Equation 2.2, we have that $\chi^{\mathrm{orb}}(\Sigma/W) = 1 - \frac{4}{4} = 0$. So $\Sigma/\operatorname{Ker}(f)$ is a genus 0 surface. In fact it is orientable and is a torus, as we see in Figure 7. Here, with K denoting the fundamental chamber of Σ and the action of the generators of W on Σ given by 180° rotations around the midpoint of each edge of K, we have K and sK making up the space $\Sigma/\operatorname{Ker}(f)$. The three identifications are given by elements of the kernel su, sr, and st with one, two, and three arrows, respectively.

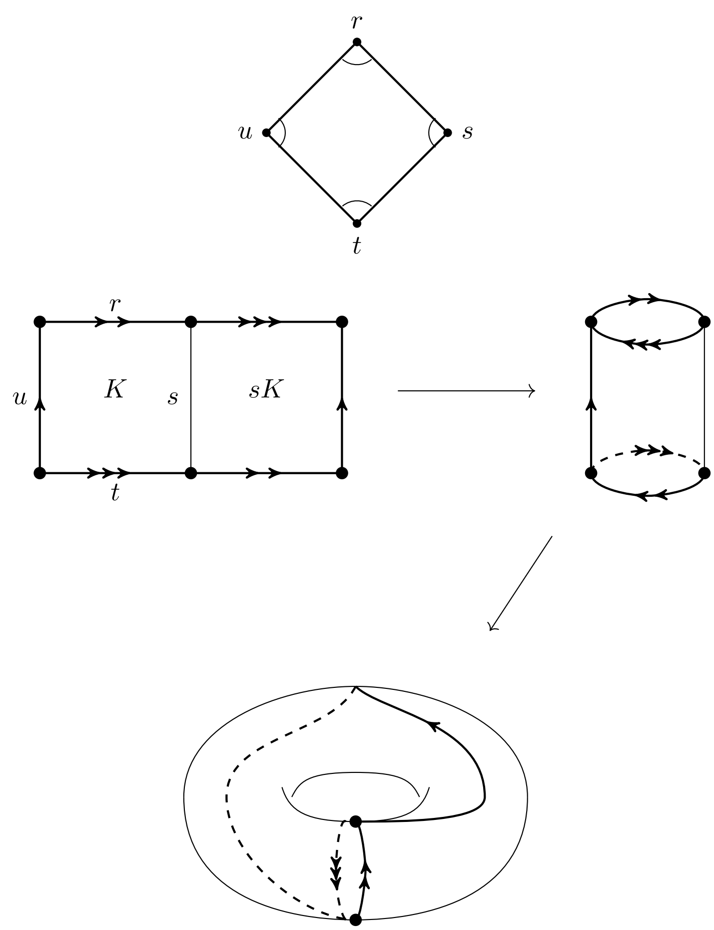


FIGURE 7. Case 1: Γ a 4-edge-set

Case 2: n is even, and Γ has $n/2$ arcs. By edge-set decomposition, Γ has an arc at every other vertex. So each 2-cell of Σ is of type $\{v, w\}$

where $\{v, w\}$ is edge of a 2-edge-set. By Lemma 2.8, the 2-cells of Σ have stabilizers conjugate to W_v for some generator v, and so our target group only needs to contain a subgroup isomorphic to $\mathbb{Z}_2$. Define $f : W(\Gamma) \to \mathbb{Z}_2$ by $f(v) = 1$ for all vertices v in Γ. $\mathrm{Ker}(f)$ is a torsion-free subgroup, and so $\chi^{\mathrm{orb}}(\Sigma/\mathrm{Ker}(f)) = \chi(\Sigma/\mathrm{Ker}(f))$, the usual Euler characteristic. Now we have, from Equation 2.2, that $\chi^{\mathrm{orb}}(\Sigma/W) = 1 - \frac{n}{4}$. But $\mathrm{Ker}(f)$ is index 2, so by Equation 2.1, we have that

$$2\chi^{\mathrm{orb}}(\Sigma/W) = \chi^{\mathrm{orb}}(\Sigma/\mathrm{Ker}(f)) = \chi(\Sigma/\mathrm{Ker}(f)) = 2 - \frac{n}{2}.$$

To see that $\Sigma/\mathrm{Ker}(f)$ is non-orientable, number the vertices of Γ so that v_1 does not have an arc, and therefore v_2 does have an arc, v_3 does not, etc. Then K and v_1K make up the space $\Sigma/\mathrm{Ker}\, f$. Moreover, $v_1v_2 \in \mathrm{Ker}(f)$ and the identification of the mirror K_{v_2} with $v_1v_2K_{v_2}$ results in a Möbius strip. So with g representing the genus of $\Sigma/\ker(f)$, we have $2 - g = 2 - n/2$. Thus, $\Sigma/\mathrm{Ker}(f)$ is a non-orientable surface of genus $n/2$.

See Figure 8 for the case $n = 6$, where the identification space for the resulting genus 3 non-orientable surface is shown.

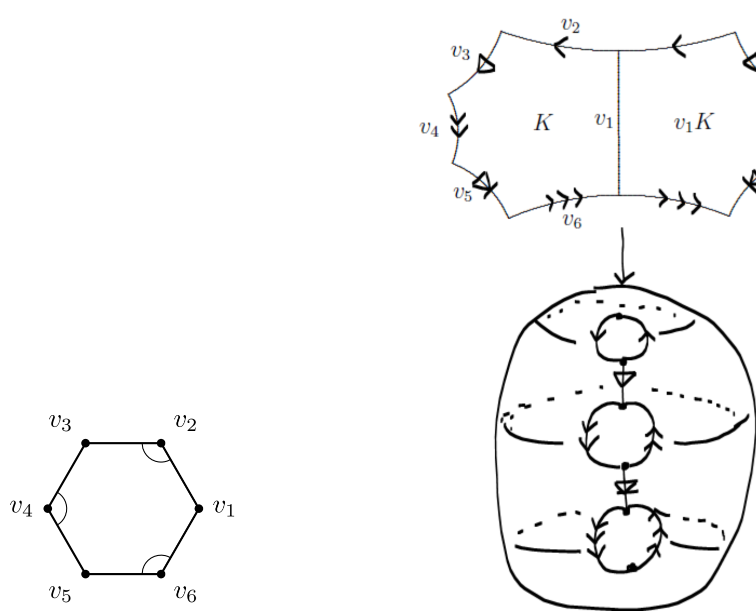


FIGURE 8. Case 2: Σ/Ker is a genus 3, non-orientable surface.

In the figure, the involutions v_2, v_4, and v_6 correspond to rotations of $180°$; the other involutions are reflections. In the figure, the identification of the mirror K_{v_2} with $v_1 K_{v_2}$ is shown with a single arrow. Other identifications are similar.

Case 3: n is even, Γ is not as in Case 2. Let $v_1, v_2, \ldots, v_n$ be the vertices of Γ. Since we do not have an arc at every other vertex, there must be a 1-edge-set in Γ. Therefore, by Lemma 2.8, there are 2-cells of Σ with stabilizers isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Here we have a map $f : W(\Gamma) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$, where $f(v_i) = (1, 0)$, where i is odd, and $f(v_j) = (0, 1)$, where j is even. It is clear that generators inject; therefore, edge-stabilizers inject. Furthermore, since the vertices of any 1-edge are mapped to different generators of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, we have that the stabilizers of the 2-cells corresponding to the 1-edge-sets inject as well. Thus, $\operatorname{Ker}(f)$ is a torsion-free subgroup of index 4. So

$$4\chi^{\text{orb}}(\Sigma/W) = \chi^{\text{orb}}(\Sigma/\operatorname{Ker}(f)) = \chi(\Sigma/\operatorname{Ker}(f)) = 4 - n.$$

If Γ has no arcs, we have a RACG and $\Sigma/\operatorname{Ker}(f)$ is orientable. (All identifications are annuli.) Then if g represents the genus of $\Sigma/\ker(f)$, we get $2 - 2g = 4 - n$, so $g = n/2 - 1$. See Figure 9.

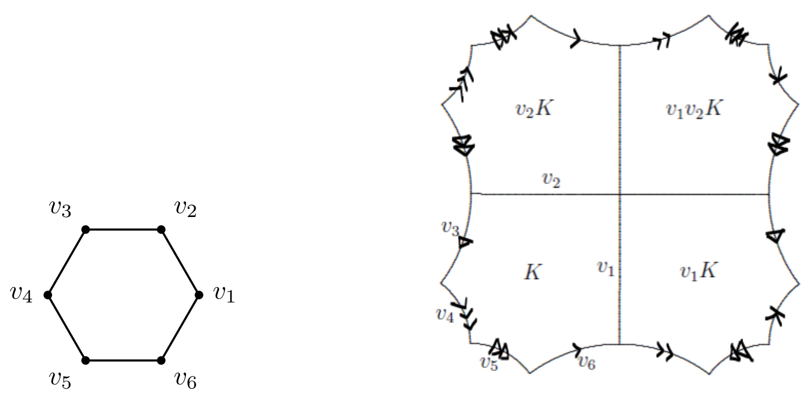


FIGURE 9. Case 3: $\Sigma/\operatorname{Ker}(f)$ an orientable surface of genus 2

If Γ does have an arc, but not at every other vertex, then number the vertices so that there is not an arc at v_1, is an arc at v_2, and is not an arc at v_3 and v_4. Then $\Sigma/\operatorname{Ker}(f)$ is comprised of K, v_1K, v_2K, and $v_1v_2K = v_2v_3K$. Moreover, $v_2v_4 \in \operatorname{Ker}(f)$ and the identification of the mirror K_{v_4} with $v_2v_4K_{v_4}$ results in a Möbius strip. So with g representing the genus of $\Sigma/\operatorname{Ker}(f)$, we have $2-g=4-n$. Thus, $\Sigma/\operatorname{Ker}(f)$ is a non-orientable surface of genus $n-2$. See Figure 10 for the case when $n=6$, and Γ contains one arc.

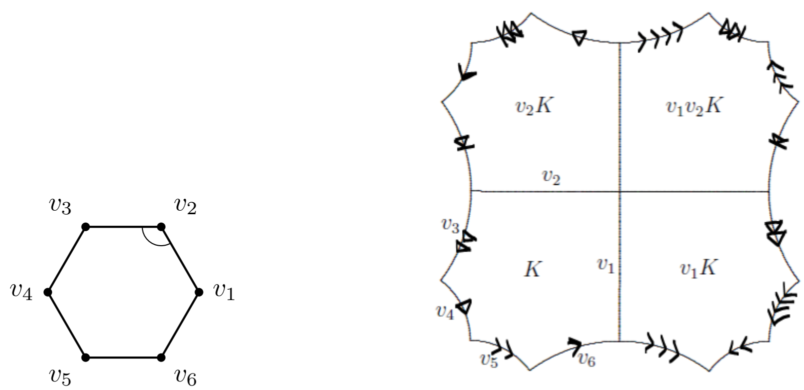


FIGURE 10. Case 3: $\Sigma/\operatorname{Ker}$ is a genus 4, non-orientable surface.

Case 4: Γ is the graph in Figure 11. First note that for such a Γ, all cell-stabilizers in Σ are isomorphic to $\mathbb{Z}_2$ except for the 2-cell spanned by edges v_1 and v_n, which is stabilized by a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action. So we need our target group to contain at least a subgroup isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. But we need more than that group in this case, because, in $W(\Gamma)$, we have that v_1 and v_3 are conjugate (by v_2), v_3 and v_5 are conjugate (by v_4), and so on. Thus, a map from $W(\Gamma)$ to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ would have to send $v_1, v_3, \ldots, v_n$ to the same element. But then the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ stabilizer of the 2-cell of type $\{v_1, v_n\}$ does not inject. So we need a target group to be non-Abelian and to contain a subgroup isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. With this in mind, we define a map f from $W(\Gamma)$ to the dihedral group of order 8, $D_4 = \langle r, s \mid r^2 = s^2 = (rs)^4 = 1 \rangle$ by

$$\begin{array}{rcl} v_1 & \mapsto & r \\ v_2 & \mapsto & srs \\ v_3 & \mapsto & r \\ v_4 & \mapsto & srs \\ v_5 & \mapsto & r \\ \vdots & & \vdots \\ v_{n-2} & \mapsto & r \\ v_{n-1} & \mapsto & s \\ v_n & \mapsto & srs. \end{array}$$

Clearly, for each v_i, $f(v_i^2) = 1$, and $f(v_1 v_n v_1 v_n) = r(srs)r(srs) = 1$. The remaining relators are all of the form $v_k v_{k-1} v_k v_{k+1}$ for even k. If $k \neq (n-1)$, we have $f(v_k v_{k-1} v_k v_{k+1}) = (srs)r(srs)r = 1$, and if $k = (n-1)$, we have $f(v_{n-1} v_{n-2} v_{n-1} v_n) = s(r)s(srs) = 1$. Thus, f defines a map from $W(\Gamma)$ onto D_4. Moreover, since each generator injects and the stabilizer of the 2-cell corresponding to v_1 and v_n injects as $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \langle r, srs \rangle \leq D_4$, we have that $\operatorname{Ker} f$ is a torsion-free subgroup of $W(\Gamma)$ of index 8. See Figure 11.

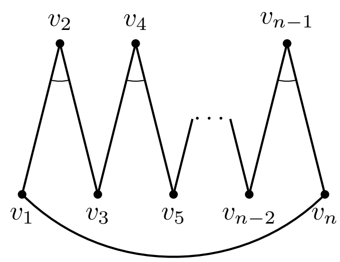


FIGURE 11. Case 4.

Here, $\Sigma / \operatorname{Ker}(f)$ is comprised of K, v_1K, v_nK, v_1v_nK, $v_{n-1}K$, $v_1v_{n-1}K$, $v_nv_{n-1}K$, and $v_nv_1v_{n-1}K$, and $\chi^{\text{orb}}(\Sigma / \operatorname{Ker}(f)) = \chi(\Sigma / \operatorname{Ker}(f)) = 8-2n$. Finally, $v_nv_2 \in \operatorname{Ker}(f)$, and the identification of K_{v_2} with $v_nv_2K_{v_2}$ results in a Möbius strip. So $\Sigma / \operatorname{Ker}(f)$ is non-orientable and $8 - 2n = 2 - g$ implies $\Sigma / \operatorname{Ker}(f)$ has genus $2n - 6$. See Figure 12 for the case $n = 5$, with $v_1, v_2 \mapsto r$, $v_2, v_5 \mapsto srs$, and $v_4 \mapsto s$.

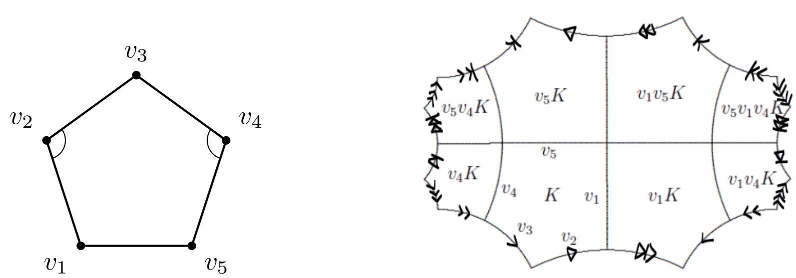


FIGURE 12. Case 4: $\Sigma / \operatorname{Ker}(f)$ a genus 4, non-orientable surface.

Case 5: n is odd, Γ is not as in Figure 11. Since Γ is not as in Figure 11, then we can number the vertices so that there are no arcs on v_1, v_3, and v_4, and if there is at least one arc in Γ, it occurs at v_2. Again, since we have 1-edge-sets, our target group must at least contain $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and this group does, in fact, suffice. Define the map $f : W(\Gamma) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ as follows:

$$\begin{array}{lcl} v_1, v_3, \ldots, v_{n-2} & \mapsto & (1,0) \\ v_2, v_4, \ldots, v_{n-1} & \mapsto & (0,1) \\ v_n & \mapsto & (1,1). \end{array}$$

Note that the stabilizers of the 2-cells of type $\{v_1, v_n\}$ and $\{v_{n-1}, v_n\}$ inject. It is clear that the stabilizers for the other 2-cells inject. So $\mathrm{Ker}(f)$ is a torsion-free subgroup of index 4, and thus $\chi^{\mathrm{orb}}(\Sigma/\mathrm{Ker}(f)) = \chi(\Sigma/\mathrm{Ker}(f)) = 4 - n$. We can realize $\Sigma/\mathrm{Ker}(f)$ with K, v_3K, v_4K, and v_4v_3K. Regardless if there are arcs in Γ or not, we have a non-orientable surface for $v_4v_3v_n \in \mathrm{Ker}(f)$, and a Möbius strip connects K_{v_n} with $v_4v_3v_nK_{v_n}$ ($v_4v_2 \in \mathrm{Ker}(f)$, and so if v_2 does have an arc, we can also find a Möbius strip connecting K_{v_2} with $v_4v_2K_{v_2}$). So with $2-g = 4-n$, we get $g = n-2$. See Figure 13 for an example with $n = 5$, where Γ has an arc at v_2.

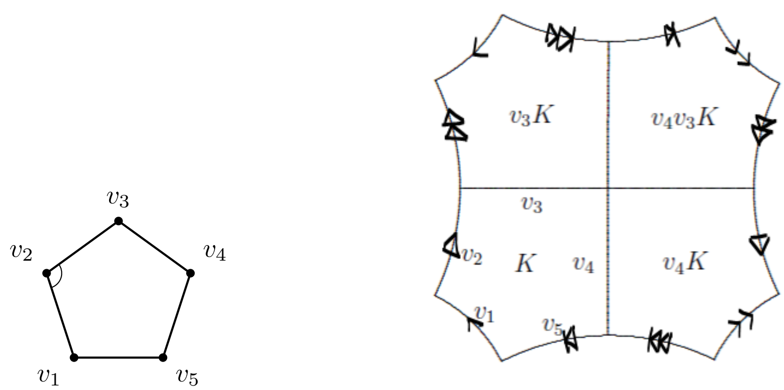


FIGURE 13. Case 5: $\Sigma/\mathrm{Ker}(f)$ a genus 3, non-orientable surface.

Note that to obtain an orientable surface, we need the corresponding multiple of $1-n/4$ to be even (for it must equal $2-2g$). So to obtain an orientable surface, we must have a map to a group of order at least 8. Indeed, for Case 5 we also have a map $g : W(\Gamma) \to \mathbb{Z}_2^3$ where $v_1, v_3, \ldots, v_{n-2} \mapsto (1,0,0)$, $v_2, v_4, \ldots, v_{n-1} \mapsto (0,1,0)$, and $v_n \mapsto (0,0,1)$. The stabilizers of the 2-cells clearly inject so $\ker(g)$ is a torsion-free subgroup of order 8. We can realize $\Sigma/\operatorname{Ker}(g)$ with K, v_1K, v_nK, v_1v_nK, $v_{n-1}K$, $v_1v_{n-1}K$, $v_nv_{n-1}K$, and $v_nv_1v_{n-1}K$, and $\chi^{\text{orb}}(\Sigma/\operatorname{Ker}(g)) = \chi(\Sigma/\operatorname{Ker}(g)) = 8-2n$. If Γ has no arcs, we have a RACG and $\Sigma/\operatorname{Ker}(g)$ is orientable (all identifications are annuli), and we get $2-2g = 8-2n$, so $g = n-3$. $\square$

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References

1. Martin R. Bridson and André Haefliger, *Metric Spaces of Non-Positive Curvature.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Berlin: Springer-Verlag, 1999.
2. Michael W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. (2) **117** (1983), no. 2, 293–324.
3. Michael W. Davis, *The Geometry and Topology of Coxeter Groups.* London Mathematical Society Monographs Series, 32. Princeton, NJ: Princeton University Press, 2008.
4. M. Davis, T. Januszkiewicz, and R. Scott, *Nonpositive curvature of blow-ups*, Selecta Math. (N.S.) **4** (1998), no. 4, 491–547.
5. M. Davis, T. Januszkiewicz, and R. Scott, *Fundamental groups of blow-ups*, Adv. Math. **177** (2003), no. 1, 115–179.
6. M. W. Davis and G. Moussong, *Notes on nonpositively curved polyhedra* in Low Dimensional Topology (Eger, 1996/Budapest, 1998). Ed. Károly Böröczky, Jr., Walter Neumann, and András Stipsicz. Bolyai Soc. Math. Stud., 8. Budapest: János Bolyai Math. Soc., 1999. 11–94.
7. Michael W. Davis and Boris Okun, *Vanishing theorems and conjectures for the ℓ^2-homology of right-angled Coxeter groups*, Geom. Topol. **5** (2001), 7–74.
8. Beno Eckmann, *Introduction to ℓ_2-methods in topology: Reduced ℓ_2-homology, harmonic chains, ℓ_2-Betti numbers. Notes prepared by Guido Mislin*, Israel J. Math. **117** (2000), 183–219.
9. M. Gromov, *Hyperbolic groups* in Essays in Group Theory. Ed. S. M. Gersten. Mathematical Sciences Research Institute Publications, 8. New York: Springer, 1987. 75–263.
10. Colin Hagemeyer and Richard Scott, *On groups with Cayley graph isomorphic to a cube*, Comm. Algebra **42** (2014), no. 4, 1484–1495.
11. Allen Hatcher, *Algebraic Topology.* Cambridge: Cambridge University Press, 2002.
12. Gabor Moussong, *Hyperbolic Coxeter groups.* Ph.D. Thesis. Ohio State University, 1988.
13. Timothy A. Schroeder, *The ℓ^2-homology of even Coxeter groups*, Algebr. Geom. Topol. **9** (2009), no. 2, 1089–1104.

14. Timothy A. Schroeder, *Geometrization of 3-dimensional Coxeter orbifolds and Singer's conjecture*, Geom. Dedicata **140** (2009), 163–174.
15. Richard Scott, *Right-angled mock reflection and mock Artin groups*, Trans. Amer. Math. Soc. **360** (2008), no. 8, 4189–4210.
16. Jacques Tits, *Groupes et géométries de Coxeter*. Institut des Hautes Études Scientifiques, Bures-sur-Yvette, 1961.

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