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TOPOLOGY PROCEEDINGS

Volume 54, 2019

Pages 163–175

http://topology.nipissingu.ca/tp/

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Electronically published on January 26, 2019

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
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E-mail:	topolog@auburn.edu
ISSN:	(Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 54 (2019)
Pages 163-175

E-Published on January 26, 2019

THE ONLY KNOT CONSTRUCTIBLE AS A MAHAVIER PRODUCT IS THE UNKNOT

MICHAEL LOCKYER

Abstract. Mahavier products can be considered as finite approximations of the inverse limits of set-valued functions. Unlike the inverse limits of single-valued continuous functions, interesting structures can occur after finitely many iterations in the set-valued case. These structures are called Mahavier products.

A general question in the study of the inverse limits of set-valued functions is whether a given continuum can be the Mahavier product of bonding spaces defined on a particular factor space, usually $[0, 1]$. In this paper we show that with the assumption of piecewise linear bonding spaces, the only knot obtainable as a Mahavier product on $[0, 1]$ is the unknot.

1. Introduction

Mahavier products originate from the study of inverse limits of set-valued functions. In contrast to inverse limits of single-valued continuous functions, when using set-valued functions interesting non-trivial structures can appear before reaching the "limit," i.e., after finitely many steps. These finite approximations of inverse limits of set-valued functions are called *Mahavier products* (they will be defined more rigorously in §2).

One of the general problems in the study of inverse limits of set-valued functions is whether a given continuum can be obtained as the inverse limit of set-valued functions on $[0, 1]$ (either with a single bonding function or allowing for different bonding functions). These are Problem 6.57 and Problem 6.59 in [4]. Some examples of results relating to these problems are in [3], where Alejandro Illanes shows that the circle is not obtainable

2010 *Mathematics Subject Classification.* Primary 54F15, 54C60, 54D80; Secondary 57M25.

Key words and phrases. knots, Mahavier product, set-valued inverse limit.

as an inverse limit (with a single bonding function on $[0,1]$), and in [10], where Van Nall shows that the only finite graph obtainable as an inverse limit (with a single bonding function on $[0,1]$) is an arc.

Due to the nontrivial nature of the finite Mahavier products, this problem can be extended to whether a given continuum can be obtainable as a Mahavier product. In this paper we consider the question of which knots can be obtained as Mahavier products. Because every knot is equivalent to the unknot when embedded in dimensions greater than 3, we will only consider the Mahavier product of two bonding spaces, so the Mahavier product is embedded in 3-space (specifically $[0,1]^3$). We also restrict the knots to be piecewise linear, i.e., consisting of finitely many straight lines. The reason for this is discussed in §4.

The main result of this paper is that any non-trivial knot is not obtainable as a Mahavier product of intervals. The intuitive reason for this is that the projection of any nontrivial knot in $[0,1]^3$ will have crossings which will need to be represented on the bonding spaces, but the Mahavier product of these bonding spaces will not be able to recreate the knot. This specific example highlights one of the key features in the structure of Mahavier products and is discussed in §4.

In §2 we start with some basic definitions and results in knot theory that will be needed, before defining Mahavier products and related concepts. We then prove some basic results needed in the next section. Section 3 begins with a series of lemmas used to prove the main result at the end of the section. Section 4 then discusses the main result and its relevance.

2. Preliminaries

We begin this section with a brief introduction to knot theory. We only require a few definitions and some basic facts about knots. There are a number of (equivalent) formulations for the basic definitions in knot theory; we will follow those in [6]. A *curve* in a space X is the image of a continuous function $f : [0,1] \to X$, and a *simple closed curve* in X is the image of a continuous function $g : [0,1] \to X$, where $g(0) = g(1)$, and g is injective elsewhere. Note that in this paper the subset symbol $\subset$ includes the possibility of equality. A *continuum* is a non-empty compact, connected metric space.

Definition 2.1. A *knot* is a subset of $\mathbb{R}^3$ that is a piecewise linear simple closed curve. For the purposes of this paper, we will assume that any knot K is embedded in $[0,1]^3$.

Knots K_1 and K_2 are said to be *equivalent* if there exists an orientation preserving homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K_1) = K_2$.

Informally, the equivalence means K_1 can be continuously deformed into K_2 without intersecting itself. The piecewise linear condition in Definition 2.1 means that a knot is the union of finitely many straight line segments, with any intersections between the line segments occurring at the endpoints. See §4 for more discussion on this. One special knot is the unknot, informally, a curve that has not been "tangled." A *disc* is a space homeomorphic to a 2-simplex.

Definition 2.2. A knot K is said to be the *unknot* if it bounds an embedded piecewise linear disc in $\mathbb{R}^3$.

The Jordan curve theorem and the Schönflies theorem can be used to show a well-known and very useful property of the unknot, presented here as Corollary 2.5. A proof of the Jordan curve theorem can be found in [7, Ch. 4], and a proof of the Schönflies theorem can be found in [7, p. 68].

Theorem 2.3 (the Jordan curve theorem)**.** *Let J be a simple closed curve in a plane P. Then $P \setminus J$ is the union of two disjoint connected sets, one bounded set, known as the interior, and one unbounded set, known as the exterior.*

Theorem 2.4 (the Schönflies theorem)**.** *Let J be a simple closed curve in a plane, and let I be the interior of J. Then $\overline{I}$ is a disc.*

Corollary 2.5. *A knot K embedded in a (piecewise linear) plane is an unknot.*

Proof. Since K is a (piecewise linear) simple closed curve embedded in a plane in $\mathbb{R}^3$, by the Jordan curve theorem, it will have an interior. Then by the Schönflies theorem, it will bound a (piecewise linear) disc in $\mathbb{R}^3$, and hence is an unknot. $\square$

That concludes the knot theory needed; we now move to some definitions and basic results in continuum theory. One notion that will be important is the notion of the order of a point, which we take from [8].

Definition 2.6. Let X be a continuum and let $A \subset X$. Let β be a cardinal. We say that A *is of order less than or equal to* β *in* X provided that for each open U such that $A \subset U$, there exists an open V such that $A \subset V \subset U$ and $|Bd(V)| \leq \beta$.

We say that A *is of order* β *in* X, written $ord(A, X) = \beta$, provided that A is of order less than or equal to β in X and A is not of order less than or equal to α in X for any $\alpha < \beta$.

Often (including all instances in this paper), the set A in the above definition will be a singleton $\{p\}$. In this case, we will simply refer to the order of p, written $ord(p, X)$.

An immediate observation from the definition is that a point p has order 0 if and only if p is its own component. It can also be seen (with the help of [8, Lemma 9.9]) that if a point p in a piecewise linear curve has order n for some $n < \aleph_0$, then p is the intersection of the endpoints of n straight lines.

The following proposition appears as Proposition 9.5 and Corollary 9.6 in [8].

Proposition 2.7. *If X is a nondegenerate continuum, then $ord(x, X) \leq 2$ for all $x \in X$ if and only if X is an arc or a simple closed curve. X is a simple closed curve if and only if $ord(x, X) = 2$ for all $x \in X$.*

Definition 2.8. If $X \subset \mathbb{R}^n$ is a continuum, then we define the *band of radius ε about X* as the set of points $\{y \in \mathbb{R}^n : d(x, y) < \varepsilon$ for some $x \in X\}$.

Lemma 2.9. *If $M \subset [0,1] \times [0,1]$ is a piecewise linear curve, then there is an $\varepsilon > 0$ such that for each pair of straight lines l_1 and l_2 in M, the bands of radius ε about l_1 and l_2 have non-empty intersection if and only if l_1 and l_2 have non-empty intersection.*

Proof. If a pair of lines l_1 and l_2 do not intersect, there is some minimum distance $d(l_1, l_2) = \min\{d(a, b) : a \in l_1, b \in l_2\}$ between the points in the lines, where $d(l_1, l_2) > 0$. Since there are only finitely many pairs of lines, there is some minimum distance over all these pairs $\partial > 0$. Setting $\varepsilon = \frac{\partial}{2}$ will make the hypothesis work. $\square$

The object of this paper is to show that the only knot that can be created as a Mahavier product is the unknot. We will now define the Mahavier product and associated concepts. Mahavier products arose from the study of inverse limits of set-valued functions, where the bonding spaces (described below) are viewed as upper semicontinuous set-valued functions. In this paper we view them as simply being closed subsets of a unit square (these are equivalent formulations).

Definition 2.10. Let $I_1 = I_2 = [0,1]$ and let $G \subset I_1 \times I_2$ be closed. We say that G is *full* if, for each $x_1 \in I_1$, there is a point $(x_1, y) \in G$ for some $y \in I_2$ and, for each $x_2 \in I_2$, there is a point $(y, x_2) \in G$ for some $y \in I_1$.

There have been various formulations of a Mahavier product, for example in [1] and [2]. The following definition is a special case of these more general definitions and is sufficient for the purposes of this paper.

Definition 2.11. Let $n \in \mathbb{N}$. For $0 \leq i \leq n$, let $I_i = [0,1]$, and for $1 \leq i \leq n$, let G_i be a closed subset of $I_{i-1} \times I_i$. Then the *Mahavier product* of $\{G_i : 1 \leq i \leq n\}$, denoted by either $\bigstar_{i=1}^{n} G_i$ or $G_1 \star G_2 \star \cdots \star G_n$, is the set $\{(x_0, x_1, \ldots, x_n) \in [0,1]^{n+1} :$ for all $1 \leq i \leq n,\ (x_{i-1}, x_i) \in G_i\}$.

In the above definition, the sets I_i are called *factor spaces* and the sets G_i are called *bonding spaces.*

One important function we will use is the projection map. There are two projection maps commonly used. In practise, when the word "projection map" is used, the context will indicate which projection map is being referred to.

Definition 2.12. Let $K = \bigstar_{j=1}^{n} G_j$ be a Mahavier product. The *projection of* K *onto the space* G_i (for $1 \leq i \leq n$) is the function $\pi_i : \bigstar_{j=1}^{n} G_j \to G_i$, defined by $\pi_i((x_0, \ldots, x_n)) = (x_{i-1}, x_i)$.

If G_i is a closed subset of $I_{i-1} \times I_i$, then the *projections of* G_i *onto* I_{i-1} *and* I_i are the functions $\rho_{i,i-1} : G_i \to I_{i-1}$ and $\rho_{i,i} : G_i \to I_i$, defined by $\rho_{i,i-1}((x_{i-1}, x_i)) = x_{i-1}$, and $\rho_{i,i}((x_{i-1}, x_i)) = x_i$.

If $A \subset G_i$, and $B \subset I_j$ for some $j \in \{i-1, i\}$, we say that A *projects onto* B if $B \subset \rho_{i,j}(A)$. We say A *projects entirely onto* B if $B = \rho_{i,j}(A)$. Note that the projection maps are continuous.

In this paper we will require Mahavier products only where $n = 2$. This is because we need the knot to be a subset of 3-space. We will assume throughout the paper that if we have a Mahavier product $K = G_1 \star G_2$, then for all $i \in \{1, 2\}$ and $j \in \{i-1, i\}$, either $\rho_{i,j}(\pi_i(K)) = [0, 1]$, or $\rho_{i,j}(\pi_i(K)) = \{a\}$ for some $a \in [0, 1]$. We will see later that if K is a non-trivial knot, then the second case is not possible. The requirement that a (non-degenerate) projection be $[0, 1]$ is harmless, since if a projection is a non-degenerate interval, we can simply rescale the knot to an equivalent knot that projects entirely onto $[0, 1]$.

The following lemma is a basic fact of Mahavier products and will be assumed for the remainder of the paper.

Lemma 2.13. *Suppose that* $I_0 = I_1 = I_2 = [0, 1]$ *and that* $G_1 \subset I_0 \times I_1$ *and* $G_2 \subset I_1 \times I_2$ *are piecewise linear curves. Suppose* l_1 *is a straight line in* G_1 *and* l_2 *is a straight line in* G_2 *such that there is an open interval* $U \subset I_1$ *such that* l_1 *and* l_2 *both project onto* U *(in their respective spaces). Then there is a straight line in* $G_1 \star G_2$ *consisting of points* (x_0, x_1, x_2)*, where* $x_1 \in \overline{U}$*.*

Proof. Let $U = (a, b)$. Since l_1 and l_2 are straight lines and U has non-empty interior, the projection maps onto U are injective, so the Mahavier product will be non-empty and there will be a unique point in the Mahavier product for each point in l_1 (also l_2). Also, since l_1 and l_2 are compact and the projection map is closed, we can extend the projection to $\overline{U}$.

To see the Mahavier product of the projections is a straight line in $G_1 \star G_2$, note that $G_1 \star G_2 = \pi_1^{-1}(G_1) \cap \pi_2^{-1}(G_2)$ (this observation appears as an observation in [4, p. 9]). So if we consider the two lines l_1 and l_2 in their respective bonding spaces, the inverse image of each line will be a (flat) plane (restricted to $\overline{U}$ in the I_1 coordinate), so their intersection will be a straight line. $\square$

3. Main Result

This section starts with a few lemmas relating to Mahavier products.

Definition 3.1. Let $n \in \mathbb{N}$. For $0 \leq i \leq n$, let $I_i = [0,1]$, and for $1 \leq i \leq n$, let G_i be a closed subset of $I_{i-1} \times I_i$, and let $K = \bigstar_{i=1}^n G_i$ be the Mahavier product of $\{G_1, \ldots, G_n\}$. Let $1 \leq j \leq n$ and let $(x_{j-1}, x_j) \in G_j$. Then we say (x_{j-1}, x_j) *contributes* to K if there is a point $(p_0, p_1, \ldots, p_n) \in K$ such that $x_{j-1} = p_{j-1}$ and $x_j = p_j$.

Lemma 3.2. *Let $n \in \mathbb{N}$. For $0 \leq i \leq n$, let $I_i = [0,1]$, and for $1 \leq i \leq n$, let G_i be a closed subset of $I_{i-1} \times I_i$, let $K = \bigstar_{i=1}^n G_i$ be the Mahavier product of $\{G_1, \ldots, G_n\}$, and suppose K is a continuum. Then for each graph G_i, there is a maximum of one component that contains points that contribute to K.*

Proof. Suppose there is $1 \leq m \leq n$ such that G_m contains two components C and D with points $(c_{m-1}, c_m) \in C$ and $(d_{m-1}, d_m) \in D$, where (c_{m-1}, c_m) and (d_{m-1}, d_m) both contribute to K. Then in G_m there are disjoint open sets U_C and U_D such that $C \subset U_C$, $D \subset U_D$, and $U_C \cup U_D = G_m$.

Then since $K \subset (([0,1]^{m-1} \times U_C \times [0,1]^{n-m}) \cup ([0,1]^{m-1} \times U_D \times [0,1]^{n-m}))$, and $K \cap ([0,1]^{m-1} \times U_C \times [0,1]^{n-m})$ and $K \cap ([0,1]^{m-1} \times U_D \times [0,1]^{n-m})$ are both open, nonempty, and disjoint, this means K is disconnected, a contradiction to K being a continuum. $\square$

Lemma 3.3. *Let $n \in \mathbb{N}$. For $0 \leq i \leq n$, let $I_i = [0,1]$, and for $1 \leq i \leq n$, let G_i be a full closed subset of $I_{i-1} \times I_i$ and let $\bigstar_{i=1}^n G_i$ be the Mahavier product of $\{G_1, \ldots, G_n\}$. Then for all $1 \leq i \leq n$, we have $\pi_i(\bigstar_{i=1}^n G_i) = G_i$.*

Proof. Consider a point $(x_{i-1}, x_i) \in G_i$. Since each G_i is full, there is a point $\mathbf{p} = (p_0, \ldots, p_n)$ in the Mahavier product such that $p_{i-1} = x_{i-1}$ and $p_i = x_i$. Therefore, $(x_{i-1}, x_i) \in \pi_i(\bigstar_{i=1}^n G_i)$, so $G_i \subset \pi_i(\bigstar_{i=1}^n G_i)$.

Consider a point $\mathbf{p} = (p_0, \ldots, p_n)$ in the Mahavier product, and, in particular, $\pi_i(\bigstar_{i=1}^n G_i) = (p_{i-1}, p_i)$. From the definition of the Mahavier product, there must be a point $(x_{i-1}, x_i) \in G_i$ such that $x_{i-1} = p_{i-1}$ and $x_i = p_i$. Therefore, $G_i \supset \pi_i(\bigstar_{i=1}^n G_i)$, so we conclude that $G_i = \pi_i(\bigstar_{i=1}^n G_i)$. $\square$

Lemma 3.4. *Let $n \in \mathbb{N}$. For $0 \leq i \leq n$, let $I_i = [0,1]$, and for $1 \leq i \leq n$, let G_i be a closed subset of $I_{i-1} \times I_i$, and let $\bigstar_{i=1}^n G_i$ be the Mahavier product of $\{G_1, \ldots, G_n\}$. Suppose $K \subset \bigstar_{i=1}^n G_i$ is a straight line. Then for any $1 \leq i \leq n$, $\pi_i(K)$ is a (possibly degenerate) straight line.*

Proof. Let the straight line $K \subset \bigstar_{i=1}^n G_i$ be described by the vector equation: $K = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid (x_0, \ldots, x_n) = (a_0, \ldots, a_n) + t(b_0, \ldots, b_n)$ for $t \in [0,1]\}$.

Then $\pi_i(K) = \{(x_{i-1}, x_i) \in \mathbb{R}^2 \mid (x_{i-1}, x_i) = (a_{i-1}, a_i) + t(b_{i-1}, b_i)$ for $t \in [0,1]\}$, which is the vector equation for a straight line (possibly degenerate). □

Lemma 3.5. *Let $I_0 = I_1 = I_2 = [0,1]$, and let $G_1 \subset I_0 \times I_1$ and $G_2 \subset I_1 \times I_2$ be full and closed. Let $K = G_1 \star G_2$ be a piecewise linear curve. Then G_1 and G_2 are piecewise linear curves.*

Proof. From Lemma 3.3, G_1 and G_2 are the projections of K. From Lemma 3.4, the projection of a straight line is a (possibly degenerate) straight line. Since K is the union of finitely many straight lines, and the projection of each of these is a (possibly degenerate) straight line, each projection is the union of finitely many straight lines, therefore piecewise linear. As projection maps are continuous, each projection will also be a continuum. □

In the next lemma we use the following terminology. If $p \in (0,1)$, the *sides* of p are $[0,p)$ and $(p,1]$. An *open interval on a side of* p is of the form (a,p) or (p,b).

Lemma 3.6. *Let $I_0 = I_1 = I_2 = [0,1]$, and let $G_1 \subset I_0 \times I_1$, $G_2 \subset I_1 \times I_2$ be full and closed. Let $K = G_1 \star G_2$ be a piecewise linear curve. Then for each factor space I_i (for $i \in \{0,1,2\}$), for each p_i in the interior of I_i, for each bonding space G_j (for $j \in \{1,2\}$) with I_i as a factor space, either*

1. *there is a point $\mathbf{p} \in G_j$ that is the intersection of the endpoints of straight lines that project onto open intervals in I_i either side of p_i (the two straight lines may be part of the same straight line), or*
2. *there is a straight line l in G_j such that the projection of l on I_i is $\{p_i\}$, and for each side S of p_i, there is a straight line k that intersects l at one of its endpoints, and k projects onto an open interval in S.*

Proof. By Lemma 3.5 each bonding space consists of a finite number of straight lines. Suppose there was a bonding space G_j with a factor space I_i and a point p_i in the interior of I_i such that for every point $\mathbf{p}$ in G_j with coordinate p_i (corresponding to the interval I_i), either $\mathbf{p}$ is the intersection of the endpoints of straight lines that project only onto open intervals on one side of p_i, or $\mathbf{p}$ is part of a straight line segment l that projects completely onto $\{p_i\}$, and any straight line connected to the endpoints of l only projects onto open intervals on one side of p_i.

Since G_j is piecewise linear, by Lemma 2.9, there is some $\varepsilon > 0$ such that for each straight line segment in G_j, the band of radius ε of a line segment intersects the band of radius ε of another line if and only if the two lines meet at their endpoints.

Take the bands of all lines that project onto intervals in $[0, p_i)$ (along with any lines attached to endpoints of intervals in this collection), and call this U, and take the bands of all lines that project onto intervals in $(p_i, 1]$ (along with any lines attached to endpoints of intervals in this collection), and call this V. Note that U and V are disjoint. Then $(U \cap G_j)$ and $(V \cap G_j)$ are open subsets of G_j, and $(U \cap G_j) \cup (V \cap G_j) = G_j$, so G_j is disconnected, contradicting the fact that K is a continuum. $\square$

We now prove four lemmas, each of which will be directly used to prove the main result. In the following lemmas we assume each Mahavier product is a continuum. We also assume either each bonding space G_i is full or G_i is a continuum. The assumption that G_i is full was discussed in §2, and the assumption that G_i is a continuum is not a loss of generality due to Lemma 3.2.

Lemma 3.7. *Let $I_0 = I_1 = I_2 = [0,1]$, and let $G_1 \subset I_0 \times I_1$ and $G_2 \subset I_1 \times I_2$ be continua. Let $K = G_1 \star G_2$ be a non-degenerate continuum. Then if G_1 or G_2 contain a point of order 0, K is an arc.*

Proof. Without loss of generality suppose G_1 has a point $\mathbf{x}_1$ of order 0. Since a point of order 0 is a component and G_1 is a continuum, G_1 is a single point. Then $K \subset (\{\mathbf{x}_1\} \times [0,1])$. Since K is a non-degenerate continuum, it must be the case that K is an arc. $\square$

Lemma 3.8. *Let $I_0 = I_1 = I_2 = [0,1]$, and let $G_1 \subset I_0 \times I_1$ and $G_2 \subset I_1 \times I_2$ be full and closed. Let $K = G_1 \star G_2$, and suppose K is a piecewise linear simple closed curve. Then if $\pi_i(K)$ for $i \in \{1,2\}$ contains a point of order $n \geq 3$, K contains a point of order $m \geq 3$.*

Proof. Let $\mathbf{x}_1 = (x_0, x_1) \in G_1$, and suppose that $\mathbf{x}_1$ is a point of order $n \geq 3$. Then $\mathbf{x}_1$ is the intersection of the endpoints of n straight lines.

Suppose $x_1 \in \{0,1\}$. If $x_1 = 1$, then since $\mathbf{x}_1$ has n straight lines connected to it, there will be a non-empty open interval $(a_1, 1) \subset I_1$ such that each of the n straight lines projects either onto $(a_1, 1)$, or entirely onto $\{1\}$. Then since G_2 is full, connected, and piecewise linear, there is a point $\mathbf{x}_2 = (1, x_2) \in G_2$ that is the endpoint of a straight line that projects onto some non-empty open interval $(a_2, 1) \subset I_1$. Then by Lemma

2.13, there are at least $n \geq 3$ straight lines connecting to $(x_0, 1, x_2)$ in K; hence, $(x_0, 1, x_2)$ has order $m \geq n \geq 3$. This case is entirely similar if $x_1 = 0$.

Otherwise, suppose $x_1 \in (0, 1)$. Then we know from Lemma 3.6 that in G_2 there is either a point $\mathbf{x}_2 = (x_1, x_2)$ that satisfies condition (1), or a line l_2 that satisfies condition (2). We will consider both cases.

Case 1: $\mathbf{x}_2$ is the intersection of the endpoints of at least two straight lines l_1 and l_2 that project onto open intervals in I_1 either side of x_1.

In this case, there is a neighbourhood N of $\mathbf{x}_1$ such that for each straight line q in N that intersects at x_1, there is a straight line l in $\{l_1\} \cup \{l_2\} \cup \{\mathbf{x}_2\}$ such that q and l project onto the same subset of I_1. By Lemma 2.13, for each such line l_1 there will be a straight line in the Mahavier product. Therefore, the point $(x_0, x_1, x_2) \in K$ is the intersection of (at least) n straight lines, and so has order $m \geq n \geq 3$.

Case 2: For each side S of x_1, there is a point $\mathbf{x}_{2,S}$ in the bonding space G_2 such that $\mathbf{x}_{2,S}$ is the intersection of the endpoints of two straight lines, one that projects entirely onto $\{x_1\}$ (this is l_2 in the condition in the lemma) and the other that projects onto an open interval in S.

Suppose in this case that there is a straight line l_1 in G_1 that has an endpoint at $\mathbf{x}_1$, and l_1 projects entirely onto $\{x_1\}$ in I_1. Then since there is a line l_2 in G_2 that also projects entirely onto $\{x_1\}$ in I_1, there will be points of the form (a, x_1, b) (where $(a, x_1) \in l_1$ and $(x_1, b) \in l_2$) in K, so there will be a 2-cell in K, a contradiction to K being a simple closed curve.

Therefore, we know that in this case every straight line that meets at $\mathbf{x}_1$ in G_1 projects onto an open interval on one side of x_1 in I_1. Since there are at least three straight lines that do this, there must be an interval $U \subset I_1$ on one side S of x_1 that has at least two straight lines projecting onto it. Then by the hypothesis for this case, there is also a point $\mathbf{x}_{2,S} = (x_1, x_2)$ that has a straight line with $\mathbf{x}_{2,S}$ as an endpoint which projects onto an open interval that is a subset of U, along with another straight line with $\mathbf{x}_{2,S}$ as an endpoint that projects entirely onto $\{x_1\}$. Then by Lemma 2.13, there will be at least two straight lines in the Mahavier product whose endpoints connect at the point $(x_0, x_1, x_2) \in K$. There will also be another line in the Mahavier product of the form (x_0, x_1, a), where (x_1, a) is a point in the line in G_2 that projects entirely onto $\{x_1\}$. Therefore, there will be at least three straight lines in K whose endpoints meet at (x_0, x_1, x_2).

In either case there is a point in K of order at least 3. The proof works in an entirely similar manner if a point $\mathbf{x}_2 \in G_2$ has order $n \geq 3$. $\square$

Lemma 3.9. *Let $I_0 = I_1 = I_2 = [0,1]$, and let $G_1 \subset I_0 \times I_1$ and $G_2 \subset I_1 \times I_2$ be full and closed. Let $K = G_1 \star G_2$, and suppose K is a piecewise linear continuum. Then if G_1 and G_2 are both simple closed curves, K is not a simple closed curve.*

Proof. Since K projects fully onto each factor space, there is at least one point $\mathbf{x}_1 = (x_0, x_1)$ in G_1 such that $x_1 = 1$ and at least one point $\mathbf{x}_2 = (x_1, x_2)$ in G_2 such that $x_1 = 1$. For the point $\mathbf{x}_1$ consider the component C_1 of $G_1 \cap ([0,1] \times \{1\})$ that contains $\mathbf{x}_1$. C_1 is either a singleton or an arc. Similarly, for the point $\mathbf{x}_2$, consider the component C_2 of $G_2 \cap (\{1\} \times [0,1])$ that contains $\mathbf{x}_2$. C_2 will also be a singleton or an arc.

There are three cases to consider.

Case 1: C_1 and C_2 are both singletons. By Lemma 3.5, each graph is piecewise linear, and we are assuming each graph is a simple closed curve, so the points C_1 and C_2 have order 2 by Proposition 2.7. In this case, there must be two straight lines $l_{1,1}$ and $l_{1,2}$ in G_1 and an interval $(a_1, 1) \subset I_1$ such that each straight line projects onto $(a_1, 1)$. Similarly, there must be two straight lines $l_{2,1}$ and $l_{2,2}$ in G_2 and an interval $(a_2, 1) \subset I_2$ such that each straight line projects onto $(a_2, 1)$. Let $a = \max\{a_1, a_2\}$.

Consider what happens in K around the point $(x_0, 1, x_2)$ (the coordinates x_0 and x_2 are taken from the points $\mathbf{x}_1$ and $\mathbf{x}_2$). For each line $l_{1,1}$ and $l_{1,2}$, by Lemma 2.13, there will be two lines which appear in K, since $l_{2,1}$ and $l_{2,2}$ in G_2 also project onto $(a, 1)$ in I_1 and meet at $(x_2, 1)$.

Therefore, there will be a total of four distinct straight lines in K that meet at $(x_0, 1, x_2)$, so this point has order 4, a contradiction to K being a simple closed curve.

Case 2: C_1 is a singleton and C_2 is a non-degenerate interval. Similar to case 1, there will be two lines $l_{1,1}$ and $l_{1,2}$ in G_1 that meet at $C_1 = \mathbf{x}_1 = (x_0, 1)$. These will project onto an interval $(a_1, 1)$ in I_1. In G_2, there is a non-degenerate straight line with second coordinate 1; this is C_2. As each point of G_2 has order 2, there will also be straight lines that connect to the endpoints of C_2, which project onto an interval $(a_2, 1)$ in I_1. Let l_2 be one of those, and suppose l_2 intersects C_2 at the point $(1, x_2)$ in G_2. Let $a = \max\{a_1, a_2\}$.

Then by Lemma 2.13, in K there will be three straight lines that intersect at the point $(x_0, 1, x_2)$: two resulting from $l_{1,1}$ and $l_{1,2}$ meeting l_2, and one of the form $(x_0, 1, \alpha)$, where $(1, \alpha) \in C_2$.

The case where C_2 is a singleton and C_1 is a non-degenerate interval is entirely similar.

Case 3: C_1 and C_2 are both non-degenerate intervals. In this case, K will contain a subset of points of the form $(x_0, 1, x_2)$, where $x_0 \in C_1$ and

$x_2 \in C_2$. Therefore, K contains a 2-cell, and so is not a simple closed curve. □

Lemma 3.10. *Let $I_0 = I_1 = I_2 = [0,1]$, and let $G_1 \subset I_0 \times I_1$ and $G_2 \subset I_1 \times I_2$ be continua. Let $K = G_1 \star G_2$, and suppose K is a piecewise linear simple closed curve. Then if either G_1 or G_2 is an arc, K is an unknot.*

Proof. Suppose, without loss of generality, that G_1 is an arc. Then $K \subset (G_1 \times [0,1])$, so K is embedded in a (piecewise linear) plane; hence, by Corollary 2.5, K is an unknot. □

The following is the main theorem.

Theorem 3.11. *Let $I_0 = I_1 = I_2 = [0,1]$, and let $G_1 \subset I_0 \times I_1$ and $G_2 \subset I_1 \times I_2$ be continua. If $K = G_1 \star G_2$ is a piecewise linear simple closed curve, then K is an unknot.*

Proof. If the projection $\rho_{j,i}(\pi_j(K))$ of K onto a factor space I_i is a singleton, then the graph G_j must be either a singleton or an arc. In the first case, by Lemma 3.7, K is an arc, so not a simple closed curve. For the second case, by Lemma 3.10, K is an unknot.

Thus, we can assume that if K is non-trivial, K projects fully onto all factor spaces.

If G_1 or G_2 contains a point of order $n \geq 3$, then by Lemma 3.8, K contains a point of order $n \geq 3$, so K is not a simple closed curve. So the order of every point in G_1 and G_2 must be either 1 or 2. Therefore, by Proposition 2.7, each is either a simple closed curve or an arc.

Suppose neither G_1 nor G_2 is an arc, then they must both be simple closed curves. Then by Lemma 3.9, K is not a simple closed curve, contradicting the fact that it is a knot. So we conclude that either G_1 or G_2 must be an arc.

Then by Lemma 3.10, K is an unknot. □

4. Discussion

The definition of a knot in Definition 2.1 requires the embedding of the knot to be piecewise linear. This is a definition commonly used in the study of knot theory (indeed, it was taken directly from [6]). The reason this is often used (instead of simply the "embedding of a simple closed curve in $\mathbb{R}^3$") is that it prevents pathological examples, for example wild knots, but allows the study of many of the properties of interest to a knot theorist. It would, however, be disingenuous to pretend that, in this context, requiring the Mahavier product to be piecewise linear does not exclude a large number of examples that are of interest. We have shown that the only piecewise linear knot that is a Mahavier product is the unknot. This does not mean that it is not possible to create non-piecewise linear knots as Mahavier products, even if they are "well behaved" in a knot theoretic sense.

The reason that knots are restricted to be piecewise linear in this paper is that it makes them much easier to work with. If each bonding space is a piecewise linear finite graph, then at any point there will be a neighbourhood consisting of finitely many straight lines connecting at that point. This means the order of points in the Mahavier product can be easily calculated, which is central to the proof of the main theorem.

An answer to the following problem would generalise Theorem 3.11 to include any knot and also go a long way to advancing the study of what continua (in particular, finite graphs) are constructible as Mahavier products.

Problem 4.1. Let $M = \bigstar_{i=1}^{n} G_i$ be a Mahavier product and suppose M is a finite graph. Let $\varepsilon > 0$. Does there exist a sequence of piecewise linear bonding spaces $G_1^*, G_2^*, \dots G_n^*$, where $d_H(\bigstar_{i=1}^{n} G_i, \bigstar_{i=1}^{n} G_i^*) < \varepsilon$?

The main result in this paper is fairly specific, relating to knots in $\mathbb{R}^3$. Intuitively, the reason that Theorem 3.11 is true is that if you have any non-trivial knot in $\mathbb{R}^3$ (any knot that is not equivalent to the unknot), then no matter how you embed it, it will always have "crossings" in any projection. Since projections of a Mahavier product are its bonding spaces, the information of these crossings is lost in the bonding spaces, so when you try to recreate the knot as a Mahavier product, it loses this information and cannot be recreated. In the language of Nall [9], we need $Cr(K) = K$, where K is the embedded knot in question and $Cr(K)$ is the set of crossovers of K. We have shown that this is impossible for a non-trivial knot.

On the question of what continua are possible to be constructed as Mahavier products, this paper does not strictly provide much of an answer. Every knot is a simple closed curve (i.e., a circle), and a circle can quite trivially be constructed as a Mahavier product in $[0,1]^3$. What it does show is that certain *embeddings* of a circle are not possible to be constructed as Mahavier products. This highlights the importance of the embedding of a continuum when considering whether it is possible to create it as a Mahavier product.

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